EMBEDDING THEOREMS FOR WEIGHTED CLASSES OF HARMONIC AND ANALYTIC FUNCTIONS

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Introduction

In many problems of analysis the question arises as to the comparative strength of various norms on a set of analytic functions. For example Carleson, in his celebrated paper on the corona problem (see [1]), rests on the following proposition.

THEOREM. Let D be the unit disk, let H^p, $p \geq 1$ be a Hardy class (see [2]) of functions regular in the unit disk, let μ be a positive measure in D, and let

$$\Delta_{\delta,\varphi_0} = \left\{ re^{i\varphi} : 1-\delta < r < 1, \quad \varphi_0 - \delta < \varphi < \varphi_0 \right\}.$$

In order for the inequality

$$\left\{ \int_D |u|^p d\mu \right\}^{1/p} \leq C \, \|u\|_{H^p}$$

to hold for all functions $u \in H^p$ it is necessary and sufficient that

$$\mu\left(\Delta_{\delta,\varphi}\right) \leq C\,\delta. \tag{1}$$

If we introduce into the discussion the class $\mathcal{H}^p(\mu)$ of functions regular in the unit disk and satisfying the condition

$$\|u\|_{\mathcal{H}^p(\mu)} \equiv \left\{ \int_D |u|^p d\mu \right\}^{1/p} < \infty,$$

then (1) is a boundedness condition on the embedding operator for H^p in $\mathcal{H}^p(\mu)$.

Refining Carleson's proof, Hörmander found necessary and sufficient conditions for boundedness of the corresponding embedding operator in spaces of functions of several variables (see [3]).

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The present article is concerned with embedding theorems for certain spaces of harmonic and analytic functions. Let $\Omega \subset R_n$ be a bounded domain of n-dimensional Euclidean space with a smooth (C^2) boundary $\partial\Omega$. We denote by d_x the distance from a point $x \in \Omega$ to $\partial\Omega$, and by D_ρ a ball (arbitrary) of radius ρ located in Ω at a distance ρ from the boundary. We denote by $S^{\ell}_{p,\alpha}(\Omega)$ ($p \geq 1$, $\alpha > -1$, and $\ell \geq 0$) is an integer) the Banach space of functions harmonic in Ω with norm

$$\|u\|_{S^{\ell}_{p,\alpha}} \equiv \left\{ \int_\Omega \left(d_x^\alpha |u|^p + d_x^{p\ell+\alpha} |D^\ell u|^p \right) dx \right\}^{1/p}.$$

We also denote $S_{p,\alpha} \equiv \overset{\circ}{S}_{p,\alpha}$.

Finally, let μ be a nonnegative measure defined on Borel subsets of Ω. Along with the Banach space $S^{\ell}_{p,\alpha}$ we consider the normed space $S_p(\mu;\Omega)$ (in general, incomplete) of functions harmonic in Ω for which the norm is finite:

$$\|u\|_{S_p(\mu)} \equiv \left\{ \int_\Omega |u|^p \, d\mu \right\}^{1/p}.$$

Following is the fundamental result of the article.

<u>THEOREM 1.</u> In order for the operator of embedding of $S^{\ell}_{p,\alpha}$ into $S_q(\mu;\Omega)$ $(1 \leq q \leq p < \infty)$ to be bounded (completely continuous) it is sufficient and, for $\alpha \geq 0$, necessary that the measure μ satisfy the condition

$$\begin{gathered} \mu(D_\rho) = O\left(\rho^{(n+\alpha)q/p}\right) \\ \left(\mu(D_\rho) = o\left(\rho^{(n+\alpha)q/p}\right)\right) \end{gathered} \tag{2}$$

uniformly with respect to all D_ρ.

Our method of proof is based on the embedding theorems of S. L. Sobolev and are essentially similar to the proof used in [4, 5] to establish certain discreteness criteria for the spectrum of differential operators.

The first section is given over to the derivation of inequalities that will enable us to estimate the norms of the derivatives of a harmonic function in terms of the norm of the function itself. In the second section we prove the fundamental theorem. In the last section we discuss the necessity of condition (2) for the boundedness (compactness) of the embedding operator for spaces of analytic functions.

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§ 1. Auxiliary Inequalities

Later on we shall have need of estimates of the norms of the derivatives of a harmonic function in terms of the norm of the function itself. This type of inequality is met (for analytic functions) in Zygmund's book [6]. For harmonic functions analogous inequalities have been obtained by Nikol'skii [7].

While the present manuscript was being prepared for publication the authors discovered a recent article by Fokht [8], in which some of the results of this section have been obtained.

Let Ω be an arbitrary n-dimensional domain with boundary $\partial\Omega$, let Ω_ε be the ε-neighborhood of $\partial\Omega$ in Ω, and let $\Omega^\varepsilon = \Omega \setminus \Omega_\varepsilon$. We prove the following proposition.

LEMMA. For an arbitrary function u harmonic in Ω,* any $\varepsilon > 0$, $\alpha > -1$, $-\infty < \beta < \infty$, $p \geqslant 1$, and integers $\ell \geqslant 0$ the following inequality holds:

$$\int_{\Omega_\varepsilon} d_x^{p\ell+\alpha} |D^\ell u|^p dx + \int_{\Omega^\varepsilon} d_x^{p\ell+\beta} |D^\ell u|^p dx \leqslant C \left\{ \int_{\Omega_\varepsilon} d_x^\alpha |u|^p dx + \int_{\Omega^\varepsilon} d_x^\beta |u|^p dx \right\}. \tag{3}$$

Proof. We note that if one of the integrals on the right-hand side of inequality (3) is divergent, the lemma is trivial. We assume, therefore, that the right member of (3) is finite-valued. We partition the entire domain Ω into "layers":

$$\Omega_\kappa = \{ x \mid x \in \Omega,\ 2^\kappa \varepsilon < d_x \leqslant 2^{\kappa+1} \varepsilon \};\quad \kappa = 0, \pm 1, \pm 2, \ldots$$

We also let $\eta(|x|)$ be a smooth finite nonnegative function: $\eta(|x|) = 0$ for $|x| > 1$. Hereinafter we use the averaging operator

$$K_\rho u(x) = \frac{1}{c_0 \rho^n} \int \eta\left(\left|\frac{x-\xi}{\rho}\right|\right) u(\xi)\, d\xi,$$

where

$$c_0 = \int \eta(|\xi|)\, d\xi.$$

We assume that in the layer Ω_κ the averaging K_ρ is carried out over a region of radius $\rho = 2^{\kappa-1}\varepsilon$. It is clear that for a harmonic function $K_\rho u = u$ and, hence, for $x \in \Omega_\kappa$ we have

$$(D^\ell u)(x) = \frac{1}{c_0 \rho^{n+\ell}} \int (D^\ell \eta)\left(\left|\frac{x-\xi}{\rho}\right|\right) u(\xi)\, d\xi.$$

The latter at once yields the estimate

$$\int_{\Omega_\kappa} |D^\ell u|^p dx \leqslant \frac{C}{\rho^{p\ell+n}} \int_{\Omega_\kappa} \int_{|\xi| \leqslant \rho} |u(x-\xi)|^p d\xi\, dx \leqslant$$

$$\leqslant \frac{C}{\rho^{p\ell+n}} \int_{\Omega_{\kappa-1} \cup \Omega_\kappa \cup \Omega_{\kappa+1}} |u|^p dx.$$

Recognizing that for $x \in \Omega_\kappa$ the condition $2^\kappa \varepsilon < d_x \leqslant 2^{\kappa+1}\varepsilon$ holds, we obtain for any $\gamma \in (-\infty, \infty)$

*In other words, harmonic in any compact subdomain of Ω.

$$\int_{\Omega_\kappa} d_x^{p\ell+\gamma} |D^\ell u|^p dx \le C \int_{\Omega_{\kappa-1}\cup\Omega_\kappa\cup\Omega_{\kappa+1}} d_x^\gamma |u|^p dx. \tag{4}$$

Summing inequality (4) on κ and assuming that $\gamma=\alpha$ for $\kappa<0$ and that $\gamma=\beta$ for $\kappa\ge 0$, we arrive at the desired inequality.

Remark 1. If Ω is a bounded domain with C^2-boundary, the lemma can be restated as follows: For a harmonic function $u\in S_{p,\alpha}$ the derivative $\mathfrak{D}^\ell u \in S_{p,\alpha+p\ell}$, and the following estimate holds:

$$\int_\Omega d_x^{p\ell+\alpha} |D^\ell u|^p dx \le C \int_\Omega d_x^\alpha |u|^p dx. \tag{5}$$

Remark 2. Estimates analogous to (5) can be obtained in spaces of harmonic functions other than $S_{p,\alpha}$, for example in spaces with norm

$$\|u\| \equiv \left\{ \overline{\lim_{\kappa\to\infty}}\, 2^\kappa \int_{\Omega_{-\kappa}} |u|^p\, dx \right\}^{1/p}.$$

Remark 3. Let Ω be the exterior of a bounded domain; then in the integral on Ω^ε in inequality (3) we can clearly replace d_x with $|x|$.

§ 2. The Embedding Theorem for $\overset{\ell}{S}_{p,\alpha}$ in $S_q\,(\mu)$.

In this section we prove Theorem 1 stated in the Introduction.

In the proof of sufficiency we are only required to verify the following inequality for some $\varepsilon>0$:

$$\left\{ \int_{\Omega_\varepsilon} |u|^q d\mu \right\}^{1/q} \le C \left\{ \int_\Omega \left(d_x^\alpha |u|^p + d_x^{p\ell+\alpha} |D^\ell u|^p \right) dx \right\}^{1/p} \tag{6}$$

To do so, we use the partition of Ω_ε into layer Ω_κ, $\kappa<0$ (see §1). We construct a special covering of Ω_ε; we cover each layer Ω_κ with a finite number of balls of radius $3\cdot 2^{\kappa-2}\varepsilon$. We place the centers of the balls on a surface located a distance $3\cdot 2^{\kappa-1}\varepsilon$ from $\partial\Omega$ and pick them in such a way that the multiplicity of the covering of Ω_κ does not exceed a fixed number (which is the same for all κ). This step is permitted by the smoothness of $\partial\Omega$ for sufficiently small ε.* We observe that the radius of each ball involved in the covering of Ω_ε is equal to the distance between the ball and the boundary. As before, we denote any such ball by D_ρ, where ρ is the radius of the ball. We now choose an integer ℓ_1 such that $p\ell_1 > n$. It follows from the embedding theorems ([9], p. 372) that the following estimate holds in every D_ρ:

$$|u(x)| \le C \left\{ \rho^{-n} \int_{\mathfrak{D}_\rho} |u|^p dx + \rho^{p\ell_1-n} \int_{\mathfrak{D}_\rho} |\mathfrak{D}^{\ell_1} u|^p dx \right\}^{1/p},\ x\in \mathfrak{D}_\rho$$

*In proving the sufficiency of condition (2) we do not even need smoothness of $\partial\Omega$, but merely the existence of a finite-fold covering of Ω_ε for some $\varepsilon>0$ by balls $\mathfrak{D}_\rho$.

Also, on account of the relation $\rho < d_x < 3\rho$ for $x \in \mathcal{D}_\rho$, we have

$$\int_{\mathcal{D}_\rho} |u|^q d\mu \leq C \rho^{-(n+\alpha)q/p} \mu(\mathcal{D}_\rho) \|u\|^q_{S^{\ell_1}_{p,\alpha}(\mathcal{D}_\rho)} .$$

Summing on all balls with regard for the finite-multiplicity of the covering and making use of the lemma from §1, we obtain

$$\int_{\Omega_\varepsilon} |u|^q d\mu \leq C \sup_{\{\mathcal{D}_\rho\}} \left\{ \rho^{-(n+\alpha)q/p} \mu(\mathcal{D}_\rho) \right\} \cdot \|u\|^q_{S^{\ell}_{p,\alpha}(\Omega)} \tag{7}$$

Inequality (6) and, hence, the boundedness of the embedding operator for $S^{\ell}_{p,\alpha}$ in $S_q(\mu)$ follow from (2) and (7).

Now we have only to verify that the condition

$$\mu(\mathcal{D}_\rho) = o\left(\rho^{(n+\alpha)q/p}\right) \tag{8}$$

implies the complete continuity of the embedding operator for $S^{\ell}_{p,\alpha}$ in $S_q(\mu)$. We note first of all that the compactness of the embedding of $S^{\ell}_{p,\alpha}(\Omega)$ in $S_q(\mu, \Omega \setminus \Omega_\varepsilon)$ for every $\varepsilon > 0$ is certainly true. In order to arrive at the proper result, we need merely (see, e.g., [4]) verify for every set bounded in $S^{\ell}_{p,\alpha}$ the equiconvergence to zero of the integral

$$\int_{\Omega_\varepsilon} |u|^p d\mu \to 0 \qquad \text{as} \quad \varepsilon \to 0 . \tag{9}$$

But (7) and (8) are at once implied by (9). This proves the sufficiency of condition (2).

By virtue of the lemma, the necessity of condition (2) for $\alpha \geq 0$ only has to be proved for $\ell = 0$. Let ρ be small, and $\mathcal{D}_\rho \subset \Omega$; let us construct a ball M_ρ of radius ρ lying outside Ω and tangent to $\partial\Omega$ at a point whose distance from $\mathcal{D}_\rho$ is exactly equal to ρ. We adopt the center of this ball as the coordinate origin. We direct the axis x_1 toward the center of $\mathcal{D}_\rho$. We consider a function u harmonic in Ω, namely:

$$u(x) = \frac{C^{n/2-1}_{[\alpha]+3}(\cos\theta_1)}{r^{n+[\alpha]+1}} ,$$

in which $C^{\lambda}_m(t)$ is a Gegenbauer polynomial, $r^2 = \sum x_i^2$, $\cos\theta_1 = x_1 r^{-1}$. Inasmuch as $C^{\lambda}_m(1) = \frac{\Gamma(2\lambda+1)}{m!\,\Gamma(2\lambda)} \neq 0$, it is clearly possible to choose M and c_0 in such a way that $C^{\lambda}_m(\cos\theta_1) \geq c_0$ when $x \in \mathcal{D}_\rho$, where M and c_0 are independent of $\mathcal{D}_\rho$. Then the following estimates hold:

$$\int_{\Omega} d_x^{\alpha} |u|^p dx \leq C \int_{M_\rho}^{\infty} \frac{r^{\alpha} r^{n-1}}{r^{p(n+[\alpha]+1)}} dr \leq C \rho^{-\gamma p}, \tag{10}$$

$\gamma = n + [\alpha] + 1 - (n+\alpha)/p$, and

$$\int_{\Omega}|u|^q d\mu \geq \int_{\mathfrak{D}_\rho}|u|^q d\mu \geq c_0 \rho^{-\gamma q-(n+\alpha)q/p}\mu(\mathfrak{D}_\rho). \tag{11}$$

By the premise of the theorem the embedding operator for $S_{p,\alpha}$ in $S_q(\mu)$ is bounded, so that, using estimates (10) and (11), we obtain

$$\rho^{-\gamma-(n+\alpha)/p}\mu^{1/q}(\mathfrak{D}_\rho) \leq \|u\|_{S_q(\mu)} \leq C\|u\|_{S_{p,\alpha}} \leq C\rho^{-\gamma}.$$

Consequently, the boundedness of the embedding operator for $S^\ell_{p,\alpha}$ in $S_q(\mu)$ for $\alpha \geq 0$ implies satisfaction of the required relation (2).

We now assume that the embedding operator for $S_{p,\alpha}$ in $S_q(\mu)$ is completely continuous. We then show that condition (8) holds for $\alpha \geq 0$.

Let us suppose that (8) does not hold, whereupon there must exist a sequence of nonintersecting balls $\mathfrak{D}_{\rho_\kappa}$, $\rho_\kappa \to 0$ as $\kappa \to \infty$ such that

$$\rho_\kappa^{-(n+\alpha)q/p}\mu(\mathfrak{D}_{\rho_\kappa}) \geq a_0 > 0. \tag{12}$$

Also, let u_κ be the harmonic function constructed above and corresponding to the ball $\mathfrak{D}_{\rho_\kappa}$. Then the sequence $v_\kappa = u_\kappa \rho_\kappa^{\gamma}$ is bounded in $S_{p,\alpha}$ on account of (10). However, this sequence is not compact in $S_q(\mu)$, because for any $\varepsilon > 0$ there is a function v_κ $\kappa = \kappa(\varepsilon)$ [see (11) and (12)] such that

$$\int_{\Omega_\varepsilon}|v_\kappa|^q d\mu \geq \int_{\mathfrak{D}_{\rho_\kappa}}|v_\kappa|^q d\mu \geq c\rho_\kappa^{-(n+\alpha)q/p}\mu(\mathfrak{D}_{\rho_\kappa}) \geq ca_0 > 0,$$

contradicting the complete continuity of the embedding operator for $S_{p,\alpha}$ in $S_q(\mu)$. This proves the theorem.

Remark 1. As evident from the proof of Theorem 1, condition (2) only has to be verified for balls $\mathfrak{D}_\rho$ forming some covering of the support, closed in Ω, of the measure μ.

Remark 2. Theorem 1 with $\ell = 0$ is also valid for the embedding operator for the corresponding classes of subharmonic functions.

§ 3. Embedding Theorems for Spaces of Analytic Functions

Let Ω be a bounded domain in C^n. We denote by $\mathcal{H}^\ell_{p,\alpha}(\Omega)$ and $\mathcal{H}_p(\mu;\Omega)$ the subspaces of $S^\ell_{p,\alpha}(\Omega)$ and $S_p(\mu,\Omega)$, respectively, consisting of functions analytic in Ω. Clearly, the satisfaction of condition (2) implies boundedness (complete continuity) of the embedding operator for $\mathcal{H}^\ell_{p,\alpha}(\Omega)$ in $\mathcal{H}_q(\mu;\Omega)$. The converse, in general, is not true (for example, if Ω is not a domain of holomorphicity), but for $n=1$, $\alpha \geq 0$ condition (2) is a necessary condition for boundedness (complete continuity) of the embedding operator. The proof of this proposition is entirely analogous to the proof of necessity of condition (2) for boundedness (complete continuity) of the embedding operator for $S^\ell_{p,\alpha}(\Omega)$ in $S_q(\mu)$

in §2. Here, instead of a function $u(x)$ harmonic in Ω it is required to analyze the function $f(z) = z^{-\kappa}$ analytic in Ω, where κ is an integer $> \alpha + 2/p$. Note that the origin is situated outside Ω.

The following is therefore valid.

THEOREM 2. Let Ω be a bounded domain of the complex plane with C^2-boundary. In order for the embedding operator for $\mathcal{H}^{l}_{p,\alpha}(\Omega)$ in $\mathcal{H}_q(\mu, \Omega)$ to be bounded (completely continuous) it is sufficient, and for $\alpha \geq 0$ necessary, that the measure μ satisfy the condition

$$\mu(\mathcal{D}_\rho) = O\left(\rho^{(\alpha+2)q/p}\right)$$
$$\left(\mu(\mathcal{D}_\rho) = o\left(\rho^{(\alpha+2)q/p}\right)\right) \tag{13}$$

uniformly on the entire $\mathcal{D}_\rho$.

We prove the following proposition as a representative application of Theorem 2.

THEOREM 3. If z_κ, $\kappa = 1, 2, \ldots$ is a sequence of (distinct) points in the open unit disk $\mathcal{D}$ and satisfies the condition

$$|z_j - z_\kappa| \geq \delta \max(\rho_j, \rho_\kappa) \quad \text{for all} \quad j, \kappa = 1, 2, \ldots, \tag{14}$$

$j \neq \kappa$, where $\rho_\kappa = 1 - |z_\kappa|$, then for any function $f \in \mathcal{H}_{p,\alpha}$ $(p \geq 1)$ we have

$$\sum_\kappa (1 - |z_\kappa|)^{2+\alpha} |f(z_\kappa)|^p \leq C \int_{\mathcal{D}} |f(x+iy)|^p (1-|z|)^\alpha \, dx\, dy. \tag{15}$$

Proof. It suffices to show that for a measure μ having mass $(1 - |z_\kappa|)^{2+\alpha}$ at points z_κ and equal to zero at all other points, condition (13) is satisfied. According to Remark 1 of §2 we confined ourselves to a set of nonintersecting [by virtue of (14)] disks $|z - z_\kappa| < 1/2\, \rho_\kappa \min(\delta, 1)$, for which condition (13) is clearly satisfied. Since the disk $\mathcal{D}_\rho$ with center z_κ can be covered with a finite number (independent of κ) of disks of the form $|z - z_0| < 1/2\, |z - z_0| \min(\delta, 1)$, condition (13) is also satisfied for all $\mathcal{D}_\rho$ with centers z_κ, Q.E.D.

Inequality (15) and condition (14) are related to the interpolation problem in $\mathcal{H}_{p,\alpha}$ and H_p (see Shapiro and Shields [10]; inequality (15) is also derived in this paper for $p = 2$ and $\alpha = 0.1$).

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