EIGENFUNCTIONS OF THE DIRICHLET AND NEUMANN BOUNDARY VALUE PROBLEMS FOR THE ELLIPTIC SINH-GORDON EQUATION ON A RECTANGLE

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The Dirichlet and Neumann zero boundary value problems on a rectangle for the equation $\Delta u + \sinh u = 0$ are considered. Exact solutions are constructed by means of finite-gap integration theory.

Let R be a rectangular domain, namely,

$$R = \{ (x,y) \mid x \in [0,X],\ y \in [0,Y] \}.$$

In the present article we construct the solutions of the boundary value problem

$$\begin{cases} \Delta u + \sinh u = 0, \qquad \Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}, & (1) \\ u\Big|_{\partial R} = \begin{Bmatrix} \mathcal{N} \\ \mathcal{D} \end{Bmatrix} & (2) \end{cases}$$

with Neumann or Dirichlet zero boundary conditions on the edges of R, such that different boundary conditions are admitted on different edges.

This problem arises in differential geometry in the construction of tori with constant mean curvature [1].

Equation (1) is one of the real reductions of the sine-Gordon equation, which is well known in the theory of solitons. Finite-gap solutions of the latter equation, which were first constructed in [2], can be used to solve the problem (1), (2).

We consider the Riemann surface C of genus g of the hyperbolic curve

$$\mu^2 = \lambda \prod_{i=1}^{g} (\lambda - c_i)(\lambda - 1/\bar{c}_i), \qquad |c_i| \neq 1. \tag{3}$$

The surface consists of two copies of the complex plane pasted together along the cuts $[0,\infty]$ and $[c_i, 1/\bar{c}_i]$. We take a contour $\mathcal{L}$ on C that encompasses the cut $[0,\infty]$, and we choose a canonical basis of cycles on C in such a way that a_i encompasses the cut $[c_i, 1/\bar{c}_i]$ and $\mathcal{L} = a_1 + a_2 + \ldots + a_g$. We fix a branch of the function $\sqrt{\lambda}$ on $C \setminus \mathcal{L}$. $U = (U_1, \ldots, U_g)$ and $V = (V_1, \ldots, V_g)$, where

$$U_n = \int_{b_n} d\Omega_1, \qquad V_n = \int_{b_n} d\Omega_2$$

are the vectors formed by the b-periods of the normalized Abelian differentials of the second kind with singularities of the form

$$d\Omega_1 = d\left(\sqrt{\lambda}\right), \quad \lambda \sim \infty, \qquad d\Omega_2 = d\left(1/\sqrt{\lambda}\right), \quad \lambda \sim 0.$$

du_n denote holomorphic differentials normalized in the chosen basis in such a way that $\int_{a_n} du_m = 2\pi i \delta_{nm}$, and $B_{nm} = \int_{b_n} du_m$ is the matrix of periods, which determines the corresponding Riemann theta function

$$\theta(p) = \sum_{k \in \mathbb{Z}^g} \exp\left(\frac{1}{2}\langle Bk, k\rangle + \langle p, k\rangle\right), \qquad p = (p_1, \ldots, p_g) \in \mathbb{C}^g.$$

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For the curve (3) there is the antiholomorphic involution $\tau : \lambda \to 1/\bar{\lambda}$, the action of which generates the following transformations:

$$\tau a_n = -a_n, \qquad \tau b_n = b_n - a_n + \sum_{i=1}^{g} a_i, \qquad \tau\sqrt{\lambda} = \frac{1}{\sqrt{\bar{\lambda}}}, \qquad \tau^* d\Omega_1 = \overline{d\Omega_2}, \tag{4}$$

from which it follows that $\bar{U} = V$.

THEOREM ([3]). *All real-valued finite-gap solutions of Eq. (1) are given by the formula*

$$u(x,y) = \ln\left(\frac{\theta(\Omega + D)}{\theta(\Omega + D + \Delta)}\right)^2, \qquad \Omega = \frac{i}{2}(\alpha x - \beta y),$$
$$\Delta = \pi i(1,1,\dots,1), \qquad iD \in \mathbb{R}^g, \qquad U = \alpha + i\beta, \qquad \alpha, \beta \in \mathbb{R}^g, \tag{5}$$

where $\theta(p)$ is the theta function of the curve (3) and D is an arbitrary vector. All of the solutions are nonsingular.

Let now $\pi : \lambda \to 1/\lambda$ be an involution of C. It is not difficult to show that the basis of cycles on C can be chosen in such a way that, apart from (4), it also undergoes the transformations

$$\pi a = a\Pi, \qquad \pi b = b\Pi, \qquad \Pi = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & ⋰ & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}, \qquad a = (a_1, \dots, a_g), \qquad b = (b_1, \dots, b_g) \tag{6}$$

under the action of π. Since $\pi^*\sqrt{\lambda} = -1/\sqrt{\lambda}$, the Abelian differentials of the second kind are transformed into each other under the action of π:

$$\pi^* d\Omega_1 = -d\Omega_2. \tag{7}$$

In turn, equalities (6) and (7) ensure that the following symmetry properties hold for the theta function and the vectors α and β:

$$\theta(p) = \theta(p\Pi) = \theta(-p\Pi), \qquad \alpha = -\alpha\Pi, \qquad \beta = \beta\Pi. \tag{8}$$

Let us note that all the arguments of the theta function in (5) are purely imaginary. The theta function is periodic with periods $2\pi i N$, where N is a g-dimensional vector with integral entries. Equality to within such periods will be written in the following way:

$$z_1 \equiv z_2 \iff z_1 = z_2 + 2\pi i N, \quad N \in \mathbb{Z}^g.$$

LEMMA. *If* $D \equiv D\Pi$, *then* $u(-x,y) = u(x,y)$.
If $D \equiv D\Pi + \Delta$, *then* $u(-x,y) = -u(x,y)$.
If $D \equiv -D\Pi$, *then* $u(x,-y) = u(x,y)$.
If $D \equiv -D\Pi + \Delta$, *then* $u(x,-y) = -u(x,y)$.

This assertion is a direct consequence of (8) and the fact that $\Delta \equiv \Delta\Pi \equiv -\Delta\Pi$. Using this assertion one can easily prove the following result.

THEOREM. *The solution (5) of Eq. (1) constructed from the Riemann surface C with the involution π satisfies the Dirichlet ($\mathcal{D}$) or Neumann ($\mathcal{N}$) zero boundary conditions on the edges of R if the following conditions are satisfied:*

∂R	$\mathcal{N}$	$\mathcal{D}$
$x = 0$	$D \equiv D\Pi$	$D \equiv D\Pi + \Delta$
$y = 0$	$D \equiv -D\Pi$	$D \equiv -D\Pi + \Delta$
$x = X$	$\alpha X \equiv D\Pi - D$	$\alpha X \equiv D\Pi - D + \Delta$
$y = Y$	$\beta Y \equiv D\Pi + D$	$\beta Y \equiv D\Pi + D + \Delta$.

In particular, if the Neumann zero boundary conditions are satisfied on the entire boundary ∂R, then we get $\alpha X \equiv \beta Y \equiv 0$, $D = \pi i(\varepsilon, \varepsilon)$ if $g = 2k$, and $D = \pi i(\varepsilon, \varepsilon_{k+1}, \varepsilon)$ if $q = 2k+1$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ with $\varepsilon_i \in \{0, 1\}$. If the condition $u|_{\partial R} = 0$ holds, we have $g = 2k$, $\alpha X \equiv \beta Y \equiv 0$, and $D = \pi i((\varepsilon, \varepsilon) + (\mathbf{1}, 0))$, where $\mathbf{1}$ is a k-dimensional row whose entries are equal to 1.

By virtue of (8), the equalities $\alpha X \equiv \beta Y \equiv 0$ represent g conditions for g free parameters (branching points) of the Riemann surface C.

The curve C is a two-sheeted covering of C/π. Using the reduction technique for theta functions [4], one can easily show that the original g-dimensional theta function can be expressed in terms of the product of two theta functions of dimension $k+1$ and k if $g = 2k+1$, or k and k if $g = 2k$ in such a way that the variables x and y are separated, i.e., they appear in the arguments of different theta functions.

It turns out that we have constructed *all* solutions of the boundary value problems posed. This fact can be proved with the aid of the following important theorem.

THEOREM. *All double-periodic nonsingular solutions of Eq. (1) are finite-gap solutions.*

PROOF: We denote by Λ the lattice of periods of the solution $u(x, y)$. We "include" the "higher" flows of Eq. (1). In this case u depends on infinitely many "higher" times t_k and $u(x, y, t_1, t_2, \ldots)$ as a function of x and y satisfies Eq. (1) as before. We have

$$(\Delta + \cosh u) u_{t_k} = 0 \tag{9}$$

for all partial derivatives u_{t_k}. Since (9) is an operator on the torus $\mathbb{R}^2/\lambda$, it has a finite-dimensional kernel. It follows that u_{t_k} are linearly dependent and there exists a "higher" time t with respect to which the solution is stationary, i.e. $u_t = 0$, which proves that u is a finite-gap solution.

Final remarks:

1. Finite-gap solutions of a boundary value problem were first constructed in [5] for the nonlinear Schrödinger equation on an interval with a general boundary condition.

2. In analogy with the case considered, one can construct all the solutions of the Dirichlet and Neumann problems on a rectangle with zero boundary conditions for other elliptic equations with important physical applications, namely, the real reductions $\Delta u = \sin u$, $\Delta u = \sinh u$, and $\Delta u = \cosh u$ of the sine-Gordon equation.

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