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On the Complexity of Diagram Testing

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Abstract. In 1987, Nešetřil and Rödl [4] claimed to have proved that the problem of finding whether a given graph G can be oriented as the diagram of a partial order is NP-complete. A flaw was discovered in their proof by Thostrup [11]. Nešetřil and Rödl [5] have since corrected the proof, but the new version is rather complex. We give a simpler and more elementary proof, using a completely different approach.

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In 1984, the Editorial Board of *Order* drew up a list of the leading problems in the theory of ordered sets. One problem on the list, posed in Ore's book [6], was that of characterising the undirected graphs which can be oriented as the diagram of an ordered set. No real progress has been made on any such characterisation, and, in the opinion of many, none is likely. The principal reason for this pessimism was that, in 1987, Nešetřil and Rödl [4] claimed to have shown that the problem of determining whether a graph G can be oriented as a diagram is NP-complete. Of course, this does not rule out illuminating characterisations in terms of other NP-complete properties of graphs, but no such characterisations have been suggested.

In 1991, a flaw in Nešetřil and Rödl's proof was discovered by Thostrup [11]. Repairing the proof turned out to be harder than originally expected, but Nešetřil and Rödl have produced a corrected version [5]. However, this new proof is rather complicated, and makes use of several earlier results, in particular a result of Lund and Yannakakis [2] on approximating the chromatic number of a graph.

The purpose of this note is to give a completely different proof that diagram testing is NP-complete. The proof is, in the author's opinion, considerably simpler and more elementary than that offered by Nešetřil and Rödl. It was produced following a talk on the subject by Oliver Pretzel at the One-day Colloquium in Combinatorics at Reading in May 1993, at which time the problem was understood by the author to be open.

Recall that the *diagram* of a partial order $P = (X, <)$ is the directed graph on X defined by directing an edge from x to y if $x < y$ and there is no z with $x < z < y$.

A directed graph G with vertex set X is a *diagram* if it is the diagram of some partial order on X. There is a simple and useful characterisation for when a *directed* graph is a diagram, for which we need a few more definitions.

Essentially, we follow the terminology of Pretzel [7, 8, 9]. A *walk* W of *length* k in a graph G is a sequence of vertices $v_1v_2\ldots v_kv_{k+1}$, where each of v_iv_{i+1} $(1 \leqslant i \leqslant k)$ is an edge of G. (Thus both repeated vertices and repeated edges are allowed.) A *circuit* C in G is a walk where the initial vertex v_1 is equal to the terminal vertex v_{k+1}. A *simple circuit* is a circuit with no repeated vertices. (The word 'circuit' rather than 'cycle' is used to emphasise that the circuit comes equipped with a direction.)

An *orientation* of a graph G is an assignment of a direction to each edge of G. Given an orientation R of G, and a circuit C in G, a *forward edge* of C is an edge v_iv_{i+1} oriented from v_i to v_{i+1} in R, and a *backward edge* is an edge oriented in the opposite direction. The *flow difference* of C (under R) is the number of forward edges of C, minus the number of backward edges.

A directed cycle in the orientation R is thus exactly a circuit C of length k and flow difference k. A circuit of length $k \geqslant 3$ and flow difference $k-2$ (i.e., with exactly one backward edge) is called a *bypass*. It is clear that, if R is a diagram orientation of G, then it gives rise to no directed cycles or bypasses. Furthermore, the converse is true: if, in an orientation R of G, there is no simple circuit of length k and flow difference k or $k-2$, then R is a diagram orientation. In particular, if R is a diagram orientation, then every simple circuit of length 4 has flow difference 0, and every simple circuit of length 5 has flow difference ± 1.

The problem we are concerned with is the following.

DIAGRAM TESTING

Instance: A graph G.

Question: Can G be oriented as a diagram?

THEOREM. *DIAGRAM TESTING is* NP-*complete.*

We shall prove the Theorem by giving a polynomial transformation from NOT-ALL-EQUAL-3-SAT, which is on the standard list of NP-complete problems in Garey and Johnson [1] as [LO3]. This problem is as follows.

NOT-ALL-EQUAL-3-SAT (NAE-3-SAT)

Instance: A set U of variables, and a set C of clauses each containing three literals from U.

Question: Is there a truth assignment for U such that each clause in C contains at least one true and one false literal.

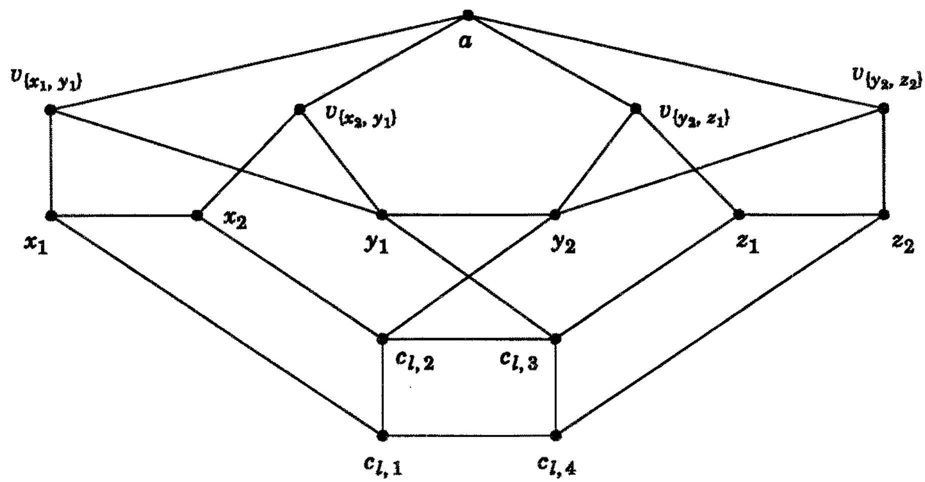

Fig. 1. The clause represented here is $(x\bar{y}z)$.

The problem NAE-3-SAT is equivalent to that of 2-colouring a 3-uniform hypergraph. There is a fairly easy transformation from 3-SAT to NAE-3-SAT. Given an instance $I = (U, C)$ of NAE-3-SAT, we call a truth assignment for U a *satisfying assignment* if each clause in C contains at least one true and one false literal.

Proof of Theorem. Given an instance $I = (U, C)$ of NAE-3-SAT, we construct a graph G_I such that G_I can be oriented as a diagram iff I has a satisfying assignment. The construction is given below; it might aid the understanding for the reader to refer to Figure 1.

(1) Take one distinguished vertex a.
(2) For each variable x in U, take a pair of vertices x_1, x_2, joined by an edge.

One piece of notation at this point: if u is a literal corresponding to the variable x (i.e., $u = x$ or $u = \bar{x}$), then define u_1, u_2 to be the two vertices x_1, x_2 corresponding to x, with $u_1 = x_1$ and $u_2 = x_2$ if $u = x$, but $u_1 = x_2$ and $u_2 = x_1$ if $u = \bar{x}$.

(3) For each pair $\{x_i, y_j\}$ with $x \neq y$, $i, j \in \{1, 2\}$, take a vertex $v_{\{x_i, y_j\}}$, joined to x_i, y_j, and a.
(4) For each clause $l = (u, v, w)$, take four vertices forming a 4-cycle

$$c_{l,1}c_{l,2}c_{l,3}c_{l,4}c_{l,1}.$$

These are joined to the rest of the graph as follows. The vertex $c_{l,1}$ is joined to u_1; $c_{l,2}$ is joined to u_2 and v_1; $c_{l,3}$ is joined to v_2 and w_1; and $c_{l,4}$ is joined to w_2.

The basic idea is as follows. The orientation of the edge x_1x_2 corresponds to the truth value of the variable x, with edge oriented from x_1 to x_2 if x is True. We shall force the edge $c_{l,1}c_{l,2}$ to be oriented the same way as u_1u_2, and similarly for the edges $c_{l,2}c_{l,3}$ and $c_{l,3}c_{l,4}$. If these three edges are all oriented in the same direction around the four-cycle corresponding to l, it will then not be possible to orient the edge $c_{l,4}c_{l,1}$ without producing a bypass or oriented cycle.

At first sight, this is not feasible, since, if a graph can be oriented as a diagram, then it is known that it may be re-oriented in many different ways as a diagram, and in particular that it can be oriented as a diagram with any two independent edges given an arbitrary direction. Perhaps the key idea of the proof is to take advantage of exactly this freedom, by using the following result of Mosesian [3], see also Pretzel [8].

LEMMA. *Let G be a connected graph that can be oriented as a diagram, and let a be any vertex of G. Then G can be oriented as the diagram of a partial order in which a is the only maximal element.*

Perhaps it should be remarked that the proof of this lemma is not difficult: a suitable orientation can be obtained by repeatedly 'pushing down' maximal vertices x other than a (i.e., reversing the directions of all the edges incident with x).

Our claim above should now be read as: in every diagram orientation of G_I in which a is the only maximal element, $c_{l,1}c_{l,2}$ has the same direction as u_1u_2.

Returning to the formal proof of the Theorem, it is clear that the construction of G_I can be carried out in time polynomial in the size of the instance I. We make the following two claims which, in the light of the lemma above, suffice to complete the proof.

(a) If G_I can be oriented as a diagram, with a the unique maximal element, then I has a satisfying assignment.
(b) If I has a satisfying assignment, then G_I can be oriented as a diagram.

Part (a) is more interesting, although perhaps part (b) is more critical. We start with (a).

Suppose then that G_I can be oriented as a diagram, and fix one diagram orientation R of G_I where a is the unique maximal element. Note that all the edges from the vertices $v_{\{x_i,y_j\}}$ to a are directed towards a by R, since otherwise a is not a maximal element. Also, there is some directed path P from each x_i to a, and if some edge $v_{\{x_i,y_j\}}x_i$ is directed towards x_i, then the circuit $x_iPav_{\{x_i,y_j\}}x_i$ is a bypass. Therefore all the edges of the form $v_{\{x_i,y_j\}}x_i$ are directed towards $v_{\{x_i,y_j\}}$ in R.

Now let $l = (u, v, w)$ be a clause, and consider the subgraph of G_I in Figure 2, with the known directions of edges $x_iv_{\{x_i,y_j\}}$ marked.

The flow difference of the simple circuits of length 4 bounding each of the six faces $F_1 \ldots F_6$ is zero, and the flow difference of the circuit bounding F_7 is ± 1. There is an obvious sense in which the external circuit is the sum of these seven circuits, and it follows that the flow difference around the external circuit is ± 1. (A more careful discussion of this principle, and several applications, are to be found

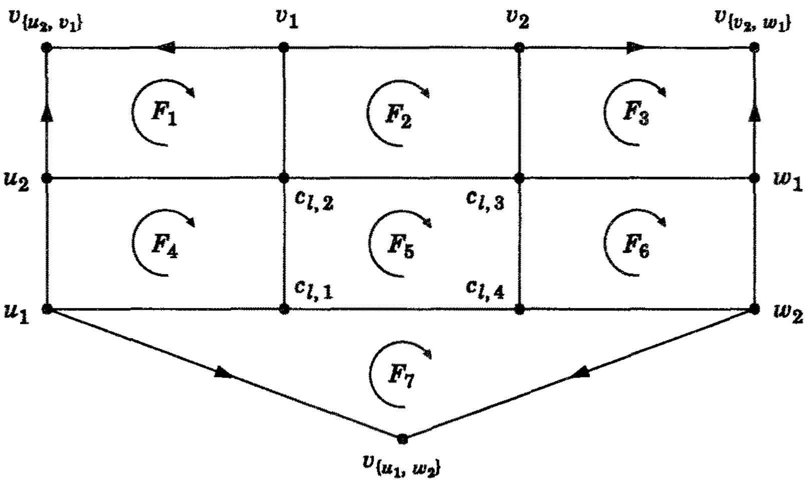

Fig. 2. A subgraph of G_I, corresponding to the clause containing literals u, v, w.

in Pretzel and Youngs [10].) This means that the three edges u_1u_2, v_1v_2, and w_1w_2 are not all oriented in the same direction.

We define a truth assignment for $I = I(R)$ by setting variable x True if the edge x_1x_2 is directed from x_1 to x_2 in R, and False otherwise. Thus literal u is True iff the edge u_1u_2 is oriented from u_1 to u_2. By construction of G_I, and the observation in the previous paragraph, no clause contains three false, or three true, literals. Hence I is a satisfying assignment.

We now move on to proving assertion (b). Suppose then that we have a satisfying assignment for I. We shall give an orientation to each edge of the graph G_I, and claim that G_I is thereby made into a diagram. Naturally, a will be the unique maximal element in this orientation.

So, given the truth assignment, we orient the edges as follows. Each edge from a vertex $v_{\{x_i,y_j\}}$ to a is directed towards a, and each edge from x_i to $v_{\{x_i,y_j\}}$ is oriented towards $v_{\{x_i,y_j\}}$. The edge between x_1 and x_2 is oriented from x_1 to x_2 if x is set True by our satisfying assignment, and in the other direction if x is set False. All the edges from vertices $c_{l,j}$ to vertices x_i are oriented in that direction. For a clause $l = (u, v, w)$, the edge $c_{l,1}c_{l,2}$ is oriented forwards if the literal u is True, and backwards if False (so the edge is oriented the same way as the edge u_1u_2). Similarly the edges $c_{l,2}c_{l,3}$ and $c_{l,3}c_{l,4}$ are oriented forwards iff v and w, respectively, are True. The edge $c_{l,4}c_{l,1}$ is oriented forwards if exactly one of u, v, w is True, and backwards if exactly two are True: since we have a satisfying assignment, this does define an orientation of G_I.

It is immediate that there are no directed cycles thus created, since there are no directed cycles inside (i) the set of all vertices $c_{l,j}$, (ii) the set of all vertices x_i, (iii) the set of all vertices $v_{\{x_i,y_j\}}$, (iv) the single vertex a, and all edges between these classes are directed (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\rightarrow$ (iv). Note in particular that the digraph

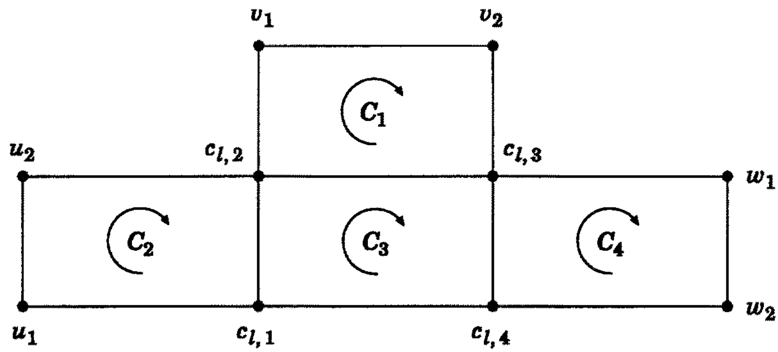

Fig. 3. The subgraph of G_I, corresponding to the clause containing literals u, v, w.

restricted to class (i) consists of independent circuits of length 4, each oriented with two forward and two backward edges. Class (ii) consists of independent edges x_1x_2 oriented according to the truth assignment.

It remains to be shown that there are no bypasses. Suppose then that there is a circuit with just one backward edge. Evidently such a circuit does not lie entirely within one of the four classes. Also, if a circuit contains vertices u and v from two nonconsecutive classes, then two backward edges are needed to get from one to the other. So our bypass is contained in some pair of consecutive classes. Classes (iii) and (iv) consist of isolated elements, and two vertices of (ii) connected to the same vertex of (iii) lie in different components, so the only remaining possibility is a circuit contained in classes (i) and (ii).

The putative bypass must consist of: a forward path inside class (i), a single (forward) edge from (i) to (ii), possibly a forward edge inside (ii), and the sole backward edge from (ii) to (i). Therefore the cycle lies entirely inside some subgraph of the form shown in Figure 3. The four circuits $C_1, \ldots, C_4$ shown in Figure 3 have flow-difference 0, by the specification of our orientation. Therefore every cycle in that subgraph has flow-difference 0, contradicting our assumption that the subgraph contains a bypass.

This completes the proof. □

Two diagram orientations of a graph G are said to be *inversion equivalent* if the flow difference of every circuit in G is the same in the two orientations. It is shown by Pretzel [8] that any two inversion equivalent orientations can be obtained from each other by a sequence of simple operations known as 'push-downs'. The proof of the result of Mosesian we used earlier implies that, if a is a vertex of a connected graph G, then every diagram orientation of G is inversion equivalent to one where a is the unique maximal element.

Given an instance I of NAE-3-SAT, let G_I be the graph constructed above. Note that different satisfying assignments of I are associated with orientations that are not inversion equivalent, since the circuit $av_{\{x_1,y_1\}}x_1x_2v_{\{x_2,y_1\}}a$ has flow difference ± 1 according to the direction of x_1x_2.

We claim that inversion equivalence classes of diagram orientations of G_I are in 1–1 correspondence with satisfying assignments of I. Indeed, given an inversion equivalence class, there is at least one orientation in the class such that a is the only maximal element; the directions of the edges x_1x_2 in such an orientation are determined by the equivalence class, and we saw above that a set of directions for these edges can occur in this way iff the associated truth assignment is a satisfying assignment for I. Furthermore, it is straightforward to check that a diagram orientation of G_I is specified uniquely by the orientations of the edges x_1x_2 corresponding to a satisfying assignment, and the requirement that a be the unique maximal element.

It is not too hard to show that counting the number of satisfying assignments of a NAE-3-SAT instance is #P-complete. The above fact then shows that counting the number of inversion equivalence classes of diagram orientations of a graph is also #P-complete.

Finally, we note that the graph G_I is clearly 4-colorable, so we have in fact proved that DIAGRAM TESTING is NP-complete even for 4-colorable graphs. This answers a question posed by Nešetřil and Rödl in [5].

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