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The Basic Theorem of Triadic Concept Analysis

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Abstract. Experiences with applications of concept lattices and the pragmatic philosophy founded by Ch. S. Peirce have suggested a triadic approach to formal concept analysis. It starts with the notion of a *triadic context* combining *objects, attributes*, and *conditions* under which objects may have certain attributes. *The Basic Theorem* of triadic concept analysis clarifies the class of structures which are formed by the *triadic concepts* of triadic contexts: These structures are exactly the *complete trilattices* up to isomorphism.

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1. Triadic Concepts

Formal Concept Analysis, as it has been developed during the last fifteen years, is based on the dyadic understanding of a concept constituted by its extension and its intension (cf. [7, 8, 13]). It starts with the primitive notion of a *formal* (or *dyadic) context* defined as a triple (G, M, I) consisting of two sets G and M and a binary relation I between G and M. A *formal* (or *dyadic) concept* of a dyadic context (G, M, I) is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$ where A' and B' result from the following *derivation operators* $(X \subseteq G, Y \subseteq M)$:

$$X \longmapsto X' := \{m \in M \mid gIm \text{ for all } g \in X\},$$

$$Y \longmapsto Y' := \{g \in G \mid gIm \text{ for all } m \in Y\}.$$

The dyadic concepts of (G, M, I) are exactly the maximal pairs (A, B) in $\mathfrak{P}(G) \times \mathfrak{P}(M)$ with $A \times B \subseteq I$ according to the component-wise set inclusion. The dyadic concepts are structured by the following order relation:

$$(A_1, B_1) \leqslant (A_2, B_2) :\Longleftrightarrow A_1 \subseteq A_2 (\Longleftrightarrow B_1 \supseteq B_2).$$

The set of all dyadic concepts together with this order relation forms a complete lattice denoted by $\underline{\mathfrak{B}}(G, M, I)$ and is called the *concept lattice* of (G, M, I).

The concept lattices of dyadic contexts are exactly the complete lattices up to isomorphism. Experiences with applications of concept lattices and the pragmatic philosophy of Ch. S. Peirce with his three universal categories have suggested a triadic approach to formal concept analysis (see [5, 9, 10]). The mathematical foundation of this approach is the theme of this paper.

As in the dyadic case, the foundation starts with the primitive notion of a formal context by defining a *triadic context* as a quadruple (G, M, B, Y) where G, M, and B are sets and Y is a ternary relation between G, M, and B, i.e., $Y \subseteq G \times M \times B$; the elements of G, M, and B are called *objects, attributes*, and *conditions*, respectively, and $(g, m, b) \in Y$ is read: the object g has the attribute m *under* the condition b (the relational notation $b(g, m)$ may also be used for $(g, m, b) \in Y$). The conditions in B are understood in a broad sense; in particular, they comprise relations, mediations, interpretations, evaluations, modalities, meanings, purposes, and reasons concerning connections between objects and attributes. For allowing shorter formulations, K_1, K_2, and K_3 are often used instead of G, M, and B, where the chosen symbols indicate that the component K_i may be viewed as a formal reference of Peirce's *i-th category*. A triadic context $\mathbb{K} := (K_1, K_2, K_3, Y)$ gives rise to numerous *dyadic contexts*; in particular, it is useful to define

$$\begin{aligned}
\mathbb{K}^{(1)} &:= \big(K_1, K_2 \times K_3, Y^{(1)}\big),\\
\mathbb{K}^{(2)} &:= \big(K_2, K_1 \times K_3, Y^{(2)}\big),\\
\mathbb{K}^{(3)} &:= \big(K_3, K_1 \times K_2, Y^{(3)}\big),
\end{aligned}$$

where $gY^{(1)}(m, b) :\Leftrightarrow mY^{(2)}(g, b) :\Leftrightarrow bY^{(3)}(g, m) :\Leftrightarrow (g, m, b) \in Y$, and, for $\{i, j, k\} = \{1, 2, 3\}$ and $A_k \subseteq K_k$,

$$\mathbb{K}_{A_k}^{ij} := (K_i, K_j, Y_{A_k}^{ij}),$$

where $(a_i, a_j) \in Y_{A_k}^{ij}$ if and only if a_i, a_j, a_k are related by Y for all $a_k \in A_k$. The *derivation operators* in $\mathbb{K}^{(i)}$ are denoted by $Z \mapsto Z^{(i)}$ and in $\mathbb{K}_{A_k}^{ij}$ by $Z \mapsto Z^{(i,j,A_k)}$. Now, a *triadic concept* of $\mathbb{K}$ is defined as a triple (A_1, A_2, A_3) with $A_i \subseteq K_i$ for $i = 1, 2, 3$ and $A_i = (A_j \times A_k)^{(i)}$ for $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$; A_1, A_2, and A_3 are called the *extent*, the *intent*, and the *modus* of the triadic concept (A_1, A_2, A_3), respectively.

PROPOSITION 1. *The triadic concepts of a triadic context* (K_1, K_2, K_3, Y) *are exactly the maximal triples* (A_1, A_2, A_3) *in* $\mathfrak{P}(K_1) \times \mathfrak{P}(K_2) \times \mathfrak{P}(K_3)$ *with* $A_1 \times A_2 \times A_3 \subseteq Y$ *according to the component-wise set inclusion.*

Proof. For $A_i \subseteq B_i \subseteq K_i (i = 1, 2, 3)$, $B_1 \times B_2 \times B_3 \subseteq Y$ implies $B_i \subseteq (A_j \times A_k)^{(i)}$ for $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$. This immediately yields the assertion of the proposition. □

The set $\mathfrak{T}(\mathbb{K})$ of all triadic concepts of $\mathbb{K} := (K_1, K_2, K_3, Y)$ is structured by the quasiorders $\lesssim_i$ and their corresponding equivalence relations $\sim_i$ defined by

$$(A_1, A_2, A_3) \lesssim_i (B_1, B_2, B_3) :\Longleftrightarrow A_i \subseteq B_i \quad \text{and}$$

$$(A_1, A_2, A_3) \sim_i (B_1, B_2, B_3) :\Longleftrightarrow A_i = B_i \quad (i = 1, 2, 3).$$

Let $[(A_1, A_2, A_3)]_i$ denote the equivalence class of $\sim_i$ represented by the triadic concept (A_1, A_2, A_3). The quasiorder $\lesssim_i$ induces an order relation $\leqslant_i$ on the set $\mathfrak{T}(\mathbb{K})/\sim_i$ of all equivalence classes of $\sim_i$.

PROPOSITION 2. *For* $\{i, j, k\} = \{1, 2, 3\}$, $(A_1, A_2, A_3) \lesssim_i (B_1, B_2, B_3)$ *and* $(A_1, A_2, A_3) \lesssim_j (B_1, B_2, B_3)$ *implies* $(A_1, A_2, A_3) \gtrsim_k (B_1, B_2, B_3)$ *for all triadic concepts* (A_1, A_2, A_3) *and* (B_1, B_2, B_3) *of* $\mathbb{K}$*; furthermore,* $\sim_i \cap \sim_j$ *is the identity on* $\mathfrak{T}(\mathbb{K})$ *for* $i \neq j$.

Proof. The first assertion follows from the fact that $A_i \subseteq B_i$ and $A_j \subseteq B_j$ implies $A_k = (A_i \times A_j)^{(k)} \supseteq (B_i \times B_j)^{(k)} = B_k$. The second assertion is a direct consequence of the definition of triadic concepts requiring that two components of a triadic concept uniquely determine the third component. □

The relational structures $\underline{\mathfrak{T}}(\mathbb{K}) := (\mathfrak{T}(\mathbb{K}), \lesssim_1, \lesssim_2, \lesssim_3)$ play an analogous role in triadic concept analysis as the concept lattices in the dyadic case. Hence a fundamental question is: What are the natural algebraic operations on $\underline{\mathfrak{T}}(\mathbb{K})$ corresponding to infima and suprema? First of all, such operations have the aim to yield triadic concepts from others. Therefore it should be first analysed how triadic concepts can be constructed within a triadic context. In the dyadic case, arbitrary concepts can be derived from single subsets $X \subseteq G$ or $Y \subseteq M$ by forming (X'', X') or (Y', Y'') (notice that $X''' = X'$ and $Y''' = Y'$). In the triadic case, one needs two subsets to generate a triadic concept:

PROPOSITION 3. *For* $X_i \subseteq K_i$ *and* $X_k \subseteq K_k$ *with* $\{i, j, k\} = \{1, 2, 3\}$, *let* $A_j := X_i^{(i,j,X_k)}$, $A_i := A_j^{(i,j,X_k)}$ *and* $A_k := (A_i \times A_j)^{(k)}$ *(if* $i < j$*) or* $A_k := (A_j \times A_i)^{(k)}$ *(if* $j < i$*). Then* (A_1, A_2, A_3) *is the triadic concept* $\mathfrak{b}_{ik}(X_i, X_k)$ *with the property that it has the smallest k-th component under all triadic concepts* (B_1, B_2, B_3) *with the largest j-th component satisfying* $X_i \subseteq B_i$ *and* $X_k \subseteq B_k$. *In particular,* $\mathfrak{b}_{ik}(A_i, A_k) = (A_1, A_2, A_3)$ *for each triadic concept* (A_1, A_2, A_3) *of* $\mathbb{K}$.

Proof. Without loss of generality we can assume $i = 1$, $j = 2$, and $k = 3$. Obviously, $X_1 \subseteq A_1$ and $X_3 \subseteq A_3$. First it has to be proved that (A_1, A_2, A_3) is a triadic concept. $A_3 = (A_1 \times A_2)^{(3)}$ is already satisfied by definition. It follows that $A_2 \subseteq A_1^{(1,2,(A_1 \times A_2)^{(3)})} = A_1^{(1,2,A_3)} \subseteq X_1^{(1,2,X_3)} = A_2$ and hence $A_2 = A_1^{(1,2,A_3)} = (A_1 \times A_3)^{(2)}$. Similarly one gets $A_1 = (A_2 \times A_3)^{(1)}$. Now, let (B_1, B_2, B_3) be a triadic concept of $\mathbb{K}$ with $X_1 \subseteq B_1$ and $X_3 \subseteq B_3$. Then

$B_2 = (B_1 \times B_3)^{(2)} = B_1^{(1,2,B_3)} \subseteq X_1^{(1,2,X_3)} = A_2$; hence $B_2 \subseteq A_2$. Let B_2 be equal A_2. Then $A_1 = A_2^{(1,2,X_3)} \supseteq B_2^{(1,2,B_3)} = B_2^{(2,1,B_3)} = (B_2 \times B_3)^{(1)} = B_1$; hence $A_1 \supseteq B_1$. It follows $A_3 = (A_1 \times A_2)^{(3)} \subseteq (B_1 \times B_2)^{(3)} = B_3$. This finishes the proof of the first assertion. If (A_1, A_2, A_3) is assumed to be a triadic concept, then $A_1^{(1,2,A_3)} = (A_1 \times A_3)^{(2)} = A_2$ and $A_2^{(1,2,A_3)} = (A_2 \times A_3)^{(1)} = A_1$; hence $\mathfrak{b}_{ik}(A_1, A_3) = (A_1, A_2, A_3)$ by the first assertion. □

It seems natural to use the described construction of triadic concepts for defining algebraic operations on $\underline{\mathfrak{T}}(\mathbb{K})$ as follows: For $i \neq k$ in $\{1,2,3\}$, the *ik-join* of two sets $\mathfrak{X}_i$ and $\mathfrak{X}_k$ of triadic concepts of $\mathbb{K}$ is defined by

$$\begin{aligned}&\nabla_{ik}(\mathfrak{X}_i, \mathfrak{X}_k)\\ &\quad := \mathfrak{b}_{ik}\Big(\bigcup\{A_i \mid (A_1, A_2, A_3) \in \mathfrak{X}_i\}, \bigcup\{A_k \mid (A_1, A_2, A_3) \in \mathfrak{X}_i\}\Big).\end{aligned}$$

In the next section, abstractions of the operations ∇_{ik} are studied on a purely order-theoretic level to derive a triadic analogue of the dyadic notion of a complete lattice.

2. Complete Trilattices

First an analogue of the dyadic notion of an ordered set is introduced (notice that an ordered set is structured by two mutually dependent relations, namely $\leqslant$ and $\geqslant$): A *triordered set* is defined as a relational structure $(S, \lesssim_1, \lesssim_2, \lesssim_3)$ for which the relations $\lesssim_i$ are quasiorders on S such that $\lesssim_i \cap \lesssim_j \subseteq \gtrsim_k$ for $\{i,j,k\} = \{1,2,3\}$ and $\sim_1 \cap \sim_2 \cap \sim_3 = id_S$ where $\sim_i := \lesssim_i \cap \gtrsim_i$ $(i = 1,2,3)$. It immediately follows that $\sim_i \cap \sim_j = id_S$ for $i \neq j$. For $x \in S$, let $[x]_i := \{y \in S \mid x \sim_i y\}$. The quasiorder $\lesssim_i$ induces on $S/\sim_i$ $(= \{[x]_i \mid x \in S\})$ an order relation $\leqslant_i$. For $\{i,j,k\} = \{1,2,3\}$ and $X_i, X_k \subseteq S$, an element u of S is called an *ik-bound* of (X_i, X_k) if $u \gtrsim_i x$ for all $x \in X_i$ and $u \gtrsim_k x$ for all $x \in X_k$; an *ik*-bound u of (X_i, X_k) is called an *ik-limit* of (X_i, X_k) if $u \gtrsim_j v$ for all *ik*-bounds v of (X_i, X_k).

PROPOSITION 4. *Let $\underline{S} := (S, \lesssim_1, \lesssim_2, \lesssim_3)$ be a triordered set. For $\{i,j,k\} = \{1,2,3\}$ and $X_i, X_k \subseteq S$, there exists at most one ik-limit u of (X_i, X_k) with $u \lesssim_k v$ for all ik-limits v of (X_i, X_k) in $\underline{S}$; the element u is called the* ik-join *of (X_i, X_k) and denoted by $\nabla_{ik}(X_i, X_k)$.*

Proof. Let u_1 and u_2 be ik-limits of (X_i, X_k) less than or equal all ik-limits of (X_i, X_k) with respect to the quasiorder $\lesssim_k$. Then, in particular, $u_1 \lesssim_k u_2$ and $u_2 \lesssim_k u_1$; hence $u_1 \sim_k u_2$. As ik-limits of (X_i, X_k), u_1 and u_2 also satisfy $u_1 \sim_j u_2$. Now, $u_1 = u_2$ follows from $\sim_j \cap \sim_k = id_S$. □

By Propositions 2 and 3, the relational structure $\mathfrak{T}(\mathbb{K})$ derived from a triadic context $\mathbb{K}$ is a triordered set in which $\mathfrak{b}_{ik}(\bigcup\{A_i \mid (A_1, A_2, A_3) \in \mathfrak{X}_i\}, \bigcup\{A_k \mid (A_1, A_2, A_3) \in \mathfrak{X}_k\})$ is always the ik-join of $(\mathfrak{X}_i, \mathfrak{X}_k)$. Therefore $\mathfrak{T}(\mathbb{K})$ is a complete trilattice which is defined as follows: A *complete trilattice* is a triordered set $\underline{L} := (L, \lesssim_1, \lesssim_2, \lesssim_3)$ in which the ik-joins exist for all $i \neq k$ in $\{1, 2, 3\}$ and all pairs of subsets of S. In a complete trilattice L, the element $0_i := \nabla_{jk}(L, L)$ $(= \nabla_{kj}(L, L))$ is uniquely determined by $0_i \lesssim_i x$ for all $x \in L$ (note that also $0_i = \nabla_{ij}(\varnothing, L) = \nabla_{ik}(\varnothing, L) = \nabla_{ji}(\varnothing, \varnothing) = \nabla_{ki}(\varnothing, \varnothing)$).

By two classes of complete trilattices it shall be indicated how complete trilattices may look like. As a first class let us consider the *complete trichains* defined as the complete trilattices $(L, \lesssim_1, \lesssim_2, \lesssim_3)$ for which $(L/\sim_i, \leqslant_i)$ is a complete chain for $i = 1, 2, 3$. In the finite case, examples are the *equilateral trichains* $\underline{TC}_n := (TC_n, \lesssim_1, \lesssim_2, \lesssim_3)$ with $TC_n := \{(x_1, x_2, x_3) \in \{0, 1, \ldots, n\}^3 \mid x_1 + x_2 + x_3 = 2n\}$ and $(x_1, x_2, x_3) \lesssim_i (y_1, y_2, y_3) :\Leftrightarrow x_i \leqslant y_i$ $(i = 1, 2, 3)$. The equilateral trichain $\underline{TC}_n$ is isomorphic to $\mathfrak{T}(\mathbb{K}_n^c)$ with $\mathbb{K}_n^c := (\{1, \ldots, n\}, \{1, \ldots, n\}, \{1, \ldots, n\}, Y_n^c)$ and $(x_1, x_2, x_3) \in Y_n^c :\Leftrightarrow x_1 + x_2 + x_3 \leqslant 2n$, where the unique isomorphism is described by $(x_1, x_2, x_3) \mapsto ([1, x_1], [1, x_2], [1, x_3])$. A *triadic diagram* of $\underline{TC}_5$ is shown in Figure 1; it makes clear how the three quasiorders give rise to three directions in the graphical representation. Surprisingly, not every finite trichain can be order embedded into an equilateral trichain; a 9-element

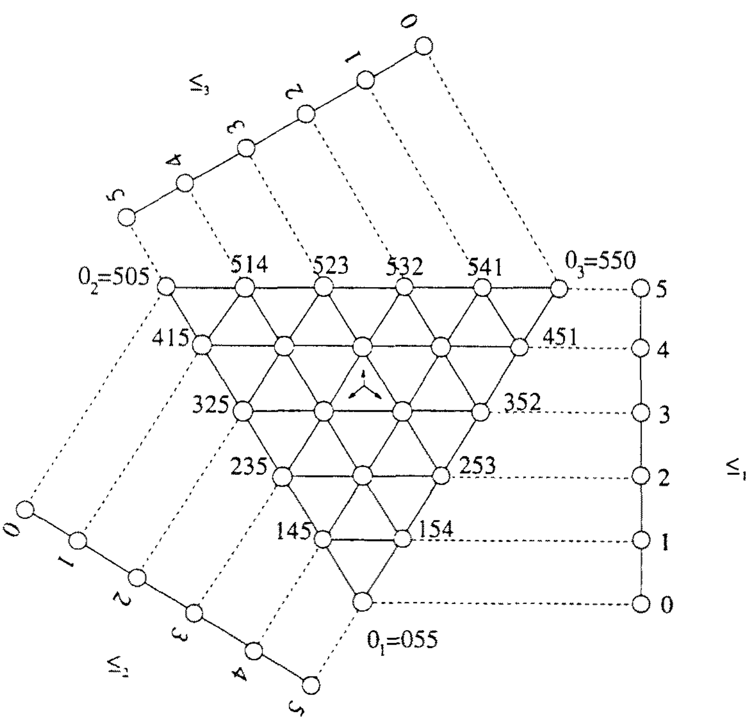

Fig. 1. The equilateral trichain $\underline{TC}_5$.

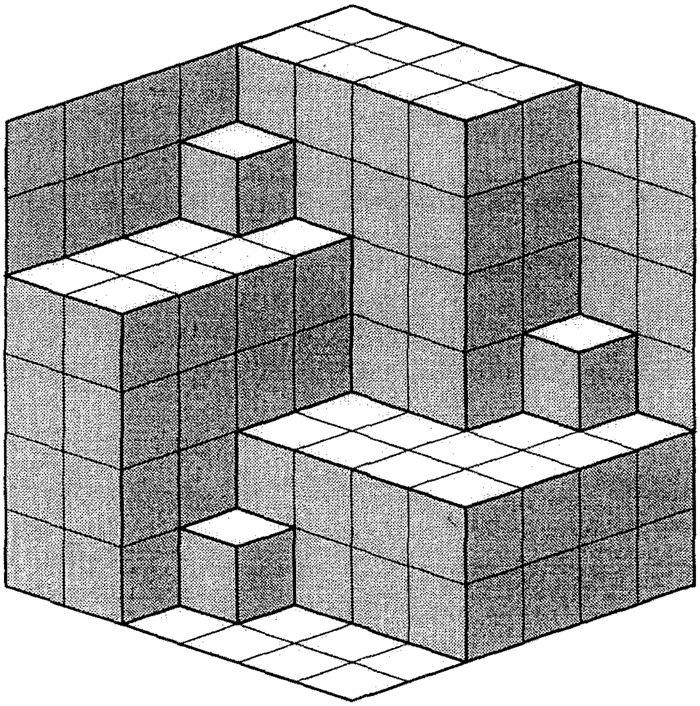

Fig. 2. 3-dimensional visualization of a triadic (6 × 6 × 6)-context.

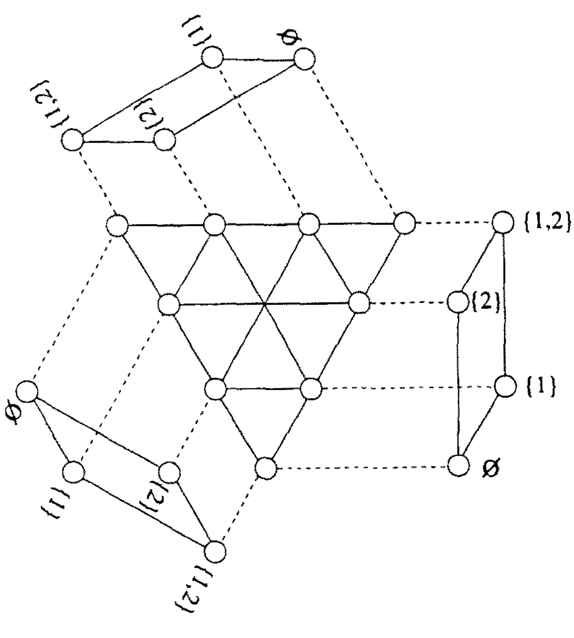

Fig. 3. The Boolean trilattice $\underline{B}(\{1,2\})$.

counterexample, which is given by the triadic concepts of the triadic context in Figure 2, was found by U. Wille [13] (for representations of triordered sets by ordered algebraic structures in general see [11] and [12]).

As a second class we consider the *complete Boolean trilattices* defined as the complete trilattices $(L, \lesssim_1, \lesssim_2, \lesssim_3)$ for which $(L/\sim_i, \leqslant_i)$ is a complete Boolean lattice for $i = 1, 2, 3$. In analogy to the dyadic case, the subsets of any set S determine a complete Boolean trilattice, namely $\underline{B}(S) := (B(S), \lesssim_1, \lesssim_2, \lesssim_3)$ with $B(S) := \{(X_1, X_2, X_3) \in \mathfrak{P}(S)^3 \mid X_1 \cap X_2 \cap X_3 = \varnothing$ and $X_i \cup X_j = S$ for $i \neq j$ in $\{1, 2, 3\}\}$ and $(X_1, X_2, X_3) \lesssim_i (Y_1, Y_2, Y_3) :\Leftrightarrow X_i \subseteq Y_i$ $(i = 1, 2, 3)$. The elements of the complete Boolean trilattice $\underline{B}(S)$ are exactly the triadic concepts of the triadic context $\mathbb{K}_S^b := (S, S, S, Y_S^b)$ with $Y_S^b := S^3 \backslash \{(x, x, x) \mid x \in S\}$; thus $\underline{B}(S) = \mathfrak{T}(\mathbb{K}_S^b)$. A triadic diagram of $\underline{B}(\{1, 2\})$ is shown in Figure 3.

In triordered sets, notions connecting the three quasiorders are of specific interest. Such a notion is given by the following definition: A triple (X_1, X_2, X_3) of subsets of a triordered set is said to be *joined* if there exists an element u with $u \gtrsim_i x_i$ for all $x_i \in X_i$ and $i \in \{1, 2, 3\}$, i.e., u is an ik-bound of (X_i, X_k) for all $i \neq k$ in $\{1, 2, 3\}$. A triple (x_1, x_2, x_3) of elements of a triordered set is *joined* if $(\{x_1\}, \{x_2\}, \{x_3\})$ is joined. In complete trilattices, joined triples of subsets can be recognized via joined triples of elements, which is the content of the following proposition.

PROPOSITION 5. *Let X_1, X_2, and X_3 be subsets of a complete trilattice. Then (X_1, X_2, X_3) is joined if and only if (x_1, x_2, x_3) is joined for all $x_i \in X_i$ and $i \in \{1, 2, 3\}$; in particular, if (X_1, X_2, X_3) is joined then $\nabla_{12}(X_1, X_2) \gtrsim_i x_i$ for all $x_i \in X_i$ and $i \in \{1, 2, 3\}$.*

Proof. Obviously, the joinedness of (X_1, X_2, X_3) implies the joinedness of each triple (x_1, x_2, x_3) in $X_1 \times X_2 \times X_3$. Conversely, let (x_1, x_2, x_3) be joined for all $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$. For $x_2 \nabla_{23} x_3 := \nabla_{23}(\{x_2\}, \{x_3\})$ and $u \gtrsim_i x_i$ $(i = 1, 2, 3)$, it follows $u \lesssim_1 x_2 \nabla_{23} x_3$ and hence $x_1 \lesssim_1 x_2 \nabla_{23} x_3$. This yields $x_2 \nabla_{23} x_3 \lesssim_2 \nabla_{13}(X_1, \{x_3\})$ and therefore $x_2 \lesssim_2 \nabla_{13}(X_1, \{x_3\})$ for all $x_2 \in X_2$ and $x_3 \in X_3$. Now it follows $\nabla_{13}(X_1, \{x_3\}) \lesssim_3 \nabla_{12}(X_1, X_2)$ and so $x_3 \lesssim_3 \nabla_{12}(X_1, X_2)$ for all $x_3 \in X_3$. Thus, (X_1, X_2, X_3) is joined. □

3. The Basic Theorem

As already mentioned in Section 2, the triadic concepts of a triadic context $\mathbb{K} := (G, M, B, Y)$ form a complete trilattice with respect to the three component-wise defined quasiorders; therefore $\mathfrak{T}(\mathbb{K})$ is called the *concept trilattice* of the triadic context $\mathbb{K}$. Conversely, every complete trilattice is isomorphic to a concept trilattice of a suitable triadic context by *The Basic Theorem*, which is the main content of this section. For formulating the theorem, some order-theoretic notions are useful: For a complete trilattice $\underline{L} := (L, \lesssim_1, \lesssim_2, \lesssim_3)$, the set of all order filters of $(L, \lesssim_i)$ is denoted by $\mathcal{F}_i(\underline{L})$ $(i = 1, 2, 3)$ where an *order filter* of the quasiordered set $(L, \lesssim_i)$ is a subset F of L for which $x \in F$ and $x \lesssim_i y$ always imply $y \in F$. A *principal filter* of $(L, \lesssim_i)$ is defined by $[x)_i := \{y \in L \mid x \lesssim_i y\}$. A subset $\mathcal{X}$ of $\mathcal{F}_i(\underline{L})$ is said to be *i-dense* with respect to $\underline{L}$ if each principal filter of $(L, \lesssim_i)$ is the intersection of some order filters from $\mathcal{X}$. Since the principal filter generated by the triadic concept (A_1, A_2, A_3) in $(\underline{\mathfrak{T}}(\mathbb{K}), \lesssim_i)$ equals $\bigcap_{a_i \in A_i} \{(B_1, B_2, B_3) \in \underline{\mathfrak{T}}(\mathbb{K}) \mid a_i \in B_i\} \in \mathcal{F}_i(\underline{\mathfrak{T}}(\mathbb{K}))$, one obtains an i-dense set $\kappa_i(K_i)$ of order filters of $(\underline{\mathfrak{T}}(\mathbb{K}), \lesssim_i)$ for $i = 1,2,3$ by defining $\kappa_i(a_i) := \{(B_1, B_2, B_3) \in \underline{\mathfrak{T}}(\mathbb{K}) \mid a_i \in B_i\}$ for $a_i \in K_i$.

THE BASIC THEOREM OF TRIADIC CONCEPT ANALYSIS. *Let* $\mathbb{K} := (K_1, K_2, K_3, Y)$ *be a triadic context. Then* $\underline{\mathfrak{T}}(\mathbb{K})$ *is a complete trilattice of* $\mathbb{K}$ *for which the ik-joins can be described as follows* ($\{i,j,k\} = \{1,2,3\}$):

$$\nabla_{ik}(\mathfrak{X}_i, \mathfrak{X}_k)$$
$$:= \mathfrak{b}_{ik}\Big(\bigcup \{A_i \mid (A_1, A_2, A_3) \in \mathfrak{X}_i\}, \bigcup \{A_k \mid (A_1, A_2, A_3) \in \mathfrak{X}_k\}\Big).$$

In general, a complete trilattice $\underline{L} := (L, \lesssim_1, \lesssim_2, \lesssim_3)$ *is isomorphic to* $\underline{\mathfrak{T}}(\mathbb{K})$ *if and only if there exist mappings* $\tilde{\kappa}_i: K_i \to \mathcal{F}_i(\underline{L})$ $(i = 1,2,3)$ *such that* $\tilde{\kappa}_i(K_i)$ *is i-dense with respect to* $\underline{L}$ *and* $A_1 \times A_2 \times A_3 \subseteq Y \Leftrightarrow \bigcap_{i=1}^{3} \bigcap_{a_i \in A_i} \tilde{\kappa}_i(a_i) \neq \varnothing$ *for all* $A_1 \subseteq K_1$, $A_2 \subseteq K_2$, *and* $A_3 \subseteq K_3$; *in particular,* $\underline{L} \cong \underline{\mathfrak{T}}(L, L, L, Y_{\underline{L}})$ *with* $Y_{\underline{L}} := \{(x_1, x_2, x_3) \in L^3 \mid (x_1, x_2, x_3)$ *is joined*$\}$.

Proof. The first assertion is covered by Proposition 3. Let $\varphi: \underline{\mathfrak{T}}(\mathbb{K}) \to \underline{L}$ be an isomorphism between complete trilattices. For $i \in \{1,2,3\}$, let us define $\tilde{\kappa}_i(a_i) := \varphi\kappa_i(a_i)$ for all $a_i \in K_i$. Since $\kappa_i(K_i)$ is i-dense with respect to $\underline{\mathfrak{T}}(\mathbb{K})$, $\tilde{\kappa}_i(K_i)$ is i-dense with respect to $\underline{L}$. Furthermore, $A_1 \times A_2 \times A_3 \subseteq Y \Leftrightarrow \bigcap_{i=1}^{3} \bigcap_{a_i \in A_i} \kappa_i(a_i) \neq \varnothing \Leftrightarrow \bigcap_{i=1}^{3} \bigcap_{a_i \in A_i} \tilde{\kappa}_i(a_i) \neq \varnothing$.

Conversely, let $\tilde{\kappa}_i: K_i \to \mathcal{F}_i(\underline{L})$ $(i = 1,2,3)$ be arbitrary maps having the desired properties. Let $\psi: L \to \mathfrak{P}(K_1) \times \mathfrak{P}(K_2) \times \mathfrak{P}(K_3)$ be given by $\psi(x) := (A_1^x, A_2^x, A_3^x)$ with $A_i^x := \{a_i \in K_i \mid x \in \tilde{\kappa}(a_i)\}$ for $i = 1,2,3$. Since $[x)_1 \cap [x)_2 \cap [x)_3 = \{x\}$ and $[x)_i = \bigcap_{a_i \in A_1^x} \tilde{\kappa}_i(a_i)$ by the assumed i-density, it follows $\bigcap_{i=1}^{3} \bigcap_{a_i \in A^x} \tilde{\kappa}_i(a_i) = \{x\}$ and, in particular, $A_1^x \times A_2^x \times A_3^x \subseteq Y$. For $\hat{A}_3^x := (A_1^x \times A_2^x)^{(3)}$ we have also $A_1^x \times A_2^x \times \hat{A}_3^x \subseteq Y$ and hence $\bigcap_{a_1 \in A_1^x} \tilde{\kappa}_1(a_1) \cap \bigcap_{a_2 \in A_2^x} \tilde{\kappa}_2(a_2) \cap \bigcap_{a_3 \in \hat{A}_3^x} \tilde{\kappa}_3(a_3) \neq \varnothing$. Because of $A_3 \subseteq \hat{A}_3$, this intersection equals $\{x\}$. Therefore $\hat{A}_3^x = A_3^x$, and the analogue holds for the indices 1 and 2. This proves $\psi(x) \in \underline{\mathfrak{T}}(\mathbb{K})$. The map $\psi: L \to \underline{\mathfrak{T}}(\mathbb{K})$ obviously preserves the quasiorders $\lesssim_1, \lesssim_2$, and $\lesssim_3$. Let $(A_1, A_2, A_3) \in \underline{\mathfrak{T}}(\mathbb{K})$ and let x be an element in the nonempty intersection $\bigcap_{i=1}^{3} \bigcap_{a_i \in A_i} \tilde{\kappa}_i(a_i)$. It immediately follows that $(A_1, A_2, A_3) = \psi(x)$. Therefore ψ is surjective and, because the intersection consists only of x, ψ is also injective. Obviously, ψ^{-1} preserves $\lesssim_1, \lesssim_2$, and $\lesssim_3$ too. Thus, ψ is the desired isomorphism. To prove $\underline{L} \cong \underline{\mathfrak{T}}(L, L, L, Y_{\underline{L}})$, we define $\tilde{\kappa}_i: L \to \mathcal{F}_i(\underline{L})$ by $\tilde{\kappa}_i(x) := [x)_i$ for $i = 1,2,3$ and $x \in L$. Clearly, $\tilde{\kappa}_i(K_i)$

is i-dense with respect to $\underline{L}$. Let $A_1 \times A_2 \times A_3 \subseteq Y_{\underline{L}}$ with $A_1, A_2, A_3 \subseteq L$. By Proposition 5, (A_1, A_2, A_3) is joined. This yields the second condition to guarantee $\underline{L} \cong \underline{\mathfrak{T}}(L, L, L, Y_{\underline{L}})$. □

For a dyadic context, the concept lattice is always isomorphic to the ordered set of its extents and antiisomorphic to the ordered set of its intents, Therefore the extent lattices and intent lattices are exactly the complete lattices up to isomorphism. Although, by *The Basic Theorem*, the concept trilattices are exactly the complete trilattices up to isomorphism, the extents, the intents, and the modi of a triadic context do not form complete lattices in general. One can even prove the following proposition:

PROPOSITION 6. *Let* $(P, \leqslant)$ *be an ordered set with smallest element* 0 *and greatest element* 1, *and let* $Y := \{(p, q, r) \in P^3 \mid 0 \neq p \leqslant q = r\}$. *Then* $(P, \leqslant) \cong (\underline{\mathfrak{T}}(P, P, P, Y)/{\sim_1}, \leqslant_1)$.

Proof. The assertion is proved if one shows that the triadic concepts of (P, P, P, Y) are the triples $((p]\backslash\{0\}, \{p\}, \{p\})$ with $p \neq 0$ in P and the triples $(\varnothing, P, P)$, $(P, \varnothing, P)$, and $(P, P, \varnothing)$. Obviously, the described triples are triadic concepts of (P, P, P, Y). Let (A_1, A_2, A_3) be an arbitrary triadic concept of (P, P, P, Y) with $\varnothing \neq A_1 \neq P$. Then also $A_2 \neq \varnothing \neq A_3$. There cannot exist $q \in A_2$ and $r \in A_3$ with $q \neq r$ or $q = 0$ or $r = 0$ because this would imply

$$A_1 \subseteq (\{q\}, \{r\})^{(1)} = \varnothing.$$

Hence $A_2 = \{p\} = A_3$ for some $p \neq 0$ in P and, consequently, $A_1 = (p]\backslash\{0\}$. □

Proposition 6 shows that every bounded ordered set may occur as the ordered set of the extents, the intents, or the modi of a triadic context and that these ordered sets given by a triadic context might be quite different. Assuming that they are complete lattices, one has the following independence result:

PROPOSITION 7. *Let* $(L, \leqslant)$ *and* $(M, \leqslant)$ *be complete lattices, and let*

$$Y := \Big\{(p, q, (r, s)) \in L \times M \times (L \times M) \mid 0 \neq p \leqslant r \text{ and } 0 \neq q \leqslant s\Big\}.$$

Then

$$(L, \leqslant) \cong \big(\underline{\mathfrak{T}}(L, M, L \times M, Y)/{\sim_1}, \leqslant_1\big)$$

and

$$(M, \leqslant) \cong \big(\underline{\mathfrak{T}}(L, M, L \times M, Y)/{\sim_2}, \leqslant_2\big).$$

Proof. The assertion is proved if one shows that the triadic concepts of $(L, M, L \times M, Y)$ are the triples $((p]\backslash\{0\}, (q]\backslash\{0\}, [p) \times [q))$ with $p \neq 0$ in L and $q \neq 0$ in M and the triples $(\varnothing, M, L \times M)$, $(L, \varnothing, L \times M)$, and $(L, M, \varnothing)$. Obviously, the described triples are triadic concepts of $(L, M, L \times M, Y)$. Let (A_1, A_2, A_3) be an arbitrary triadic concept of $(L, M, L \times M, Y)$ with $\varnothing \neq A_1 \neq L$. Then also $A_2 \neq \varnothing \neq A_3$; furthermore, $0 \notin A_1$ and $0 \notin A_2$. Using the derivation operators (1) and (2), it follows that $p := \bigvee A_1 \in A_1$ and $q := \bigvee A_2 \in A_2$ so that $A_1 = (p]\backslash\{0\}$ and $A_2 = (q]\backslash\{0\}$. Consequently, one obtains $A_3 = [p) \times [q)$ by applying the derivation operator (3). $\square$

Let us finally remark that one can construct a concept trilattice from its underlying triadic context by a nested use of *Ganter's Algorithm* for dyadic contexts (see [1]). This is explicitely discussed in [4] and implemented in [2].

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