EMERSON, W.R., and F.P. GREENLEAF
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Covering Properties and Følner Conditions for Locally Compact Groups

W. R. EMERSON and F. P. GREENLEAF

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1.1. Summary

Let G be a locally compact group with left Haar measure m_G on the Borel sets $\mathbb{B}(G)$ (generated by open subsets) and write $|E|=m_G(E)$. Consider the following geometric conditions on the group G.

(FC) If $\varepsilon>0$ and compact set $K\subset G$ are given, there is a compact set U with $0<|U|<\infty$ and $|xU\triangle U|/|U|<\varepsilon$ for all $x\in K$.

(A) If $\varepsilon>0$ and compact set $K\subset G$, which includes the unit, are given there is a compact set U with $0<|U|<\infty$ and $|KU\triangle U|/|U|<\varepsilon$.

Here $A\triangle B=(A\backslash B)\cup(B\backslash A)$ is the symmetric difference set; by regularity of m_G it makes no difference if we allow U to be a Borel set. It is well known that $(A)\Rightarrow(FC)$ and it is known that validity of these conditions is intimately connected with "amenability" of G: the existence of a left invariant mean on the space $CB(G)$ of all continuous bounded functions. We show, for arbitrary locally compact groups G, that (amenable)$\Leftrightarrow(FC)\Leftrightarrow(A)$. The proof uses a covering property which may be of interest by itself: we show that every locally compact group G satisfies.

(C) For at least one set K, with $\operatorname{int}(K)\neq\emptyset$ and $\bar{K}$ compact, there is an indexed family $\{x_\alpha:\alpha\in J\}\subset G$ such that $\{Kx_\alpha\}$ is a covering for G whose covering index at each point g (the number of $\alpha\in J$ with $g\in Kx_\alpha$) is uniformly bounded throughout G.

1.2. Preliminaries

Let $L^\infty(G)$ be the Banach space (with ess. sup norm) of all bounded Borel functions, identifying functions which differ only on a locally m_G-null set in G (as in HEWITT-ROSS [3], section 11). A linear functional m on L^∞ is a *mean* if

$$m(\bar{f})=\overline{m(f)}; \tag{1}$$

$$f\geqq 0 \;\Rightarrow\; m(f)\geqq 0; \tag{2}$$

and

$$m(1)=1, \tag{3}$$

where 1 is the constant function. Obviously $\|m\|=1$ and the means on L^∞ form a weak $*$ compact set $\Sigma\subset(L^\infty)^*$. A mean is *left invariant* (m is a *LIM*) if $m({}_x f) = m(f)$ for all $x \in G$, where we define ${}_x f(t) = f(x^{-1} t)$, and we say that G is *amenable* if there is at least one LIM on L^∞. Similar considerations hold for other left invariant spaces X of functions on G, such as $X = CB(G)$, and for these spaces there is a corresponding notion of amenability; however, all of the standard variants of the definition of amenability are now known to be equivalent, see [2], section 1.

In an early paper Følner showed that for *discrete* groups we have (amenable)$\Leftrightarrow(FC)$; recently the same result has been proved [2] for a large class of locally compact groups: those which have an open normal subgroup H which is *almost connected* (H/H_0 is compact, where H_0 is the identity component). At the same time Leptin [7, 8] considered the group invariant $I(G)$:

$$I(G) = \sup \left\{ \inf \left\{ \frac{|K U|}{|U|} : U \in \mathscr{K}, |U| > 0 \right\} : K \in \mathscr{K} \right\} \geqq 1,$$

where $\mathscr{K}$ is all compacta in G; it is easily verified that $I(G) = 1 \Leftrightarrow (A)$ holds. In [8] Leptin shows that, for the class of groups just described, G is amenable$\Leftrightarrow(A)$ holds. The methods in [2] and [7] depend on structure theory of locally compact groups and do not seem to extend to general locally compact groups.

For any G we have $(A) \Rightarrow (FC)$: for if (ε, K) is given let U be chosen to satisfy (A) for $(\varepsilon/2, K_1)$ where $K_1 = K \cup K^{-1} \cup K K^{-1}$. If $k \in K$ then

$$\begin{aligned} |k U \triangle U| &= |k U \setminus U| + |U \setminus k U| = |k U \setminus U| + |k^{-1} U \setminus U| \leqq 2 |K_1 U \setminus U| \\ &= 2 |K_1 U \triangle U| < \varepsilon |U| \end{aligned}$$

as required. Conversely if G is discrete we have $(FC) \Rightarrow (A)$: let (ε, K) be given, with $K = \{x_1, \ldots, x_N\}$ including the unit, and choose U to satisfy (FC) for $(\varepsilon/N, K K^{-1})$. Then

$$|K U \triangle U| = |K U \setminus U| = \left| \bigcup_{i=1}^{N} (x_i U \setminus U) \right| \leqq \sum_{i=1}^{N} |x_i U \triangle U| < \varepsilon |U|.$$

One of our main theorems, based on the covering property (C), asserts:

Theorem. *If G is any locally compact group then $(FC) \Rightarrow (A)$ (thus $(FC) \Leftrightarrow (A)$).*

We shall also include, for completeness, a proof of the following result which is based on some unpublished remarks of Ryll-Nardzewski.

Theorem. *If G is any locally compact group then (amenable)$\Rightarrow(FC)$.*

It will be a straightforward matter to see that $(FC) \Rightarrow$(amenable), so we conclude that (amenable)$\Leftrightarrow(FC)\Leftrightarrow(A)$.

Remarks. In (A) our compact set K is required to include the unit $e \in G$; if we indicate by (A') the same condition with $K \neq \emptyset$ the only restriction on K it is helpful to observe that $(A) \Leftrightarrow (A')$. Implication $(\Leftarrow)$ is trivial; conversely let $K \neq \emptyset$ be given, let $K' = K \cup \{e\}$. Then by (A) there is a compact set U with

$0<|U|<\infty$ and $|K'U\triangle U|=|K'U\setminus U|<\frac{1}{2}\varepsilon|U|$. Now $|KU\triangle U|=|KU\setminus U|+|U\setminus KU|$, but note that

$$|U\cap KU|+|U\setminus KU|=|U|\leqq|KU|=|KU\setminus U|+|KU\cap U|,$$

which implies $|U\setminus KU|\leqq|KU\setminus U|$. Thus

$$|KU\triangle U|\leqq 2|KU\setminus U|\leqq 2|K'U\setminus U|<\varepsilon|U|$$

since $K'\supset K$.

1.3. A Localization Problem

Present methods of proving (amenable)$\Rightarrow$(FC) for general groups demonstrate only the existence of a set U satisfying (FC) for a pair (ε, K) if G is amenable, but provide no hint of how such sets U are located in G or how they may be constructed. Various "localization problems" dealing with such questions are considered in [10] for condition (FC) and there are similar problems for condition (A). If (P) is one of the conditions (FC), (A) the simplest localization problem asks whether we may choose U to satisfy (P) and *simultaneously* include a prescribed compact set E:

(P_{loc}) Let (ε, K) be given, as in (P), along with any other compact set $E\subset G$. Then there is a compact set U satisfying (P) for (ε, K) such that $U\supset E$.

This localization problem is resolved as follows.

Lemma 1.3.1. *For any locally compact group,* (P)$\Rightarrow$(P_{loc}) *for either of the conditions* (P)=(A), (FC).

Proof. We deal the case (P)=(A), the proof for (FC) is much the same. Let $\varepsilon>0$ and compact set K, including the unit, be given along with a compact set E. If G is compact we may take $U=G$ and obviously satisfy (A_{loc}), so assume G is not compact. Suppose we can find compacta $\{U_n\colon n=1,2,\ldots\}$ with $|U_n|\geqq n$ such that

$$|KU_n\triangle U_n|<\frac{\varepsilon}{2}|U_n| \qquad \text{for } n=1,2\ldots;$$

then for large n we may take $U=U_n\cup E\supset E$ and satisfy (A) for (ε, K). In fact we have $|U|\geqq|U_n|$, so

$$\frac{|KU\triangle U|}{|U|}\leqq\frac{|KU\triangle U|}{|U_n|}\leqq\frac{|KU\triangle KU_n|}{|U_n|}+\frac{|KU_n\triangle U_n|}{|U_n|}+\frac{|U_n\triangle U|}{|U_n|},$$

where

$$\frac{|KU\triangle KU_n|}{|U_n|}=\frac{|(KU_n\cup KE)\setminus KU_n|}{|U_n|}\leqq\frac{|KE|}{|U_n|}\longrightarrow 0,$$

$$\frac{|U\triangle U_n|}{|U_n|}\leqq\frac{|E|}{|U_n|}\longrightarrow 0,$$

while

$$|KU_n\triangle U_n|<\frac{\varepsilon}{2}|U_n| \qquad \text{for all } n.$$

To see we can find such a sequence $\{U_n\}$ let U be any compact set satisfying (A) for the pair $(\varepsilon/2, K)$. As G is non-compact there is some $g_1 \in G$ such that $Ug_1 \cap K^{-1}KU = \emptyset$. We may assume $\Delta(g_1) \geqq 1$ where Δ is the modular function on G: this is trivial if G is unimodular, otherwise let $H = \{x \in G\colon \Delta(x) = 1\}$ – a closed normal subgroup with G/H identified as a subgroup in $\mathbb{R}$ (the real numbers). If $\pi\colon G \to G/H$ is the quotient homomorphism, $\pi(K^{-1}KU)$ is compact and there is some $g_1 \in G$ with $\Delta(g_1) \geqq 1$ [i.e. $\pi(g_1) \geqq 0$ if we identify $G/H \subset \mathbb{R}$] such that $\pi(U)\,\pi(g_1) \cap \pi(K^{-1}KU) = \emptyset$, which implies $Ug_1 \cap K^{-1}KU = \emptyset$. Let $U_1 = U \cup Ug_1$, so

$$|U_1| = |U| + |U g_1| = [1 + \Delta(g_1)]\,|U| \geqq 2\,|U|;$$

then $KU \cap Ug_1 = \emptyset$ and $KUg_1 \cap U = \emptyset$ so that

$$|K U_1 \,\triangle\, U_1| = |(K U \cup K U g_1) \backslash (U \cup U g_1)| = |K U \backslash U| + |(K U \backslash U)\, g_1|$$

$$= [1 + \Delta(g_1)]\,|K U \backslash U| < \frac{\varepsilon}{2}\,|U| \cdot [1 + \Delta(g_1)] = \frac{\varepsilon}{2}\,|U_1|.$$

Next take $g_2 \in G$ so that $U_1 g_2 \cap K^{-1}KU_1 = \emptyset$ and $\Delta(g_2) \geqq 1$, and define $U_2 = U_1 \cup U_1 g_2$, so that $|U_2| = [1 + \Delta(g_2)]\,|U_1| \geqq 2^2 |U|$. The same computations show

$$|K U_2 \,\triangle\, U_2| < \frac{\varepsilon}{2}\,|U_2|.$$

Continuing inductively we get the desired sequence $\{U_n\}$ with $|U_n| \geqq 2^n |U|$. Q.E.D.

1.4. Amenability and (FC)

Theorem 1.4.1. *For any locally compact group G, (amenable)$\Leftrightarrow(FC)$.*

Proof. To see that $(FC) \Rightarrow$(amenable): direct the system $J = \{(\varepsilon, K)\}$ in the obvious way, pick U_j to satisfy (FC) for the pair $j = (\varepsilon, K)$, and write φ_j for the normalized characteristic function of U_j. For each $x \in G$ we have $\|{}_x\varphi_j - \varphi_j\|_1 = |x U_j \triangle U_j| / |U_j| \to 0$; but each φ_j determines a function $\langle \varphi_j, f \rangle = \int \varphi_j f \, dm_G$ which is a mean on $CB(G)$ and, since the set Σ of all means on CB is weak-$*$ compact, there is at least one weak $*$ limit point m for the net $\{\varphi_j\} \subset \Sigma$. It is trivial to check that m is left invariant.

We prove the converse in two lemmas: the first is adapted from [10] and the second is an unpublished result due to RYLL-NARDZEWSKI.

Lemma 1.4.2. *Any amenable locally compact group satisfies the following measure theoretic Følner condition.*

(FC^*) Let $\varepsilon > 0$, $\delta > 0$ and a compact set $K \subset G$ be given. Then there is a compact set U with $0 < |U| < \infty$, and a Borel set $N \subset K$ with $|N| < \delta$, such that $|x U \triangle U| / |U| < \varepsilon$ for all $x \in K \backslash N$.

Proof. If $|K| = 0$ there is nothing to prove (take $N = K$) so assume $|K| > 0$. It is well known (see [4]) that G is amenable$\Leftrightarrow$for every (ε', K') there is some

$\varphi \in P(G) = \{f \in L^1(G): f \geqq 0, \int f\, dm_G = 1\}$ with $\|{}_x\varphi - \varphi\|_1 < \varepsilon'$, all $x \in K'$. Choose $\varphi \in P(G)$ corresponding to $K' = K$, $\varepsilon' = \varepsilon\delta/|K|$. We may assume that φ is a simple function of the form

$$\varphi = \sum_{i=1}^{N} \lambda_i \varphi_{A_i} \quad \text{where } \lambda_i \geqq 0, \quad \sum_{i=1}^{N} \lambda_i = 1,$$

the A_i are compacta with $0 < |A_i|$ and $A_1 \supset \cdots \supset A_N$, and $\varphi_{A_i} \in P(G)$ is the normalized characteristic function of A_i. It is a straightforward matter to see that

$$\|{}_x\varphi - \varphi\|_1 = \sum_{i=1}^{N} \lambda_i \|{}_x\varphi_{A_i} - \varphi_{A_i}\|_1 = \sum_{i=1}^{N} \lambda_i \frac{|x A_i \triangle A_i|}{|A_i|} \leqq \frac{\varepsilon\delta}{|K|}$$

since $A_1 \supset \cdots \supset A_N$. Integrating over $x \in K$ we get

$$\varepsilon\delta > \sum_{i=1}^{N} \lambda_i \int_K \frac{|x A_i \triangle A_i|}{|A_i|}\, dm_G(x)$$

and since this is a convex sum we must have

$$\varepsilon\delta > \int_K \frac{|x A \triangle A|}{|A|}\, dm_G(x)$$

for at least one $A = A_i$. The integrand can be $\geqq \varepsilon$ only on a set $N \subset K$ with $|N| < \delta$, so we have $|xA \triangle A|/|A| < \varepsilon$ all $x \in K \backslash N$. Q.E.D.

Lemma 1.4.3. *For any locally compact group,* $(FC^*) \Rightarrow (FC)$.

Proof (Ryll-Nardzewski, unpublished). It suffices to prove (FC) for (ε, K) with $|K| > 0$. Let $A = K \cup KK$ so $|kA \cap A| \geqq |kK| = |K|$, all $k \in K$. If $\delta = \frac{1}{2}|K|$, then for *any* subset $N \subset A$ with $|A \backslash N| < \delta$ we have $K \subset NN^{-1}$; in fact if $k \in K$ we have

$$2\delta = |K| \leqq |kA \cap A| \leqq |kN \cap N| + |A \backslash N| + |k(A \backslash N)| < |kN \cap N| + 2\delta,$$

so that $kN \cap N \neq \emptyset$ and $k \in NN^{-1}$. Apply (FC^*) to $(\varepsilon/2, A)$ and $\delta = \frac{1}{2}|K|$. There is a compact set U with $0 < |U| < \infty$, and Borel set $N \subset A$ with $|A \backslash N| < \delta$, such that $|xU \triangle U| < \frac{1}{2}\varepsilon|U|$, all $x \in N$. For $x_1, x_2 \in N$ this means

$$\begin{aligned} |x_1 x_2^{-1} U \triangle U| &= |x_2^{-1} U \triangle x_1^{-1} U| \leqq |U \triangle x_2^{-1} U| + |U \triangle x_1^{-1} U| \\ &= |x_2 U \triangle U| + |x_1 U \triangle U| < \varepsilon |U|, \end{aligned}$$

so $|xU \triangle U| < \varepsilon|U|$ for all $x \in K \subset NN^{-1}$. Q.E.D.

2.1. The Covering Property (C)

The sets K considered in (C) are precisely the relatively compact sets with interior. It suffices to consider K which are relatively compact neighborhoods of the unit, for if K is such a set and $\mathscr{X} = \{x_\alpha\}$ makes $\{Kx_\alpha\}$ a covering whose index is uniformly bounded by $n < \infty$, and if $p \in G$ is given, then the neighborhood Kp of p and translations $\{p^{-1}x_\alpha\}$ give a similar covering. If K and $\mathscr{X} = \{x_\alpha\}$ are given, with $\{Kx_\alpha\}$ a covering of G, we denote the index of this covering at $y \in G$ by $I(K, \mathscr{X}, y)$. Finally, we may replace the phrase "For at least one set K..." in our definition of property (C) with "For every set K...", as the following lemma shows.

Lemma 2.1.1. *Let K and $\mathscr{X} = \{x_\alpha : \alpha \in J\}$ be given with $\operatorname{int}(K) \neq \emptyset$, $\bar{K}$ compact, and $\{Kx_\alpha\}$ a covering of G with covering index uniformly bounded throughout G. Let K' be any other set with $\operatorname{int}(K') \neq \emptyset$, $\bar{K}'$ compact. Then there is a family $\mathscr{Y} = \{y_\beta : \beta \in I\}$ such that $\{K' y_\beta\}$ is a covering whose covering index is uniformly bounded throughout G.*

Proof. Assume $I(K, \mathscr{X}, x) \leqq N < \infty$ in G. Choose $\{g_1, \ldots, g_n\} \subset G$ so

$$K'' = \bigcup_{i=1}^{n} K' g_i \supset K;$$

then $\operatorname{int}(K'') \neq \emptyset$ and $\bar{K}''$ is compact so we may choose $\{h_1, \ldots, h_m\} \subset G$ such that

$$\bigcup_{j=1}^{m} h_j K \supset K''.$$

Let $I = \{(i, \alpha) : i = 1, 2, \ldots, n;\ \alpha \in J\}$ and define indexed family $\mathscr{Y} = \{y_{(i,\alpha)} = g_i x_\alpha\}$. We assert that $\{K' y_{(i,\alpha)}\}$ is a covering whose index is $\leqq N \cdot m \cdot n$ throughout G; evidently

$$\bigcup K' y_{(i,\alpha)} = \bigcup K' g_i x_\alpha \supset \bigcup K x_\alpha = G.$$

Assume there is some $x \in G$ with $I(K', \{y_{(i,\alpha)}\}, x) > Nmn = N'$. Then for more than N' indices (i, α) we have $x \in K' y_{(i,\alpha)} = K' g_i x_\alpha$; but there are only n choices of i, so there is some $1 \leqq i_0 \leqq n$ such that $x \in K' g_{i_0} x_\alpha$ for more than $N'/n = Nm$ choices of index $\alpha \in J$. For these α we have

$$x \in K' g_{i_0} x_\alpha \subset K'' x_\alpha \subset \bigcup_{j=1}^{m} h_j K x_\alpha,$$

and there are only m choices for j, so there is some $1 \leqq j_0 \leqq m$ such that $x \in h_{j_0} K x_\alpha$ for more than $Nm/m = N$ choices of $\alpha \in J$. This implies that $h_{j_0}^{-1} x \in K x_\alpha$ for more than N choices of $\alpha \in J$: i.e. $I(K, \{x_\alpha\}, h_{j_0}^{-1} x) > N$, which is a contradiction. Q.E.D.

Remark. By using the inversion symmetry $x \to x^{-1}$ of G we see there is a one to one correspondence between coverings by right translates $\{Kx_\alpha\}$ and coverings by left translates $\{x_\alpha^{-1} K^{-1}\}$. For convenience we consider only the right handed situation.

If G is a vector group ($G = \mathbb{R}^n$) or a toroid ($G = \mathbb{T}^n$) for $n \geqq 1$ the covering property (C) is evidently valid; moreover it is also clear for such G that there are relatively compact neighborhoods of the unit $\{K_i : i = 1, 2, \ldots\}$ which form a neighborhood basis, and corresponding families of translations $\mathscr{X}_i = \{x_\alpha^i : \alpha \in J_i\}$, which give coverings of index $I(K_i, \mathscr{X}_i, y) = 1$ for all $y \in G$ – i.e. $\{K_i x_\alpha^i\}$ is a non-overlapping *partition* of G by translates of the increasingly small neighborhoods K_i. All discrete groups have this stronger property (take $K=\{e\}$). If G is compact it obviously has property (C) and in fact the trivial choice of $K=G$, $\mathscr{X}=\{e\}$ gives a partition; however it is not at all clear that there are *arbitrarily small* neighborhoods K of the unit which partition G for a suitable family of right translates – the three dimensional real orthogonal group $G=SO(3, \mathbb{R})$ presents an interesting unsolved problem in this respect. With these examples in mind we define the group invariant $\rho(G)$, the *infinitesimal covering index of* G by directing the family $\mathscr{F}$ of all relatively compact neighborhoods of the unit downwards by inclusion and defining

$$I(K,\mathscr{X})=\sup\{I(K,\mathscr{X},y)\colon\ y\in G\}, \qquad \text{all } K,\mathscr{X},$$

$$\rho(K,G)=\inf\{I(K,\mathscr{X})\colon \text{translates of } K \text{ by } \mathscr{X} \text{ cover } G\}, \qquad \text{all } K,$$

$$\rho(G)=\lim\inf\{\rho(K,G)\colon K\in\mathscr{F}\}.$$

We shall inquire which groups have $\rho(G)=1$; note that $\rho(G)$ is integer valued (possibly $+\infty$) and that, by the above remarks, all discrete groups, toroids, and vector groups have $\rho(G)=1$. Although $\rho(G)<\infty\Rightarrow(C)$ it seems possible that $\rho(G)=+\infty$ even if G has property (C). We shall demonstrate that (at least) all solvable Lie groups have $\rho(G)=1$, in the course of proving the lemmas which establish the main theorem of this section:

Theorem 2.1.2. *Every locally compact group has property* (C).

Lemma 2.1.3. *If G is a locally compact group, K a compact normal subgroup such that G/K has property (C), then G has property (C).*

Proof. Let $V\subset G/K$ be a compact neighborhood of the unit $e'\in G/K$ and select $\{\xi_\alpha\}$ so $\{V\xi_\alpha\}$ covers G/K with covering index $\leqq n<\infty$ at every point $\xi\in G/K$. Pick $x_\alpha\in G$ so $\pi(x_\alpha)=\xi_\alpha$, where $\pi\colon G\to G/K$ is the quotient map, and set $U=\pi^{-1}(V)$, a compact neighborhood of the unit in G. Then $\pi(Ux_\alpha)=V\xi_\alpha$ all $\alpha\in J$ and $I(U,\{x_\alpha\},x)=I(V,\{\xi_\alpha\},\pi(x))$ for all $x\in G$. Q.E.D.

Lemma 2.1.4. *Let G be locally compact and H an open (not necessarily normal) subgroup with property (C). Then G has property (C) and $\rho(G)=\rho(H)$.*

Proof. Let $K\subset H$ and $\mathscr{X}=\{x_\alpha\}\subset H$ satisfy (C) for the group H with covering index $I_H(K,\mathscr{X},y)\leqq n<\infty$. Let $\{y_\beta\}$ be a (discrete) transversal for the right cosets $G/H=\{Hy\colon y\in G\}$ – a set with one element in each coset, an let $z_{(\alpha,\beta)}=x_\alpha y_\beta$. Then

$$G=\bigcup_\beta H\,y_\beta=\bigcup_{\alpha,\beta} K\,x_\alpha\,y_\beta$$

so $\{Kz_{(\alpha,\beta)}\}$ covers G and its covering index at any point $x\in G$ is $\leqq n$. Slight modifications show $\rho(G)=\rho(H)$. Q.E.D.

Lemma 2.1.5. *Let G be a Lie group and N a closed normal subgroup. If N and G/N have property (C), so does G; furthermore $\rho(G)\leqq\rho(G/N)\cdot\rho(N)$.*

Proof. The estimate on $\rho(G)$ will be clear from our constructions. Let $\pi\colon G\to G/N$ be the (continuous, open) canonical homomorphism. As we are dealing with Lie groups it is clear that there is a continuous (indeed analytic) local cross section for the cosets of N; i.e. if we consider any sufficiently small (compact) neighborhood K_1 of the unit $e' \in G/N$, there is a continuous "cross section map" $\tau: K_1 \to G$ [τ can be taken analytic, but is *not* a homomorphism] such that $\pi \circ \tau = id$. Write $T = \tau(K_1) \subset G$; evidently the maps π, τ are homeomorphisms between the compact set $T \subset G$ and $K_1 \subset G/N$, T meets each coset of $\pi^{-1}(K_1)$ precisely once, and $\pi^{-1}(K_1)$ is a closed N-saturated neighborhood of the unit $e \in G$ with $\pi^{-1}(K_1) = T \cdot N$. Obviously we may assume $T \supset \{e\}$.

As G/N has property (C) we may choose $\{\xi_\alpha: \alpha \in J\} \subset G/N$ such that $\{K_1 \xi_\alpha\}$ covers G/N with covering index $\leqq n < \infty$ throughout G/N [we may take $n = \rho(G/N)$ if K_1 is very small and suitably chosen]. We may assume, by making slight alterations of $K_1, \{\xi_\alpha\}$ which do not affect the bound on the covering index, that the unit $e' \in \{\xi_\alpha\}$, say $\xi_0 = e'$. For each $\alpha \in J$ take $x_\alpha \in G$ with $\pi(x_\alpha) = \xi_\alpha$ (take $x_0 = e$). Next let U_1 be any compact neighborhood of the unit in N; as N has property (C) we may find $\{y_\beta: \beta \in I\} \subset N$ such that $\{U_1 y_\beta\}$ covers N with $I_N(U_1, \{y_\beta\}, y) \leqq m < \infty$ at each point $y \in N$ [if U_1 is small we may arrange that $m = \rho(N)$].

Define $U = T \cdot U_1 \subset G$. This is evidently a compact set in G; it is actually a neighborhood of the unit [so $\text{int}(U) \neq \emptyset$]: in fact we have $U = TU_1 \subset TN = \pi^{-1}(K_1)$ and for every $x \in T \cdot N$ there is a *unique* factorization $x = \sigma(x) \cdot \eta(x)$ with $\sigma(x) \in T$, $\eta(x) \in N$. These maps σ, η are continuous: in fact $\sigma(x) = \tau(\pi(x))$, which is continuous since $\tau: K_1 \to G$ is continuous, and so $\eta(x) = \sigma(x)^{-1} x$ is also continuous, giving a continuous bijection

$$\pi^{-1}(K_1) = TN \xrightarrow{\sigma \times \eta} T \times N \qquad \text{(Cartesian product space)}.$$

If $\alpha: G \times G \to G$ is the (continuous) product mapping, we have $\alpha \circ (\sigma \times \eta) = id$, so α and $\sigma \times \eta$ are homeomorphisms between $T \cdot N$ and $T \times N$. Now $T \times U_1$ is clearly a neighborhood of $(e, e) \in T \times N$, so $\alpha(T \times U_1) = TU_1 = U$ is a neighborhood of the unit in G. Finally, note that $\pi^{-1}(K_1) = TN \subset \bigcup \{TU_1 y_\beta: \beta \in I\} \subset TN$, so these sets are equal.

Let $x_{(\beta,\alpha)} = y_\beta x_\alpha$ for all $(\beta, \alpha) \in I \times J$. We assert that $\{Ux_{(\beta,\alpha)}\}$ covers G with covering index $\leqq m \cdot n$ throughout G. To see this is a covering, let $x \in G$; then there is some $\alpha \in J$ with $\pi(x) \in K_1 \xi_\alpha$, which implies

$$x \in \pi^{-1}(K_1) \cdot x_\alpha = T N x_\alpha = \bigcup \{T U_1 y_\beta x_\alpha: \beta \in I\} = \bigcup_{\alpha,\beta} \{U y_\beta x_\alpha\}$$

as required. For $x \in G$ there are at most n indices $\alpha \in J$, say $J_x = \{\alpha_1, \ldots, \alpha_n\}$, such that $\pi(x) \in K_1 \xi_\alpha$, and these are precisely the indices $\alpha \in J$ for which

$$x \in \pi^{-1}(K_1 \xi_\alpha) = \pi^{-1}(K_1) x_\alpha = \bigcup_{\beta \in I} \{T U_1 y_\beta x_\alpha\}.$$

If, for each $\alpha \in J_x$, there are at most m indices $\beta \in I$ with $x \in TU_1 y_\beta x_\alpha = Ux_{(\beta,\alpha)}$ then the covering index at x is $\leqq mn$ as required; otherwise there is some $x \in G$ and some $\alpha \in J_x$ with $x \in TU_1 y_\beta x_\alpha$ for more than m indices $\beta \in I$, and this happens if and only if $z = x x_\alpha^{-1} \in TU_1 y_\beta$ for more than m indices $\beta \in I$. Writing the unique factorization $z=\sigma(z)\cdot\eta(z)$ this means $\eta(z)\in U_1 y_\beta$ for more than m indices $\beta\in I$, contradicting the definition of m. Q.E.D.

Proof (2.1.2). If G is any locally compact group there is an open (not necessarily normal) subgroup $H\subset G$ which is almost connected (H/H_0 compact), see [9], pp. 56–58, and by 2.1.4 it suffices to show any almost connected group G has property (C). But connected, and even almost connected, groups are known to be approximable by Lie groups; there exist arbitrarily small compact normal subgroups $K\subset G$ with G/K a Lie group (see [9]). Let K be any one of these subgroups: in view of 2.1.3 it suffices to show that Lie groups have property (C), and another application of 2.1.4 reduces our considerations to connected Lie groups.

As is well known, every connected Lie group G has a unique maximal solvable connected normal subgroup; this subgroup $r(G)$ (the *radical* of G) is closed, G is semi-simple$\Leftrightarrow r(G)$ is trivial, and $G/r(G)$ is a connected semi-simple Lie group. In any connected solvable Lie group H there is a sequence of closed subgroups $H=H_n\supset\cdots\supset H_0=\{e\}$ with H_{i-1} normal in H_i and H_i/H_{i-1} either a vector group or a toroid (see [9], section 3); applying 2.1.5 several times we see that such groups satisfy (C) and in fact have $\rho(H)=1$. Applying the remark and 2.1.5 to the exact sequence $e\to r(G)\to G\to G/r(G)\to e$ we are reduced to considering connected semi-simple Lie groups. We may also assume G is center free, for if Z is the (discrete) center of G we may apply 2.1.5 to the sequence $e\to Z\to G\to G/Z\to e$ (it is well-known that the semi-simple group G/Z is center free).

If $\mathfrak{G}$ is the Lie algebra of a center free semi-simple Lie group G and $\tilde{\mathfrak{G}}$ is the group of invertible linear operators on the real vector space $\mathfrak{G}$ generated by the operators $\{\mathrm{Exp}(ad_X)\colon X\in\mathfrak{G}\}$, where $ad_X(Y)=[X,Y]$ for $X,Y\in\mathfrak{G}$, there is a natural Lie group structure on $\tilde{\mathfrak{G}}$ and there is an analytic isomorphism $G\cong\tilde{\mathfrak{G}}$. It is a basic consequence of the structure theory of semi-simple Lie algebras such as $\mathfrak{G}$ (see [9], 3.11) that $\tilde{\mathfrak{G}}$, and hence also G, has maximal compact subgroups, all of which are connected and conjugate under inner automorphisms of G; furthermore if a maximal compact subgroup K is specified, there is a closed solvable simply connected subgroup $S\subset G$ such that every $x\in G$ has a unique factorization $x=k(x)\,s(x)$ with $k(x)\in K$, $s(x)\in S$, and the map $k\times s\colon G\to K\times S$ (Cartesian product space) is bicontinuous (although not an isomorphism). In particular we have $G=KS$. Now S has property (C), so if $V\subset S$ is a compact neighborhood of the unit in S there is a family $\{s_\alpha\}\subset S$ such that $\{Vs_\alpha\}$ covers S with index $I_S(V,\{s_\alpha\},t)\leqq n<\infty$ on S. Let $U=KV\subset G$; since $k\times s\colon G\to K\times S$ is bicontinuous, the compact set U is a neighborhood of the unit in G. Evidently $G=KS=\bigcup\{KVs_\alpha\colon \alpha\in J\}$, so $\{Us_\alpha\}$ covers G. Furthermore if $x=k(x)\,s(x)\in G$ then $x\in Us_\alpha=KVs_\alpha\Leftrightarrow s(x)\in Vs_\alpha$, and there are at most n indices $\alpha\in J$ for which this happens. Thus $I(U,\{s_\alpha\},x)\leqq n$ all $x\in G$. This proves 2.1.2 in full. Q.E.D.

Corollary 2.1.6. *If G is any locally compact group, there is a set K with $\mathrm{int}(K)\neq\emptyset$ and $\bar{K}$ compact and a corresponding set $\{x_\alpha\}\subset G$ such that $\{Kx_\alpha\}$*

partitions G; i.e.

$$I(K, \{x_\alpha\}, y) = 1 \qquad \textit{for all } y \in G.$$

Note. It may not be true that G has *arbitrarily small* sets K which may be translated to the right to partition G [this is the same as saying $\rho(G)=1$]. For example if $G=SO(3, R)$, then $K=G$ satisfies 2.1.6, but we conjecture that $\rho(G) \neq 1$. We comment on this at the end of this section.

Proof. This follows from the discussion of 2.1.2 if we strengthen 2.1.5 as follows.

Lemma 2.1.7. *If G is a connected Lie group and N a closed normal subgroup and if G/N, N have relatively compact neighborhoods of the unit which may be right translated to give partitions, then G has this property too.*

Proof (Sketch). Let $K_1 \subset G/N$ be a compact neighborhood of the unit with boundary of measure zero and continuous cross section map $\tau_1 \colon K_1 \to G$. By covering G/N with translates of K_1 we get a sequence of local cross sections

$$\{(K_i, \tau_i) \colon i=1, 2, \ldots\} \quad \text{with} \quad \bigcup_{i=1}^{\infty} K_i = G/N,$$

each compact set in G/N meets only finitely many K_i, and we may define a "piecewise continuous" global cross section $\tau \colon G/N \to G$ inductively:

$$\begin{cases} \tau(\xi) = \tau_1(\xi) & \xi \in K_1 \\ \tau(\xi) = \tau_i(\xi) & \xi \in \left(K_i \setminus \bigcup_{j=1}^{i-1} K_j\right). \end{cases}$$

It is a straightforward matter to see that if U is a compact neighborhood of the unit in N and if K is a relatively compact neighborhood of the unit in G/N, then $\tau(K)U$ is a relatively compact neighborhood of the unit in G. Furthermore, if U, K may be translated on the right to partition N, G/N respectively, the same arguments as in 2.1.5 show that $\tau(K)U$ has the same property in G. Q.E.D.

Remark. We obtain another group invariant $\lambda(G)$ which we might call *the covering index in the large* by directing the relatively compact neighborhoods of the unit $\mathscr{F}$ upwards rather than downwards and defining

$$\lambda(G) = \lim\inf\{\rho(K, G) \colon K \in \mathscr{F}\}.$$

Now it is trivial that $\lambda(G)=1$ for compact G, and the considerations in 2.1.2 (and combinatory Lemma 2.1.7) may be adapted with little difficulty to show $\lambda(G)=1$ for almost connected groups G, but it is not at all clear which discrete groups have $\lambda(G)=1$ [although this is easily seen true for abelian G].

Remark. If S is the unit sphere in $\mathbb{R}^3$ with $G=SO(3, \mathbb{R})$ acting transitively let a point $p \in S$ be fixed. Then the action $G \times S \to S$ is equivalent to the canonical action $G \times G/H \to G/H$ where $H=\{x \in G \colon x(p)=p\}$ (a 1-dimensional toroid) and $G/H = \{xH \colon x \in G\}$ is the left coset space with its usual topology [which makes

G/H homeomorphic to S under φ: $xH \to x(p)$]. We may ask whether there are arbitrary small neighborhoods V of p, and corresponding families $\mathscr{X} = \{x_\alpha\} \subset G$, such that $\{x_\alpha(V)\}$ partitions S. Since $\rho(H)=1$, an affirmative answer to this implies an affirmation answer to the corresponding question for G: i.e. $\rho(G)=1$; however, we conjecture that the answer is negative in both cases. Even the simpler question for transformation group (G, S) seems difficult to resolve.

3.1. Property (A) and Amenable Groups

We use property (C) to establish

Theorem 3.1.1. *In any locally compact group G, $(FC) \Rightarrow (A)$.*

This is an immediate consequence of the following result.

Proposition 3.1.2. *Let G be a locally compact group and $K \subset G$ any compact set which includes the unit and* $\mathrm{int}(K) \neq \emptyset$, *and let $K' = KKK^{-1}$; then there is an $\varepsilon_0 > 0$ (depending only on K) with the following property. For any $0 < \varepsilon < \varepsilon_0$ and any Borel set $U \subset G$ which satisfies (FC) relative to (ε, K'), so that*

$$0 < |U| < \infty \quad \textit{and} \quad \frac{|xU \triangle U|}{|U|} < \varepsilon \qquad \textit{for all } x \in K',$$

there is a corresponding Borel set $U' \subset U$ such that

(i) $0 < |U'| < \infty$,

(ii) $|U \backslash U'| < \sqrt{\varepsilon}\, |U|$,

(iii) $\dfrac{|KU' \triangle U'|}{|U'|} < 4MN\sqrt{\varepsilon}\, \dfrac{|K'|}{|K|}$.

Here $N = \rho(K, G)$ [as in the definition of the infinitesimal covering index $\rho(G)$] and $M = \max\{\Delta(x)\colon x \in K\}$ *where Δ is the modular function of G.*

Proof. Let $\varepsilon > 0$ be given and consider a set U which satisfies (FC) relative to (ε, K'). For any $0 < \delta < \infty$ define $E(\delta) = \{x \in U\colon |K_x| > \delta\}$, where $K_x = K' \backslash Ux^{-1}$, and define $U(\delta) = U \backslash E(\delta)$. We regard $\delta > 0$ as a parameter independent of $\varepsilon > 0$ for the time being and prove several estimates which are valid for *any* U satisfying (FC) with respect to (ε, K') if (ε, δ) lies in the region $D \subset \mathbb{R}^2$ determined by:

$$0 < \varepsilon < \min\left\{\frac{1}{4}, \frac{|K|^2}{M^2 |K'|^2}\right\} = \varepsilon_1,$$

$$\tfrac{1}{2}\varepsilon\, |K'| < \delta < \frac{|K|}{M}.$$

Then we shall demonstrate that if we require $0 < \varepsilon < \varepsilon_0$ (ε_0 to be chosen) and take $\delta(\varepsilon) = \frac{1}{2}\, |K'|\, (\varepsilon + \sqrt{\varepsilon})$, the set $U(\delta(\varepsilon))$ satisfies (i)...(iii) as required.

For $x \in K'$, $\delta > 0$ define $S(x, \delta) = \{y \in E(\delta)\colon x \in K_y\}$. Then $S(x, \delta) \subset U$ and $xS(x, \delta) \cap U = \emptyset$, for $t \in S(x, \delta) \Rightarrow x \in K_t = K' \backslash Ut^{-1}$, which$\Rightarrow$ $xt \in K't \backslash U$, which is disjoint from U. Now $|xU\triangle U|<\varepsilon|U|$ for all $x\in K'$, by hypothesis, so we have $2|U|=2|xU\cap U|+|xU\triangle U|$, which implies

$$|xU\cap U|>\left(1-\frac{\varepsilon}{2}\right)|U|.$$

Therefore $|S(x,\delta)|<\frac{1}{2}\varepsilon|U|$ since

$$\left(1-\frac{\varepsilon}{2}\right)|U|<|xU\cap U|=|x(U\setminus S(x,\delta))\cap U|\leqq|U\setminus S(x,\delta)|=|U|-|S(x,\delta)|.$$

By FUBINI's Theorem we see

$$\int_{E(\delta)}|K_t|\,dt=\int_{E(\delta)}\left[\int_{K'}\chi_{K_t}(x)\,dx\right]dt=\int_{K'}\left[\int_{E(\delta)}\chi_{K_t}(x)\,dt\right]dx=\int_{K'}|S(x,\delta)|\,dx,$$

where χ_E is the characteristic function of any set $E\subset G$. By definition of $E(\delta)$, $|K_t|>\delta$ for all $t\in E(\delta)$, so that

$$(*)\qquad \delta|E(\delta)|\leqq\int_{E(\delta)}|K_t|\,dt=\int_{K'}|S(x,\delta)|\,dx<\tfrac{1}{2}\varepsilon|U|\,|K'|.$$

If $\delta>\frac{1}{2}\varepsilon|K'|$ we see that $|E(\delta)|<|U|$, which $\Rightarrow|U(\delta)|=|U|-|E(\delta)|>0$, so condition (i) is satisfied for any set $U(\delta)$ if $(\varepsilon,\delta)\in D$.

Let $\mathscr{X}=\{x_\alpha:\alpha\in J\}$ be chosen so $\{Kx_\alpha\}$ covers G with $\sup\{I(K,\mathscr{X},y):y\in G\}=\rho(K,G)=N<\infty$, and for each $\delta>0$ let $\mathscr{X}(\delta)=\{x_\alpha\in\mathscr{X}:Kx_\alpha\cap U(\delta)\neq\emptyset\}$, which we write as $\mathscr{X}(\delta)=\{x_\alpha:\alpha\in J(\delta)\subset J\}$: then $U(\delta)\subset\bigcup\{Kx_\alpha:\alpha\in J(\delta)\}$. Next pick $p_\alpha\in Kx_\alpha\cap U(\delta)$ for each $\alpha\in J(\delta)$ and observe that for all $x_\alpha\in\mathscr{X}(\delta)$ we have

$$p\in Kx_\alpha\cap U(\delta)\ \Rightarrow\ \begin{cases}\Delta(p)\leqq M\Delta(x_\alpha)\\ |K'p\setminus U|\leqq\delta\Delta(p).\end{cases}$$

In fact,

$$p\in Kx_\alpha\ \Rightarrow\ p=kx_\alpha\ \Rightarrow\ \Delta(p)=\Delta(k)\Delta(x_\alpha)\leqq M\Delta(x_\alpha),$$

and

$$p\in U(\delta)\ \Rightarrow\ |K_p|=|K'\setminus Up^{-1}|\leqq\delta\ \Rightarrow\ |K'p\setminus U|=\Delta(p)|K'\setminus Up^{-1}|\leqq\delta\Delta(p).$$

Furthermore

$$\begin{aligned}p\in Kx_\alpha\ &\Rightarrow\ x_\alpha\in K^{-1}p\ \Rightarrow\ Kx_\alpha\subset KK^{-1}p\subset K'p\\ &\Rightarrow\ |Kx_\alpha\setminus U|\leqq|K'p\setminus U|\leqq\delta\Delta(p)\leqq\delta M\Delta(x_\alpha),\end{aligned}$$

which implies that $|Kx_\alpha\cap U|\geqq[|K|-\delta M]\,\Delta(x_\alpha)$ for all $x_\alpha\in\mathscr{X}(\delta)$. For any Borel set $V\subset G$ with $|V|<\infty$ we have $\sum\{|Kx_\alpha\cap V|:\alpha\in J\}\leqq N\cdot|V|$ since $\{Kx_\alpha\}$ covers G; in fact for any finite collection $\{x_1,\ldots,x_m\}\subset\mathscr{X}$ we have

$$\begin{aligned}\sum_{i=1}^{m}|Kx_i\cap V|&=\sum_{i=1}^{m}\int\chi_V(t)\,\chi_{Kx_i}(t)\,dt\\ &=\int\chi_V(t)\left[\sum_{i=1}\chi_{Kx_i}(t)\,dt\right]\leqq N\int\chi_V(t)\,dt=N|V|,\end{aligned}$$

since the covering index of $\{Kx_\alpha\}$ is, by hypothesis, $\leqq N$ at each point $t\in G$. Therefore we conclude, taking $V=U$, that

$$(**)\quad \begin{aligned} N\,|U| &\geqq \sum\{|Kx_\alpha\cap U|:\ \alpha\in J\}\\ &\geqq \sum\{|Kx_\alpha\cap U|:\ \alpha\in J(\delta)\}\geqq[|K|-\delta M]\sum\{\Delta(x_\alpha):\ \alpha\in J(\delta)\}.\end{aligned}$$

Now if $(\varepsilon,\delta)\in D$ we have, in particular, $\delta<|K|/M$ so $|K|-\delta M>0$.

By definition of $\{p_\alpha:\ \alpha\in J(\delta)\}$ with $p_\alpha\in Kx_\alpha\cap U(\delta)$ we have

$$U(\delta)\subset\bigcup\{Kx_\alpha:\ \alpha\in J(\delta)\}\subset\bigcup\{KK^{-1}p_\alpha:\ \alpha\in J(\delta)\}$$

which implies that

$$\begin{aligned} KU(\delta)&\subset\bigcup\{KKK^{-1}p_\alpha:\ \alpha\in J(\delta)\}=\bigcup\{K'p_\alpha:\ \alpha\in J(\delta)\}\\ &\subset U\cup\big(\bigcup\{K'p_\alpha\backslash U:\ \alpha\in J(\delta)\}\big)\end{aligned}$$

hence by $(**)$ we see that:

$$\begin{aligned} |KU(\delta)|&\leqq|U|+\sum\{|K'p_\alpha\backslash U|:\ \alpha\in J(\delta)\}\\ &\leqq|U|+\delta M\sum\{\Delta(x_\alpha):\ \alpha\in J(\delta)\}\\ &\leqq|U|+\frac{\delta MN}{|K|-\delta M}\,|U|.\end{aligned}$$

Defining ε_1 as above, let $0<\varepsilon_2\leqq\varepsilon_1$ be chosen so $(\varepsilon,\delta(\varepsilon))\in D$ for all $0<\varepsilon<\varepsilon_2$: this choice depends only on K (through the parameters M, N, $|K|$, $|K'|$) and the above estimates are valid for $\delta=\delta(\varepsilon)$, $0<\varepsilon<\varepsilon_2$. In particular if $0<\varepsilon<\varepsilon_2$ we have $|U(\delta(\varepsilon))|>0$, so (i) holds, and

$$\begin{aligned} |U\backslash U(\delta(\varepsilon))|&=|E(\delta(\varepsilon))|\leqq\tfrac{1}{2}\varepsilon\cdot\delta(\varepsilon)^{-1}\,|U|\,|K'|\\ &=\varepsilon(\varepsilon+\sqrt{\varepsilon})^{-1}\,|U|=\sqrt{\varepsilon}(1+\sqrt{\varepsilon})^{-1}\,|U|\leqq\sqrt{\varepsilon}\,|U|,\end{aligned}$$

which proves (ii) for $U(\delta(\varepsilon))$. Write $U'=U(\delta(\varepsilon))$; then

$$\frac{|KU'\,\Delta\,U'|}{|U'|}=\frac{|KU'\backslash U'|}{|U'|}=\frac{|KU'|}{|U'|}-1,$$

since K includes the unit, and

$$\begin{aligned} \frac{|KU'|}{|U'|}=\frac{|KU'|}{|U|}\,\frac{|U|}{|U'|}&\leqq\left(1+\frac{\delta(\varepsilon)MN}{|K|-\delta(\varepsilon)M}\right)\frac{|U|}{|U|-|E(\delta(\varepsilon))|}\\ &\leqq\left(1+\frac{(\varepsilon+\sqrt{\varepsilon})MN\,|K'|}{2\,|K|-(\varepsilon+\sqrt{\varepsilon})M\,|K'|}\right)(1+\sqrt{\varepsilon}).\end{aligned}$$

Note that the right side of this inequality depends only on (ε, K) and not on the particular set U which we started with. We now demonstrate that if $0<\varepsilon<\varepsilon_0$

for suitably chosen $\varepsilon_0>0$, then

$$\frac{|K\,U'|}{|U'|}<1+4MN\sqrt{\varepsilon}\,\frac{|K'|}{|K|},$$

so that

$$\frac{|K\,U'\,\triangle\,U'|}{|U'|}<4MN\sqrt{\varepsilon}\,\frac{|K'|}{|K|}.$$

Hence (i)...(iii) hold for all $0<\varepsilon<\varepsilon_0$ if we take $U'=U(\delta(\varepsilon))$.

For small $\varepsilon>0$, say $0<\varepsilon<\varepsilon_3\leqq\varepsilon_2$, the denominator in the first term of the above estimate is $\geqq|K|$ (recall $|K|>0$ by hypothesis), so for these values of $\varepsilon>0$ we have

$$\frac{|K\,U'|}{|U'|}\leqq 1+\varepsilon^{\frac{1}{2}}\left[MN\frac{|K'|}{|K|}+1\right]+\varepsilon\left[MN\frac{|K'|}{|K|}\right]+\varepsilon^{\frac{3}{2}}\left[MN\frac{|K'|}{|K|}\right].$$

But for all small $\varepsilon>0$, say $0<\varepsilon<\varepsilon_4\leqq\varepsilon_3$ we have: ($\varepsilon^{\frac{1}{2}}$ term)$\geqq$(sum of higher order terms), giving

$$\frac{|K\,U'|}{|U'|}\leqq 1+\left[2MN\frac{|K'|}{|K|}+1\right]\sqrt{\varepsilon}\leqq 1+4MN\sqrt{\varepsilon}\,\frac{|K'|}{|K|}.$$

Take $\varepsilon_0=\varepsilon_4$. Q.E.D.

3.2. Amenable σ-Compact Groups

Using the result of the preceding section and our earlier comment that $(P)\Rightarrow(P_{\mathrm{loc}})$ for $(P)=(A)$ or (FC) we may prove a strong characterization theorem for amenable σ-compact groups.

Theorem 3.2.1. *A σ-compact group G is amenable $\Leftrightarrow$ there is a sequence $\{U_n\}$ of compact sets with $0<|U_n|<\infty$ and*

(i) $$U_n\subset U_{n+1},$$

(ii) $$G=\bigcup_{n=1}^{\infty} U_n,$$

and for every nonempty compact set $K\subset G$ we have

(iii) $$\lim_{n\to\infty}\frac{|K\,U_n\,\triangle\,U_n|}{|U_n|}=0.$$

Proof. Implication ($\Leftarrow$) trivially gives (A), hence amenability; conversely, let $\{K_n\}$ be an increasing sequence of compact neighborhoods of the unit which fill up G and have $K_n\subset\mathrm{int}(K_{n+1})$. We construct $\{U_n\}$ inductively using the fact (amenable)$\Rightarrow(A)\Rightarrow(A_{\mathrm{loc}})$: choose U_1 to satisfy (A_{loc}) for $(1, K_1)$ with $K_1\subset U_1$, and for $n>1$ chose U_n to satisfy (A_{loc}) for $(1/n, K_n)$ with $K_n\cup U_{n-1}\subset U_n$. Since any compact set $K\subset G$ lies within one of the K_n, then $K_m\supset K$ for $m\geqq n$. Then, as in Remarks at end of 1.2, $|U \backslash KU| \leqq |KU \backslash U|$, so for $m \geqq n$:

$$|KU_m \triangle U_m| = |KU_m \backslash U_m| + |U_m \backslash KU_m| \leqq 2\,|KU_m \backslash U_m| \leqq 2\,|K_m U_m \backslash U_m|$$
$$= 2\,|K_m U_m \triangle U_m| \leqq 2/m \cdot |U_m|,$$

as required. Q.E.D.

From the considerations in 3.1.2 we could derive the following result; for a detailed proof of this and other properties of σ-compact amenable groups we direct the reader to [1].

Theorem 3.2.2. *Let $\{U_n\}$ be a sequence of compacta in locally compact group G with positive measure such that*

$$F_n(x) = \frac{|xU_n \triangle U_n|}{|U_n|} \to 0$$

uniformly on compacta in G. Then there exist compacta $U_n' \subset U_n$ such that $|U_n'| > 0$ for large n and

(i) $$\lim_{n\to\infty} \frac{|U_n \backslash U_n'|}{|U_n|} = 0.$$

(ii) *For every nonempty compact subset $K \subset G$*

$$\lim_{n\to\infty} \frac{|KU_n' \triangle U_n'|}{|U_n'|} = 0.$$

Note. We are assuming that $\{U_n\}$ eventually satisfies (FC) for every pair (ε, K), and conclude that we may obtain a sequence $\{U_n'\}$ which eventually satisfies the (formally) stronger condition (A) for every pair (ε, K) by doing minor surgery on the sets U_n.

Bibliography

1. EMERSON, W. R.: Sequences of sets with ratio properties in locally compact groups and asymptotic properties of a class of associated integral operators. Berkeley: PH.D. Dissertation 1967.
2. GREENLEAF, F. P.: Følner's condition for locally compact groups (to appear).
3. HEWITT, E., and K. ROSS: Abstract harmonic analysis (v. I). New York: Academic Press 1963.
4. HULANICKI, A.: Means and Følner conditions on locally compact groups. Studia Math **27**, 87–104 (1966).
5. LEPTIN, H.: Faltungen von Borelschen Massen mit L_p-Funktionen auf lokal kompakten Gruppen. Math. Ann. **163**, 111–117 (1966).
7. — On a certain invariant of a locally compact group. Bull. Amer. Math. Soc. **72**, 870–874 (1966).
8. — On locally compact groups with invariant means (to appear).
9. MONTGOMERY, D., and L. ZIPPIN: Topological transformation groups. New York: Interscience 1955.
10. NAMIOKA, I.: Følner's condition for amenable semigroups. Math. Scand. **15**, 18–28 (1964).

Dept. of Math., University of California, Berkeley, California 94720, U.S.A.