THE COHOMOLOGY RING OF THE COLORED BRAID GROUP

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The cohomology ring is obtained for the space of ordered sets of n different points of a plane.

Artin's colored braid group of the space M_n of ordered sets of n pairwise different points of a plane.† It is not difficult to show that M_n is the space $K(\pi, 1)$ for the group I(n):

$$\pi_1(M_n) = I(n), \quad \pi_i(M_n) = 0 \quad \text{for} \quad i > 1.$$

From this it follows that the cohomologies of I(n) coincide with those of M_n (what we have in mind is the trivial action of Z):

$$H^*(I(n)) \cong H^*(M_n, \mathbf{Z}).$$

In the present note a description is given of this cohomology ring. We use a realization of M_n in the form of a complex affine space $C^n = \{z = (z_1, \ldots, z_n)\}$ with "eliminated diagonals:"

$$M_n = \{\mathbf{Z} \in \mathbf{C}^n;\ z_k \neq z_l\ \forall k \neq l\}.$$

We shall denote by A(n) the external graduated ring C_n^2 generated by one-dimensional elements $\omega_{k,l} = \omega_{l,k}$, $1 \leq k \neq l \leq n$, C_n^3 satisfying the relationships

$$\omega_{k,l}\omega_{l,m} + \omega_{l,m}\omega_{m,k} + \omega_{m,k}\omega_{k,l} = 0. \tag{1}$$

THEOREM. The homology ring of the colored braid group is isomorphic to A(n). The isomorphism $H^*(M_n Z) \cong A(n)$ is set up by the formulas

$$\omega_{k,l} = \frac{1}{2\pi i}\frac{dz_k - dz_l}{z_k - z_l}. \tag{2}$$

In other words, the one-dimensional generators $\omega_{k,l}$ correspond to circuits around the diagonals $z_k = z_l$.

COROLLARY 1. The cohomology groups of the colored braid group are torsion-free.

COROLLARY 2. The Poincaré polynomial of the manifold M_n is

$$p(t) = (1 + t)(1 + 2t)\ldots(1 + (n-1)t).$$

In other words, the cohomology groups of the manifold M_n [or of the group I(n)] are the same as for the direct product of a circle, a bouquet of two circles, . . ., a bouquet of (n−1) circles.

COROLLARY 3. The additive basis of the ring A(n) consists of all products of the form

$$\omega_{k_1, l_1}\omega_{k_2 l_2}\ldots\omega_{k_p, l_p}, \text{ where } k_s < l_s,\ l_1 < l_2 < \ldots < l_p. \tag{3}$$

COROLLARY 4. The subring of the ring of external differential forms C_n^2 generated by the forms (2) is isomorphic to A(n).

COROLLARY 5. An external polynomial in the differential forms (8) is cohomologous to zero in M_n if and only if it is equal to zero.

†The name is explained by the other definition: I(n) is the kernel of the natural homomorphism B(n) → S(n) of the group of braids consisting of n strands onto the symmetric group of permutations of the ends of the braid. In other words, I(n) consists of braids each strand of which is individualized (tinted in its own color) and ends where it begins.

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COROLLARY 6. The symmetrization of an arbitrary external polynomial of degree greater than 1 in the differential forms (2) is equal to zero.

Example. The non-obvious identity

$$\sum_{120} \omega_{1,2} \wedge \omega_{2,3} \wedge \omega_{3,4} \wedge \omega_{4,5} = 0,$$

holds, where the summation is carried out over all 120 permutations of the digits 1, . . ., 5.

It is easy to prove

LEMMA 1. There exists a stratification $M_n \xrightarrow{p} M_{n-1}$; its stratum is a plane lacking n−1 points. The action of the fundamental group of the base M_{n-1} in a cohomology of the stratum is trivial. The stratification p has a secant.

In fact, let us assume $p(z_1, \ldots, z_n) = z_1, \ldots, z_{n-1}$. Then the stratum $F_{n-1} = \{z \in C : z \neq z_1, \ldots, z_{n-1}\}$. The stratum F_{n-1} is homotopically equivalent to a bouquet of n−1 circles. The group of one-dimensional (co)homologies for the stratum is isomorphic to Z + . . . + Z (n−1 times). The fundamental group of the base is the colored braid group resulting from n−1 strands, I(n−1). Its action in the stratum is the ordinary action of a braid group in a plane with eliminated points. But the braids in I(n−1) are colored, and they do not permute the eliminated points. Consequently, I(n−1) acts trivially in a (co)homology of the stratum. The secant may be given by the formula

$$z_n = \frac{z_1 + \ldots + z_{n-1}}{n-1} + 2 \max_{1 \leqslant i,\, j \leqslant n-1} |z_i - z_j| + 1.$$

The simple proof of Theorem 1 given above is due to D. B. Fuks.

We shall consider a cohomological spectral sequence of the stratification $M_n \to M_{n-1}$. Since $\pi_1(M_{n-1})$ acts trivially in a cohomology of the stratum F_{n-1}, the term $E_2^* = H^*(M_{n-1}, H^*(F_{n-1}))$ is the same as in the direct product. The only possible differential d_2 is in fact zero (this easily follows from the existence of the secant of the surface). Thus, $E_2 = E_\infty$. So the (co)homology groups of M_n are the same as in the direct product of M_{n-1} and F_{n-1}. Putting in succession n = 2, 3, . . . (M_1 = C), we find that the (co)homologies of M_n are the same as in the direct product of a circle, a lemniscate, . . ., a bouquet of n−1 circles. Corollaries 1 and 2 are proved.

We shall construct an additive basis for $H^*(M_n, Z)$. It follows from our spectral sequence that it can be obtained from the image of the additive basis of $H^*(M_{n-1}, Z)$ under the map p^* by adding the products of its elements by n−1 one-dimensional classes of cohomologies which transform into the generators $H^1(F_{n-1}, Z)$ under the map i^* (where $i: F_{n-1} \to M_n$). We note that we may take as these one-dimensional classes cohomology classes of the differential forms $\omega_{1,n}, \omega_{2,n}, \ldots, \omega_{n-1,n}$ of (2). Putting in succession n = 2, 3, . . ., we see that the products of the type (3) of the differential forms (2) form the additive bases of $H^*(M_n, Z)$.

The differential forms (2) satisfy the relationships (1). This can be verified by direct substitution. The cohomology classes of the differential forms (2) in the ring $H^*(M_n, Z)$ a fortiori satisfy the relationships (1). We can therefore construct the ring homomorphism φ: $A(n) \to H^*(M_n, Z)$ by associating with the generators $\omega_{k,l} \in A(n)$ the differential forms of $H^*(M_n, Z)$ in accordance with formula (2). We have shown above that φ has no kernel. It is easy to prove

LEMMA 2. The ring A(n) is generated additively by the products (3).

For it follows from the anticommutative property that A(n) is generated by the products $\omega_{k_1,l_1} \cdot \ldots \cdot \omega_{k_p l_p}$, where $k_s < l_s$, $l_s \leq l_{s+1}$. The relationship (1) enables us to get rid of equal l. For example,

$$\omega_{k_1, l}\, \omega_{k_2, l} = \omega_{k_1, k_2} \omega_{k_2, l} - \omega_{k_1, k_2} \omega_{k_1, l}.$$

In both the summands the greater index of the first factor is strictly less than l. Thus all the products $\omega_{k,l}$ can be expressed additively in terms of products in which $k_s < l_s$, $l_s < l_{s+1}$. The lemma is proved.

It follows from this that the ring homomorphism φ: $A(n) \to H^*$ has no kernel. For the products (3) which generate A(n) additively transform into independent elements of H^* (we have established above that they form in H^* an additive basis). Consequently φ has no kernel; so φ is a ring isomorphism. Theorem 1 is proved.

We have at the same time proved Corollary 3, since we already know that in the ring H^* the products (3) form an additive basis. Corollaries 4 and 5 follow from the fact that, on the one hand, the cohomology classes of the forms generated by the forms (2) form the ring $H^*(M_n, Z)$, isomorphic to A(n); but on the other hand, the differential forms (2) themselves satisfy the relationships (1).

Corollary 6 follows from Corollary 5 and the finiteness of the cohomology groups $H^i(B(n))$, $i > 1$ (B(n) is the braid group formed from n strands [1].

Note. Let M be the manifold obtained from $\mathbf{C}^n$ by discarding an arbitrary number of hyperplanes

$$M = \{\mathbf{z} \in \mathbf{C}^n : a_k(\mathbf{z}) \neq 0,\ k = 1, \ldots, N\}.$$

Probably, the ring $H^*(M, \mathbf{z})$ is torsion-free and is generated by the one-dimensional classes $\omega_k = (1/2\pi i)(da_k/a_k)$, an external polynomial in ω_k being cohomologous to 0 in H^* only when it is zero.

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LITERATURE CITED

1. V. I. Arnol'd, "Skew algebraic functions and swallowtail cohomologies," Uspekhi Matem. Nauk, 23, No. 4, 247-248 (1968).