Math. Z. 123, 290–300 (1971)

Weak Holonomy Groups

ALFRED GRAY

1. Introduction

The well-known holonomy theorem [2] implies, among other things, that the holonomy group G of a Riemannian manifold M determines certain identities satisfied by the curvature operator of M. These identities, in turn, are very useful for proving the theorems about the topology of compact Riemannian manifolds with holonomy group G. For example if M is a compact Kähler manifold (that is, the holonomy group of M is $U(n)$) then the curvature operator of M satisfies identities which imply that the second Betti number of M is nonzero. A similar theorem for the case of holonomy group G_2 has been proved by Bonan [6] (see also [8]).

The following question naturally arises: *If a certain group G is not the holonomy group of M, are there some weaker conditions on G and M which imply that the curvature operator of M satisfies useful identities?* In this paper we show that there is indeed such a condition, namely the property that G is a *weak holonomy group* of M. We now define this notion, together with the auxiliary notion of *special subspace.*

Let M be a pseudo-Riemannian manifold and assume the structure group of the tangent bundle of M can be reduced from $O(p,q)$ to a connected Lie group G. For $m \in M$ we denote by M_m the tangent space to M at m.

Definition. A subspace $P \subseteq M_m$ is said to be *special* provided

(i) there exists a proper subspace $P' \subset P$ such that for all $g \in G$, $g|P$ is determined by $g|P'$;

(ii) if $P' \subset P \subseteq P''$ and $g(P')$ determines $g(P'')$ for all $g \in G$, then $P = P''$.

Definition. We say that G is a *weak holonomy group* of M provided the following condition is satisfied: for each $m \in M$, and differentiable loop γ in M with $\gamma(0) = \gamma(1) = m$ there exists $g \in G$ such that $\tau_\gamma | P = g | P$ whenever P is a special subspace of M_m of minimal dimension with $\gamma'(0) \in P$. Here τ_γ denotes parallel translation along γ.

In this paper we investigate weak holonomy groups in the case when the group G has a compact real form which acts transitively on a sphere. Berger [3] has classified the holonomy groups of manifolds having an affine connection with zero torsion. Either from this classification or directly from Simons [12], it follows that the restricted holonomy group of an irreducible Riemannian manifold which is not a symmetric space acts transitively on a sphere. Thus the curvature operator of a Riemannian manifold with weak holonomy group G will satisfy certain identities which generalize those satisfied by the curvature operator of a Riemannian manifold with holonomy group G.

A pseudo-Riemannian manifold M does not necessarily have a unique weak holonomy group. Thus if M has weak holonomy group G and H is a connected Lie group such that $G \subseteq H \subseteq O(p, q)$, then H is also a weak holonomy group of M. Of course holonomy groups are unique.

According to [11] the compact connected Lie groups which act effectively and transitively on spheres are the following: $SO(n)$, $U(n)$, $SU(n)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot SO(2)$, $\mathrm{Sp}(n)\cdot \mathrm{Sp}(1)$, G_2, $\mathrm{Spin}(7)$, and $\mathrm{Spin}(9)$. (Here $\mathrm{Sp}(n)\cdot SO(2)$ and $\mathrm{Sp}(n)\cdot \mathrm{Sp}(1)$ denote Lie groups locally isomorphic to $\mathrm{Sp}(n)\times SO(2)$ and $\mathrm{Sp}(n)\times \mathrm{Sp}(1)$, respectively, in which (I, I) and $(-I, -I)$ have been identified.)

If a Riemannian manifold has weak holonomy group $U(n)$, then M is a nearly Kähler manifold in the sense of [9]. In [9] it is shown that the curvature operator of a nearly Kähler manifold satisfies certain identities which generalize those for Kähler manifolds; consequently a great deal can be said about the topology and geometry of nearly Kähler manifolds. (If G and K are compact Lie groups and K is the fixed point set of an automorphism of G of order 3, then the coset space G/K is a nearly Kähler manifold.)

In §2 we prove that if $SO(n)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot \mathrm{Sp}(1)$, or $\mathrm{Spin}(7)$, is a weak holonomy group of M then the holonomy group of M is a subgroup of $SO(n)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot \mathrm{Sp}(1)$ or $\mathrm{Spin}(7)$, respectively. Furthermore we show that if M has weak holonomy group $U(n)$, and the structure group of M can be reduced to $SU(n)$, then M also has weak holonomy group $SU(n)$. We also prove that if M has weak holonomy group $\mathrm{Sp}(n)\cdot SO(2)$, then the holonomy group of M is a subgroup of $\mathrm{Sp}(n)$.

Bonan [6] has proved that a Riemannian manifold with holonomy group G_2 has vanishing Ricci curvature. We generalize this result by proving in §3 that a Riemannian manifold with weak holonomy group G_2 is an Einstein manifold. We conjecture that a compact Riemannian manifold with weak holonomy group G_2 which has positive sectional curvature is a spherical space form. The corresponding results for $U(n)$ [4, 9], and $\mathrm{Sp}(n)\cdot \mathrm{Sp}(1)$ [7] are known to be true.

We have been unable to say anything about Riemannian manifolds with weak holonomy group $\mathrm{Spin}(9)$ except that we have computed the dimension of a minimal special subspace (relative to $\mathrm{Spin}(9)$). Alekseevskij [1] has stated without proof that if $\mathrm{Spin}(9)$ is the holonomy group of a Riemannian manifold M then M is locally symmetric (and hence an open submanifold of the Cayley plane). Probably if $\mathrm{Spin}(9)$ is a weak holonomy group of M, the same conclusion holds.

In §4 we define the notion of *special curvature*; this is a generalization of the notion of holomorphic sectional curvature. Three theorems concerning Riemannian manifolds with constant special curvature are proved.

2. Special Subspaces Associated with Weak Holonomy Groups

Suppose M is a pseudo-Riemannian manifold with weak holonomy group G where G has a compact form which acts transitively on some sphere.

In this section we determine which subspaces of a tangent space M_m to M are special with respect to G.

Theorem 1. *Let M be a pseudo-Riemannian manifold with weak holonomy group $U(p,q)$. Then:*

(i) *a holomorphic section of a tangent space M_m (i.e., a subspace spanned by x and Jx for some nonzero $x \in M_m$) is a special subspace of minimal dimension.*

(ii) *M has weak holonomy group $SU(p,q)$ if and only if the structure group of the tangent bundle of M can be reduced to $SU(p,q)\,(p+q>1)$.*

Proof. (i) and the necessity of (ii) are obvious. For the sufficiency of (ii) let τ_γ be parallel translation along a loop γ. By hypothesis if $\gamma'(0)$ is in the holomorphic section P, then there exists $g \in U(p,q)$ such that $\tau_\gamma|P = g|P$. It is easy to see that there exists $g' \in SU(p,q)$ such that $g'|P = g|P = \tau_\gamma|P$. Hence G has weak holonomy group $SU(p,q)$.

Special subspaces associated with other groups can be regarded as generalizations of the notion of holomorphic section.

Next we consider pseudo-Riemannian manifolds with weak holonomy group Spin(7) or Spin(4, 3). If M is an 8-dimensional pseudo-Riemannian manifold such that the structure group of the tangent bundle can be reduced to Spin(7) or Spin(4, 3), then this reduction gives rise to a 3-fold vector cross product on M in the sense of [8].

Theorem 2. *Let M be an (8-dimensional) pseudo-Riemannian manifold with weak holonomy group Spin(7) or Spin(4, 3). Then:*

(i) *the holonomy group of M is a subgroup of Spin(7) or Spin(4, 3);*

(ii) *any 4-dimensional subspace of a tangent space M_m of M which is closed under the 3-fold vector cross product is a special subspace of minimal dimension.*

Proof. This is a consequence of [8, Theorem (4.6)].

Next we turn to the groups $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot SO(2)$, $\mathrm{Sp}(n)\cdot \mathrm{Sp}(1)$, and their noncompact forms. If M is a pseudo-Riemannian manifold such that the structure group of the tangent bundle of M can be reduced to G where $\mathrm{Sp}(p,q) \subseteq G \subseteq \mathrm{Sp}(p,q)\cdot \mathrm{Sp}(1)$, then it is possible to define, at least locally, almost complex structures I, J, and K which preserve the metric of M such that $IJ = -JI = K$ (see [7]).

Theorem 3. *Let M be a $4(p+q)$-dimensional pseudo-Riemannian manifold with weak holonomy group G where $Sp(p,q) \subseteq G \subseteq Sp(p,q)\cdot Sp(1)$. Then:*

(i) *A quaternionic section of a tangent space M_m (i.e., subspace spanned by x, Ix, Jx, Kx for nonzero $x \in M_m$) is a special subspace of minimal dimension.*

(ii) *If $G = Sp(p,q)\cdot Sp(1)$, then the holonomy group of M is a subgroup of $Sp(p,q)\cdot Sp(1)$.*

(iii) *If $G = Sp(p,q)$ or $Sp(p,q)\cdot SO(2)$, then the holonomy group of M is a subgroup of $Sp(p,q)$.*

Proof. That (i) holds amounts to checking that a quaternionic section satisfies the definition of special subspace. We prove (ii); we first show that if $\mathrm{Sp}(p,q)$ is a weak holonomy group, then the holonomy group of M is a subgroup of $\mathrm{Sp}(p,q)$. If the structure group of the tangent bundle of a pseudo-Riemannian manifold M can be reduced to $\mathrm{Sp}(p,q)$, then M has globally defined almost complex structures I, J, and K which preserve the metric of M and such that $IJ=-JI=K$. Denote by ∇ the Riemannian connection of M. The condition that M has weak holonomy group $\mathrm{Sp}(p,q)$ is equivalent to the conditions

$$\nabla_x(I)(x)=\nabla_x(J)(x)=\nabla_x(K)(x)=0 \tag{1}$$

for $x\in M_m$. Furthermore we have

$$\begin{aligned}&\nabla_x(I)(Iy)=-I\nabla_x(I)(y),\\&\nabla_x(I)(Jy)=\nabla_x(K)(y)-I\nabla_x(J)(y),\quad \text{etc.}\end{aligned} \tag{2}$$

From (1) and (2) we obtain

$$\nabla_{Ix}(I)(Iy)=\nabla_{Jx}(I)(Jy)=\nabla_{Kx}(I)(Ky)=-\nabla_x(I)(y),\quad \text{etc.}$$

On the other hand we have

$$\nabla_{Kx}(I)(Ky)=\nabla_{IJx}(I)(IJy)=-\nabla_{Jx}(I)(Jy),\quad \text{etc.}$$

It follows that $\nabla_x(I)(y)=\nabla_x(J)(y)=\nabla_x(K)(y)=0$ for all $x,y\in M_m$, and so the holonomy group of M is a subgroup of $\mathrm{Sp}(p,q)$.

If a pseudo-Riemannian manifold M has weak holonomy group $\mathrm{Sp}(p,q)\cdot SO(2)$, it can be proved in a similar fashion that the holonomy group of M is a subgroup of $\mathrm{Sp}(p,q)\cdot SO(2)$. According to Berger [5] this implies that the holonomy group of M is a subgroup of $\mathrm{Sp}(p,q)$.

The proof that if $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ is a weak holonomy group then the holonomy group of M is a subgroup of $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ is similar to that for $\mathrm{Sp}(p,q)$, but more complicated. If the structure group of the tangent bundle of a pseudo-Riemannian manifold M can be reduced to $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$, then there is a tensor field Q globally defined on M as follows [7]. Each point $m\in M$ has a neighborhood on which there are defined three almost complex structures I, J, and K which preserve the metric of M such that $IJ=-JI=K$. Set $Qx=Ix\wedge Jx\wedge Kx$ for $x\in M_m$. Then Q can be linearized so that it becomes a tensor field of type $(3,3)$. Further Q is independent of the choice of I, J, and K so that it is globally defined on M. If M has weak holonomy group $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$, then for $x\in M_m$

$$\begin{aligned}0&=\nabla_x(Q)(x)\\&=\nabla_x(I)(x)\wedge Jx\wedge Kx+Ix\wedge\nabla_x(J)(x)\wedge Kx+Ix\wedge Jx\wedge\nabla_x(K)(x)\end{aligned} \tag{3}$$

for $x\in M_m$. From (3) it follows that

$$\nabla_x(I)(x)=0 \bmod Jx \text{ and } Kx,\quad \text{etc.} \tag{4}$$

Eq. (2) still holds, and so from (2) and (4) we obtain

$$\nabla_x(I)(y)=0 \bmod Jy \text{ and } Ky \text{ for all } x,y\in M_m. \tag{5}$$

From (5) it follows easily that $\nabla_x(Q)=0$ for $x\in M_m$, and so the holonomy group of M is a subgroup of $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$.

Finally we consider Spin(9).

Theorem 4. *Let M be a 16-dimensional Riemannian manifold and assume that the structure group of the tangent bundle of M can be reduced from $O(16)$ to the 16-dimensional irreducible representation of Spin(9). Let $m \in M$. If $P \subseteq M_m$ is a special subspace of minimal dimension, then* $\dim P = 8$.

Proof. Let $x \in M_m$, $x \neq 0$. The subgroup of Spin(9) which leaves x fixed is isomorphic to Spin(7). The induced representation of Spin(7) on $\{x\}^{\perp}$ is the sum of an 8-dimensional faithful irreducible representation of Spin(7) and a 7-dimensional 2-fold irreducible representation of Spin(7). Thus the subspace of M_m which is the sum of $\{x\}$ and the image of the 7-dimensional representation of Spin(7) is a special subspace of M_m. It has minimal dimension. Also, since x is arbitrary and Spin(9) is transitive on S^{15}, it follows that any special subspace of dimension 8 must arise in this fashion.

3. Pseudo-Riemannian Manifolds with 2-fold Vector Cross Products

Let M be a 7-dimensional pseudo-Riemannian manifold such that the structure group of the tangent bundle of M can be reduced from $O(7)$ or $O(4,3)$ to G_2 or its non-compact form G_2^*. This reduction on M is equivalent [8, 10] to the existence on M of a globally defined 2-fold vector cross product P. (In this section P does *not* denote a special subspace.) Denote by $\langle\ ,\ \rangle$ the metric tensor of M. Then on each tangent space M_m, P is a map $P: M_m \times M_m \to M_m$ characterized by the conditions $P(x,y) = -P(y,x)$, $\langle P(x,y), y\rangle = 0$, and $\langle P(x,y), P(x,y)\rangle = \langle x \wedge y, x \wedge y\rangle$ for $x, y \in M_m$. The following theorem follows easily from the definition of weak holonomy group.

Theorem 5. *Let M be a 7-dimensional pseudo-Riemannian manifold with a 2-fold vector cross product P. Then M has G_2 or G_2^* as a weak holonomy group if and only if $\nabla_x(P)(x,y) = 0$ for all $x, y \in M_m$ for all $m \in M$. A special subspace of M_m of minimal dimension is any 3-dimensional subspace closed under P.*

We shall need the following formulas. Let R_{xy} ($x, y \in M_m$) be the curvature operator of M.

Theorem 6. *Let M be a 7-dimensional pseudo-Riemannian manifold with a 2-fold vector cross product P, and assume that G_2 or G_2^* is a weak holonomy group of M. Then for $x, y, z \in M_m$ we have*

(i) $\nabla_x(P)(y, P(x,y)) = 0$;

(ii) $\nabla_x(P)(P(x,y), P(x,y)) = -\langle x,x\rangle\, \nabla_x(P)(y,z)$;

(iii) $$\langle \nabla_x(P)(y,z), \nabla_x(P)(y,z)\rangle = \langle R_{xy}x, y\rangle \langle z,z\rangle - \langle R_{xy}x, z\rangle \langle y,z\rangle - \langle P(x, P(y,z)), R_{xy}z\rangle - \langle R_{xy}P(x,z), P(y,z)\rangle;$$

(iv) $$3\langle \nabla_x(P)(y,z), \nabla_x(P)(y,z)\rangle + 2\langle R_{yz}x, z\rangle \langle y,x\rangle - 2\langle R_{yz}x, y\rangle \langle z,x\rangle = \mathfrak{S}\{\langle R_{yz}y, z\rangle \langle x,x\rangle - \langle R_{yz}P(x,y), P(x,z)\rangle\},$$

where $\mathfrak{S}$ denotes the cyclic sum over x, y and z.

Proof. In [8] it is shown that for any 2-fold vector cross product P on a Riemannian manifold M we have

$$\langle \nabla_x(P)(y,z), P(y,z)\rangle = 0 \tag{6}$$

for $x, y, z \in M_m$. Then (6) together with Theorem 5 implies (i). Linearization of (i) yields

$$\nabla_x(P)(y, P(x, z)) = \nabla_x(P)(P(x,y), z). \tag{7}$$

For any 2-fold vector cross product P we always have

$$P(x, P(y, z)) + P(P(x, y), z) = 2\langle x, z\rangle y - \langle x, y\rangle z - \langle y, z\rangle x. \tag{8}$$

In (7) we replace y by $P(x, y)$. Using (8) we obtain

$$\begin{aligned} \nabla_x(P)(P(x, y), P(x, z)) &= \nabla_x(P)(P(x, P(x, y)), z) \\ &= \nabla_x(P)(\langle x, y\rangle x - \langle x, x\rangle y, z) \\ &= -\langle x, x\rangle \nabla_x(P)(y, z). \end{aligned}$$

(iii) is a special case of [8, Theorem 5.7]. (iv) follows from a calculation from (iii) which we omit.

We shall also need the following fact.

Lemma. *Let M be a pseudo-Riemannian manifold such that the structure group of the tangent bundle of M can be reduced to G_2 or G_2^*. Then the complexification $M_m \otimes C$ of each tangent space M_m has an orthonormal basis $e_0, \ldots, e_6$ such that $P(e_{i+1}, e_{i+2}) = e_{i+4}$ where we take $i \in Z_7$.*

For a proof see [8].

We are now ready to prove the main result of this section. Denote by Ω_{ij} the curvature forms and by ω_i the 1-forms dual to the e_i, and write $R_{ijkl} = \Omega_{ij}(e_k, e_l)$, $K_{ij} = R_{ijij}$. Also, let k denote the Ricci curvature of M.

Theorem 7. *Let M be a 7-dimensional pseudo-Riemannian manifold with weak holonomy group G_2 or G_2^*. Then*

(i) *there exists a constant η such that for $i \in Z_7$*

$$\Omega_{i+1,i+3} + \Omega_{i+2,i+6} + \Omega_{i+4,i+5} = \eta(\omega_{i+1} \wedge \omega_{i+3} + \omega_{i+2} \wedge \omega_{i+6} + \omega_{i+4} \wedge \omega_{i+5});$$

(ii) *we have $k(x, y) = 6\eta\langle x, y\rangle$ for all $x, y \in M_m$ and $m \in M$; thus M is an Einstein manifold;*

(iii) *we have*

$$\begin{aligned} K_{i+1,i+3} + K_{i+2,i+6} - K_{i+4,i+5} &= \eta - 2R_{i+1,i+3,i+2,i+6}, \\ K_{i+1,i+3} - K_{i+2,i+6} + K_{i+4,i+5} &= \eta - 2R_{i+1,i+3,i+4,i+5}, \\ -K_{i+1,i+3} + K_{i+2,i+6} + K_{i+4,i+5} &= \eta - 2R_{i+2,i+6,i+4,i+5} \end{aligned}$$

for $i \in Z_7$;

(iv) *we have for $i \in Z_7$*

$$\begin{aligned} K_{i+1,i+3} + K_{i+2,i+6} - K_{i+4,i+5} + K_{i+1,i+2} + K_{i+3,i+6} - K_{i,i+5} \\ + K_{i+2,i+3} + K_{i+1,i+6} - K_{i,i+4} = 3\eta. \end{aligned}$$

Proof. Suppose that e_i, e_j, e_k, and $P(e_i, e_j)$ are linearly independent. Write $\alpha_{ijk} = \langle \nabla_{e_i}(P)(e_j, e_k), \nabla_{e_i}(P)(e_j, e_k)\rangle$. Because of Theorem 5 and Eq. (6) there exist numbers ρ_1, ρ_2, and ρ_3 such that $\nabla_{e_0}(P)(e_1, e_2) = \rho_1 e_5$, $\nabla_{e_0}(P)(e_3, e_6) = \rho_2 e_5$, and $\nabla_{e_0}(P)(e_1, e_6) = \rho_3 e_4$. Repeated applications of these equations and Theorem 6(ii) show that

$$\alpha_{012} = \alpha_{036} = \alpha_{034} = \alpha_{025} = \alpha_{046} = \alpha_{035} = \alpha_{014} = \alpha_{056} = \alpha_{024} = \alpha_{016} = \alpha_{023}.$$

A similar argument works for any α_{ijk} with e_i, e_j, e_k, and $P(e_i, e_j)$ linearly independent and $0 \leq i \leq 6$. Let η be the common value of the α_{ijk}. On the other hand if e_i, e_j, e_k, and $P(e_i, e_j)$ are linearly dependent then by Theorem 6(i), $\nabla_{e_i}(P)(e_j, e_k) = 0$. It follows from Theorem 6(iii) that both sides of Theorem 7(i) have the same value on all tangent vectors to M.

The proof of (ii) is similar to the proof of the corresponding result for Riemannian manifolds with holonomy group G_2 due to Bonan [6]. We have, using part (i) and the Bianchi identities, that

$$\begin{aligned} k(e_0, e_0) &= K_{01} + K_{02} + K_{03} + K_{04} + K_{05} + K_{06} \\ &= 6\eta - R_{0146} - R_{0152} - R_{0215} - R_{0234} - R_{3024} - R_{3056} \\ &\quad - R_{0461} - R_{0423} - R_{5063} - R_{5012} - R_{6041} - R_{6035} \\ &= 6\eta. \end{aligned}$$

Similarly $k(e_i, e_i) = 6\eta$ for $i = 1, \ldots, 6$. Linearization of these equations yields (ii).

Part (iii) is proved by using part (i) twice. Finally (iv) follows from (iii) and the Bianchi identities.

4. The Special Curvature of a Riemannian Manifold with Respect to a Weak Holonomy Group

In this section we limit ourselves to Riemannian manifolds, although some of our results also hold for pseudo-Riemannian manifolds.

In view of the importance of the concept of holomorphic sectional curvature for Kähler manifolds, it is natural to seek a corresponding curvature for other Riemannian manifolds whose holonomy group or weak holonomy group has a compact form which is transitive on some sphere. There are probably several generalizations of the notion of holomorphic sectional curvature, but we choose the following.

Definition. Let M be a Riemannian manifold with weak holonomy group G. Let $m \in M$ and let $P \subseteq M_m$ be a special subspace (not necessarily of minimal dimension). The *special curvature* $r(P)$ of P is $\frac{1}{2}$ the Ricci scalar curvature of P, i.e.,

$$r(P) = \sum_{i<j} K_{ij}$$

where $\{e_1, \ldots, e_p\}$ is an orthonormal frame spanning P and K_{ij} is the sectional curvature of the plane spanned by e_i and e_j. It is easy to prove that this definition is independent of the choice of $\{e_1, \ldots, e_p\}$.

Thus if M has weak holonomy group $SO(n)$, then $r(P)$ is $\frac{1}{2}$ the ordinary Ricci scalar curvature of M. Similarly if M has weak holonomy group $U(n)$

(or $SU(n)$) and $\dim P=2$, then $r(P)$ is the holomorphic sectional curvature of P. More generally if $\dim P=2q$ $(1\leqq q\leqq n)$, then

$$r(P)=\sum_{i=1}^{q} K_{ii*}+\sum_{i<j}(K_{ij}+K_{ij*})$$

where $\{e_1, \dots, e_q, e_{1*}, \dots, e_{q*}\}$ is a unitary frame spanning P and $Je_i=e_{i*}$.

Not much else of a general nature can be said about the special curvature of Riemannian manifolds with weak holonomy group $SO(n)$ or $U(n)$, and so we turn our attentions to other groups. We investigate the situation when $r(P)$ is constant.

First we consider the case of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$. By Theorem 3 we may assume that M is a Riemannian manifold whose holonomy group is a subgroup of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$. The next theorem is essentially due to Berger [3], but our proof is perhaps a little simpler.

Theorem 8. *Let M be a Riemannian manifold whose holonomy group is a subgroup of $Sp(n)\cdot Sp(1)$ where $n>1$. Let $m\in M$ and denote by $x, y\in M_m$ unit tangent vectors which lie in different quaternionic sections. Then*

(i) *M has constant special curvature $r(P)=r$ for any (4-dimensional) quaternionic section P;*

(ii) $K_{xIx}+\langle R_{xIx}, Jx, Kx\rangle=\frac{1}{6}r$;

(iii) $K_{uv}=K_{IuIv}=K_{JuJv}=K_{KuKv}$ *for linearly independent $u, v\in M_m$;*

(iv) $K_{xIx}+K_{xJx}+K_{xKx}=\frac{1}{2}r$;

(v) $K_{xy}+K_{xIy}+K_{xJy}+K_{xKy}=\frac{1}{6}r$;

(vi) *M is an Einstein manifold with Ricci curvature $k(u,v)=\frac{r}{6}(n+2)\langle u,v\rangle$ for $u, v\in M_m$;*

(vii) $[R_{xIx}, J]=\frac{r}{6}K$, $[R_{xJx}, K]=\frac{r}{6}I$, $[R_{xKx}, I]=\frac{r}{6}J$;

(viii) $\langle[R_{uv}, I]x, y\rangle=\langle[R_{uv}, J]x, y\rangle=\langle[R_{uv}, K]x, y\rangle=0$;

(ix) $[R_{xIx}, I]=[R_{xJx}, J]=[R_{xKx}, K]=0$.

Proof. For $u, v\in M_m$ we may write

$$R_{uv}=\alpha(u,v)I+\beta(u,v)J+\gamma(u,v)K+A_{uv}$$

where A_{uv} commutes with I, J, and K and α, β, and γ are 2-forms defined on a neighborhood of m. Then

$$\begin{aligned}
[R_{uv}, I] &= \qquad\qquad \gamma(u,v)J-\beta(u,v)K,\\
[R_{uv}, J] &= -\gamma(u,v)I \qquad\qquad +\alpha(u,v)K,\\
[R_{uv}, K] &= \beta(u,v)I-\alpha(u,v)J.
\end{aligned}\tag{9}$$

Now (viii) is immediate from (9). Furthermore by (viii) we have

$$\begin{aligned}\langle [R_{xIx}, J]\, y, Iy\rangle &= \langle R_{xIx} Jy, Iy\rangle - \langle R_{xIx} y, Ky\rangle\\ &= -\langle R_{xIy} Ix, Jy\rangle - \langle R_{xJy} Iy, Ix\rangle + \langle R_{xKy} Ix, y\rangle + \langle R_{xy} Ky, Ix\rangle\\ &= \langle R_{xIy} x, Ky\rangle + \langle R_{Ixy} x, Ky\rangle + \langle R_{xy} x, Jy\rangle + \langle R_{xy} Jy, x\rangle\\ &= 0.\end{aligned}$$

Hence $\gamma(x, Ix) = \gamma(x, Jx) = 0$, and similarly $\alpha(x, Jx) = \alpha(x, Kx) = \beta(x, Ix) = \beta(x, Kx) = 0$. This proves (ix). Therefore (9) becomes

$$\begin{aligned}[R_{xKx}, I] &= \gamma(x, Kx) J,\\ [R_{xIx}, J] &= \alpha(x, Ix) K,\\ [R_{xJx}, K] &= \beta(x, Jx) I.\end{aligned} \tag{10}$$

Moreover, we have

$$\alpha(x, Ix) = \langle [R_{xIx}, J]\, x, Kx\rangle = \langle R_{xIx} Jx, Kx\rangle + K_{xIx}, \tag{11}$$

and

$$\begin{aligned}\alpha(x, Ix) &= \langle [R_{xIx}, J]\, y, Ky\rangle\\ &= \langle R_{xIx} Jy, Ky\rangle + \langle R_{xIx} y, Iy\rangle\\ &= -\langle R_{xKy} Ix, Jy\rangle - \langle R_{xJy} Ky, Ix\rangle - \langle R_{xIy} Ix, y\rangle - \langle R_{xy} Iy, Ix\rangle\\ &= K_{xy} + K_{xIy} + K_{xJy} + K_{xKy},\end{aligned} \tag{12}$$

by (viii). Thus $\alpha(x, Ix) = \beta(x, Jx) = \gamma(x, Kx) = \alpha(y, Jy)$, etc. Hence $\alpha(Jx, Kx) = \beta(x, Jx) = \alpha(x, Ix)$, etc. From (11) it follows that

$$K_{xIx} = K_{JxKx}, \quad K_{xJx} = K_{IxKx}, \quad K_{xKx} = K_{IxJx}. \tag{13}$$

Then (13) and (viii) imply (iii). Furthermore from (11) and the first Bianchi identity we have

$$3\alpha(x, Ix) = K_{xIx} + K_{xJx} + K_{xKx}. \tag{14}$$

From (12) and (14) we get

$$k(x, x) = (n+2)\,\alpha(x, Ix), \tag{15}$$

and so M is an Einstein manifold. Thus we may write $k(u, v) = \lambda \langle u, v\rangle$ for all $u, v \in M_m$, where λ is a constant. Moreover by (12), (13), (14), and (15) we have

$$r(P) = 6\alpha(x, Ix) = \frac{6\lambda}{n+2}. \tag{16}$$

Therefore (i) and (vi) follows from (16), (ii) follows from (11) and (16), (iv) follows from (14) and (16), (v) follows from (12) and (16), and (vii) follows from (10) and (16). This completes the proof.

We also have the following result about the special curvature of Riemannian manifolds with weak holonomy group G_2.

Theorem 9. *Suppose M is a Riemannian manifold with weak holonomy group G_2 and assume that for all $m\in M$ the special curvature $r(P)$ of M has the same value r on all special subspaces $P\subseteq M_m$. Then*

(i) $r=3\eta$, *where η is defined in Theorem 7, and so r is a constant function on M;*

(ii) $K_{i+1,i+3}+K_{i+2,i+6}+K_{i+4,i+5}=r$, *for* $i\in Z_7$.

(iii) $K_{i+1,i+3}=\eta+2R_{i+2,i+6,i+4,i+5}$,
$K_{i+2,i+6}=\eta+2R_{i+1,i+3,i+4,i+5}$,
$K_{i+4,i+5}=\eta+2R_{i+1,i+3,i+2,i+6}$, *for* $i\in Z_7$.

Proof. By assumption for $i\in Z_7$, we have

$$K_{i,i+1}+K_{i,i+3}+K_{i+1,i+3}=r. \tag{17}$$

(We use the same notation as that of Theorem 7; since the metric of M is assumed to be positive definite, the frame $\{e_0,\ldots,e_6\}$ used in Theorem 7 may be taken to be a basis of M_m, $m\in M$.) From (17) and Theorem 7(ii) we get (i) and (ii). Also (iii) follows from (ii) and Theorem 7(iii).

Finally we consider the special curvature of a Riemannian manifold whose holonomy group is a subgroup of Spin(7).

Lemma. *Let M be a pseudo-Riemannian manifold such that the structure group of the tangent bundle of M can be reduced to Spin(7) or Spin(4, 3). Then the complexification $M_m\otimes C$ of each tangent space M_m has an orthonormal basis $e_0,\ldots,e_7$ such that $\tilde{P}(e_7,e_{i+1},e_{i+2})=e_{i+4}$, where $\tilde{P}$ is the 3-fold vector cross product determined by the reduction to Spin(7) or Spin(4, 3).*

For a proof see [8].

Theorem 10. *Let M be a Riemannian manifold whose holonomy group is a subgroup of Spin(7). We use the basis of a tangent space M_m described by the preceding lemma. Then*

(i) $\Omega_{i,7}+\Omega_{i+1,i+3}+\Omega_{i+2,i+6}+\Omega_{i+4,i+5}=0$ *for* $i\in Z_7$;

(ii) *M has zero Ricci curvature;*

(iii) *if M has constant special curvature r, then $r=0$.*

Proof. (i) and (ii) are due to Bonan [6]. For (iii) we have for each $i\in Z_7$,

$$K_{i,7}+K_{i+1,7}+K_{i+3,7}+K_{i,i+1}+K_{i,i+3}+K_{i+1,i+3}=r,$$
$$K_{i,7}+K_{i+2,7}+K_{i+6,7}+K_{i,i+2}+K_{i,i+6}+K_{i+2,i+6}=r,$$
$$K_{i,7}+K_{i+4,7}+K_{i+5,7}+K_{i,i+4}+K_{i,i+5}+K_{i+4,i+5}=r.$$

Upon adding these three equations and using (ii) we obtain

$$2K_{i,7}+K_{i+1,i+3}+K_{i+2,i+6}+K_{i+4,i+5}=3r.$$

Thus

$$21r=\sum_{i=0}^{6}(K_{i+1,i+3}+K_{i+1,i+6}+K_{i+4,i+5})$$
$$=\sum_{0<i<j<7}K_{ij}-\sum_{i=0}^{6}K_{i,7}.$$

Each of the sums on the right hand side of this equation vanishes and so $r=0$.

Remark. It is easy to see that analogs of Theorems 7 and 8 hold for the covariant derivative of the curvature operator. In the statements of these theorems one replaces the curvature operator by its covariant derivative and the Ricci scalar curvature by 0.

References

1. Alekseevskij, D.V.: On holonomy groups of Riemannian manifolds. Ukrain. Math. Žurn. **19**, 100–104 (1967).
2. Ambrose, W., Singer, I.M.: A theorem on holonomy. Trans. Amer. Math. Soc. **75**, 428–433 (1953).
3. Berger, M.: Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniens. Bull. Soc. Math. France **83**, 279–330 (1955).
4. – Sur les variétés d'Einstein compactes. C.R. III^e Reunion Math. Expression Latine, Namur (1965), 35–55.
5. – Trois remarques sur les variétés riemannienes à courbure positive. C.R. Acad. Sci. Paris **263**, 76–78 (1966).
6. Bonan, E.: Sur des variétés riemanniennes à groupe d'holonomie G_2 ou Spin (7). C.R. Acad. Sci. Paris **262**, 127–129 (1966).
7. Gray, A.: A note on Riemannian manifolds with holonomy group $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$. Michigan Math. J. **16**, 125–128 (1969).
8. – Vector cross products on manifolds. Trans. Amer. Math. Soc. **141**, 465–504 (1969).
9. – Nearly Kähler manifolds. J. Differential Geometry **4**, 283–310 (1970).
10. – Green, P.: Sphere transitive structures and the triality automorphism. Pacific J. Math. **34**, 83–96 (1970).
11. Montgomery, D., Samelson, H.: Transformation groups of spheres. Ann. of Math. **44**, 454–470 (1943).
12. Simons, J.: On transitivity of holonomy systems. Ann. of Math. **76**, 213–234 (1962).

Dr. Alfred Gray
Department of Mathematics
University of Maryland
College Park, Maryland 20742
USA

(Received March 24, 1971)