ON THE SOLUTION OF STATIONARY PROBLEMS FOR THE THERMAL CONDUCTIVITY OF HEAT-SENSITIVE BODIES IN CONTACT

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A method is given for the solution of stationary problems for the heat conductivity of thermoelastic bodies in contact. The method is illustrated by an example of an n-layer conduit.

In this work we propose a method for solving stationary problems for the thermal conductivity of heat-sensitive bodies in contact, which is illustrated by the solution of the problem of the thermal conductivity of an n-layer conduit. We consider an n-layer conduit the interior surface $r = r_0$ and the external surface $r = r_n$ of which are held at constant temperatures t_b and t_H, respectively. Between the layers of the conduit, the coefficients of thermal conductivity of which depend on the temperature, there is ideal thermal contact. The conduit is related to a cylindrical system of coordinates r, φ, z, the z axis of which coincides with the axis of the conduit. The thermal field of such a conduit is determined from the system of thermal conductivity equations

$$\frac{1}{r}\frac{d}{dr}\left[r\lambda_t^{(i)}(t_i)\frac{dt_i}{dr}\right] = 0, \quad i = \overline{1, n}, \tag{1}$$

with boundary conditions

$$t_1|_{r=r_0} = t_b, \; t_n|_{r=r_n} = t_{\text{н}}, \tag{2}$$

$$\left.\begin{aligned} t_i &= t_{i\text{н}}, \\ \lambda_t^{(i)}(t_i)\frac{dt_i}{dr} &= \lambda_t^{(i+1)}(t_{i+1})\frac{dt_{i+1}}{dr} \end{aligned}\right\} \text{ for } r = r_i, \quad i = \overline{1, n-1}, \tag{3}$$

where $\lambda_t^{(i)}(t_i)$ are the coefficients of thermal conductivity for the layers of the conduit. For convenience in performing calculations we also introduce dimensionless values

$$\rho = \frac{r}{r_0}, \; T_i = \frac{t_i}{t_0}, \tag{4}$$

the coefficients of thermal conductivity of the layers of the conduit are given in the form

$$\lambda_t^{(i)}(t_i) = \lambda_{t0}^{(i)}\bar{\lambda}_t^{(i)}(T_i), \tag{5}$$

where the zero-index factors are constants having the corresponding dimension and the factors with an overbar are dimensionless functions of the temperature; t_0 is any convenient choice of supporting temperature.

In terms of the dimensionless coordinates introduced above, problem (1)-(3) assumes the form

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\bar{\lambda}_t^{(i)}(T_i)\frac{dT_i}{d\rho}\right] = 0, \quad i = \overline{1, n}; \tag{6}$$

$$T_1|_{\rho=1} = T_b, \; T_n|_{\rho=\rho_n} = T_{\text{н}}; \tag{7}$$

$$\left.\begin{aligned} T_i &= T_{i+1}, \\ \lambda_{t0}^{(i)}\bar{\lambda}_t^{(i)}(T_i)\frac{dT_i}{d\rho} &= \lambda_{t0}^{(i+1)}\bar{\lambda}_t^{(i+1)}(T_{i+1})\frac{dT_{i+1}}{d\rho} \end{aligned}\right\} \text{ for } \rho = \rho_i, \quad i = \overline{1, n-1}. \tag{8}$$

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We assume that the dependence of the coefficients of thermal conductivity on the temperature of each of the layers of the conduit is linear, i.e., $\lambda_t^i(t_i) = \lambda_{t0}^{(i)}(1 + k_i T_i)$, where k_i are certain constants that occur in many practical situations [2, 3].

Using Kirchhoff variables $\theta_i = \int_0^{T_i} \bar{\lambda}_t^{(i)}(T)\,dT$ problem (6)-(8) assumes the form

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\theta_i}{d\rho}\right) = 0, \quad i = \overline{1, n}; \tag{9}$$

$$\theta_1|_{\rho=1} = \theta_b, \ \theta_n|_{\rho=\rho_n} = \theta_н; \tag{10}$$

$$\left.\begin{aligned} &\frac{\sqrt{1+2k_i\theta_i}-1}{k_i} = \frac{\sqrt{1+2k_{i+1}\theta_{i+1}}-1}{k_{i+1}}, && (11)\\ &\lambda_{t0}^{(i)}\frac{d\theta_i}{d\rho} = \lambda_{t0}^{(i+1)}\frac{d\theta_{i+1}}{d\rho} && (12) \end{aligned}\right\} \text{ for } \rho = \rho_i, \quad i = \overline{1, n-1},$$

where

$$\theta_b = \int_0^{T_b} \bar{\lambda}_t^{(1)}(T)\,dT; \quad \theta_н = \int_0^{T_н} \bar{\lambda}_t^{(n)}(T)\,dT.$$

The direct solution of the boundary-value problem (9)-(12) does not result in a closed-form analytic solution. Thus we approach the problem in the following way: If in conditions (11) we expand the root in a series and truncate the series after the first two terms of the expansion, then in place of (11) we obtain the following approximate conditions:

$$\theta_i = \theta_{i+1} \text{ for } \rho = \rho_i, \quad i = \overline{1, n-1}. \tag{13}$$

Thus we consider problem (9)-(12), changing conditions (11) to the following:

$$(1 + \mu_i)\theta_i = (1 + \mu_{i+1})\theta_{i+1} \text{ for } \rho = \rho_i, \quad i = \overline{1, n-1}, \tag{14}$$

where μ_i are certain as yet unknown constants.

In problem (9), (10), (12), (14) we implement the following substitution:

$$\theta_i^* = (1 + \mu_i)\theta_i. \tag{15}$$

Then the problem under consideration has the form

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\theta_i^*}{d\rho}\right) = 0; \tag{16}$$

$$\theta_1^*|_{\rho=1} = \theta_b^*, \ \theta_n^*|_{\rho=\rho_n} = \theta_н^*; \tag{17}$$

$$\left.\begin{aligned} &\theta_i^* = \theta_{i+1}^*,\\ &\gamma_i\frac{d\theta_i^*}{d\rho} = \gamma_{i+1}\frac{d\theta_{i+1}^*}{d\rho} \end{aligned}\right\} \text{ for } \rho = \rho_i, \quad i = \overline{1, n-1}, \tag{18}$$

where

$$\theta_b^* = (1 + \mu_1)\theta_b; \ \theta_н^* = (1 + \mu_n)\theta_н, \ \gamma_i = \frac{\lambda_{t0}^{(i)}}{1 + \mu_i} \quad i = \overline{1, n}.$$

It is possible to show [1] that the boundary-value problem (16)-(18) is equivalent to the following boundary-value problem:

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\gamma(\rho)\frac{d\theta^*}{d\rho}\right] = 0; \tag{19}$$

$$\theta_1^*|_{\rho=1} = \theta_b^*, \ \theta_n^*|_{\rho=\rho_n} = \theta_n^*, \tag{20}$$

where

$$\gamma(\rho)=\gamma_1+\sum_{j=1}^{n-1}(\gamma_{j+1}-\gamma_j)S_-(\rho-\rho_j),\quad S_-(\xi)=\begin{cases}1, & \xi\geqslant 0,\\ 0, & \xi<0.\end{cases}$$

Integrating Eq. (19), we obtain

$$\theta^*=C_1\left[\frac{\ln\rho}{\gamma(\rho)}-\sum_{j=1}^{n-1}\left(\frac{1}{\gamma_{j+1}}-\frac{1}{\gamma_j}\right)\ln\rho_j S_-(\rho-\rho_j)\right]+C_2. \tag{21}$$

Satisfying Eq. (21) with boundary condition (20), we find that

$$\theta^*=\frac{\theta_н^*-\theta_b^*}{\dfrac{\ln\rho_n}{\gamma_n}-\sum\limits_{j=1}^{n-1}\left(\dfrac{1}{\gamma_{j+1}}-\dfrac{1}{\gamma_j}\right)\ln\rho_j}\left[\frac{\ln\rho}{\gamma(\rho)}-\sum_{j=1}^{n-1}\left(\frac{1}{\gamma_{j+1}}-\frac{1}{\gamma_j}\right)\ln\rho_j S_-(\rho-\rho_j)\right]+\theta_b^* \tag{22}$$

or

$$\theta_i^*=A_i^*\ln\rho+B_i^*, \tag{23}$$

where

$$A_i^*=\frac{\theta_н^*-\theta_b^*}{\gamma_i\left[\dfrac{\ln\rho_n}{\gamma_n}-\sum\limits_{j=1}^{n-1}\left(\dfrac{1}{\gamma_{j+1}}-\dfrac{1}{\gamma_j}\right)\ln\rho_j\right]};$$

$$B_i^*=\theta_b^*-A_i^*\gamma_i\sum_{j=1}^{i-1}\left(\frac{1}{\gamma_{j+1}}-\frac{1}{\gamma_j}\right)\ln\rho_j.$$

Transforming to Kirchhoff variables, we have

$$\theta_i=A_i\ln\rho+B_i, \tag{24}$$

where

$$A_i=\frac{(1+\mu_n)\theta_н-(1+\mu_1)\theta_b}{\lambda_{i0}^{(i)}\left[\dfrac{1+\mu_n}{\lambda_{i0}^{(n)}}\ln\rho_n-\sum\limits_{j=1}^{n-1}\left(\dfrac{1+\mu_{j+1}}{\lambda_{i0}^{(j+1)}}-\dfrac{1+\mu_j}{\lambda_{i0}^{(j)}}\right)\ln\rho_j\right]};$$

$$B_i=\frac{1}{1+\mu_i}\left[(1+\mu_1)\theta_b-A_i\lambda_{i0}^{(i)}\sum_{j=1}^{i-1}\left(\frac{1+\mu_{j+1}}{\lambda_{i0}^{(j+1)}}-\frac{1+\mu_j}{\lambda_{i0}^{(j)}}\right)\ln\rho_j\right].$$

Solutions (24), containing arbitrary constants μ_i, are satisfied by Eqs. (9) and conditions (10), (12). We choose the constants μ_i to satisfy conditions (11). Taking one of the constants μ_i, say μ_1, equal to zero, we have an (n − 1)-th order system of equations determining the remaining n − 1 constants:

$$\frac{\sqrt{1+2k_i\theta_i|_{\rho=\rho_i}}-1}{k_i}=\frac{\sqrt{1+2k_{i+1}\theta_{i+1}|_{\rho=\rho_i}}-1}{k_{i+1}},\quad i=\overline{1,\,n-1}. \tag{25}$$

If we know the expressions for the Kirchhoff variables, then the temperature in the layers of the conduit is determined by the formula

$$T_i=\frac{\sqrt{1+2k_i\theta_i}-1}{k_i}. \tag{26}$$

For example, in the case of a two-layer conduit (n = 2) the expressions for the Kirchhoff variables have the form

$$\theta_1=K_\lambda\frac{(1+\mu)\theta_н-\theta_b}{(1+\mu)\ln\dfrac{\rho_2}{\rho_1}+K_\lambda\ln\rho_1}\ln\rho+\theta_b,$$

TABLE 1

ρ	Thermosensitive layer				Thermally insensitive layer			
	$\mu = 0{,}0249$		$\mu = 0$		$\lambda_{tc}^{(1)}, \lambda_{tc}^{(2)}$		$\lambda_{t0}^{(1)}, \lambda_{t0}^{(2)}$	
	T	t °C	T	t °C	T	t °C	T	t °C
1	1	700	1	700	1	700	1	700
1,34	0,7945	556,1	0,7924	554,7	0,8369	585,9	0,8314	582,0
1,69	0,6500	455,0	0,6466	452,6	0,7077	495,4	0,6978	488,5
2,03	0,5395	377,7	0,5352	374,6	0,6055	423,9	0,5922	414,6
2,37	0,4506	315,4	0,4455	311,9	0,5193	363,5	0,5031	352,1
$e-0$	0,3764	263,5	0,3707	259,5	0,4429	310,0	0,4241	296,9
$e+0$	0,3765	263,6	0,3810	266,7	0,4429	310,0	0,4241	296,9
3,65	0,2570	179,9	0,2600	182,0	0,3124	218,6	0,2991	209,4
4,59	0,1701	119,1	0,1720	120,4	0,2109	147,6	0,2019	141,3
5,52	0,1023	71,6	0,1037	72,4	0,1292	90,4	0,1237	86,6
6,49	0,0468	32,8	0,0473	33,1	0,0602	42,1	0,0576	40,4
e^2	0	0	0	0	0	0	0	0

$$\theta_2 = \frac{(1+\mu)\,\theta_н - \theta_b}{(1+\mu)\ln\frac{\rho_2}{\rho_1} + K_\lambda \ln \rho_1} \ln \frac{\rho}{\rho_2} + \theta_н. \tag{27}$$

Here $K_\lambda = \lambda_{t0}^{(2)}/\lambda_{t0}^{(1)}$, μ_1 is taken to be zero, and μ_2 is denoted by μ.

The unknown μ is determined by the equation

$$\frac{1}{k_1}\left[\sqrt{1 + 2k_1\left(K_\lambda \frac{(1+\mu)\,\theta_н - \theta_b}{(1+\mu)\ln\frac{\rho_2}{\rho_1} + K_\lambda \ln\rho_1}\ln\rho_1 + \theta_b\right)} - 1\right] = \\ = \frac{1}{k_2}\left[\sqrt{1 + 2k_2\left(\frac{(1+\mu)\,\theta_н - \theta_b}{(1+\mu)\ln\frac{\rho_2}{\rho_1} + K_\lambda \ln\rho_1}\ln\frac{\rho_1}{\rho_2} + \theta_н\right)} - 1\right]. \tag{28}$$

If we assume that the coefficients of thermal conductivity of the layers $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$ are constants, then the temperature in each of the layers of the conduit is determined by the formula

$$T_1 = NK_\lambda^* \ln\rho + T_b, \quad T_2 = N \ln\rho + T_н, \tag{29}$$

where

$$N = \frac{T_н - T_b}{(K_\lambda^* - 1)\ln\rho_1 + \ln\rho_2}, \quad K_\lambda^* = \frac{\lambda_t^{(2)}}{\lambda_t^{(1)}}.$$

Let the first layer of the conduit, of thickness $e - 1$ ($\rho_1 = e$), be prepared from U12 steel and the second layer, of thickness $e^2 - e$ ($\rho_2 = e^2$) of steel 8[2]. Assume that the temperature of the internal surface of the conduit is $t_b = 700$°C, and the temperature of the external surface is $t_H = 0$. We take the supporting temperature t_0 to be 700°C. The coefficients of thermal conductivity of the layers of the conduit in the temperature range 0-700°C are described with sufficient precision by the following linear relations $\lambda_t^{(1)} = 47.5\,(1 - 0.37T)$ W/m·deg, $\lambda_t^{(2)} = 64.5(1 - 0.49T)$ W/m·deg. Thus $\lambda_{t0}^{(1)} = 47.5$ W/(m·deg); $k_1 = -0.37$, $\lambda_{t0}^{(2)} = 64.5$ W/(m·deg); $k_2 = -0.49$, $K_\lambda = 1.36$, $T_b = 1$, $T_H = 0$, $\theta_b = \int_0^1 (1 - 0.37T)\,dT = 0.815$, $\theta_H = 0$. For the chosen values, the constant μ determined from Eq. (28) is equal to 0.0249.

The table shows the values for the temperature along the radius of a two-layer conduit. In the first four columns we give the values of the dimensionless T and the real t temperatures in the layers of the heat-sensitive conduit: The first two columns are the temperature values obtained by the proposed method [formulas (26)-(28)], and the second two columns are the approximate values for the temperatures when we retain the first two terms of the series expansion for the roots of the quadratic in conditions (11) [formulas (26) and (27) for $\mu = 0$]. Clearly, the maximal divergence between the exact values for the temperature obtained by the proposed method and the above-mentioned approximate values is about 1.5%; the approximate solution has a jump of 7.2°C at the transition through the contact boundary for the layers of the conduit, i.e., there is no satisfaction of the condition of equality of temperature on the contact surface of the layers. In the last four columns we present the values of the dimensionless and real temperatures in a conduit that does not possess thermal sensitivity. The numbers in the fifth and sixth columns correspond to the case in which the coefficients of thermal conductivity were taken to equal their mean values in the temperature range 0–700°C, i.e., $\lambda_{tc}^{(1)} = \frac{1}{700}\int_0^{700} \lambda_t^{(1)}(t)\,dt = 38{,}7$ W/(m·deg), $\lambda_{tc}^{(2)} = \frac{1}{700}\int_0^{700} \lambda_t^{(2)}(t)\,dt = 48.7$ W/(m·deg),

and the numbers in the last two columns correspond to the maximal values of the coefficients of thermal conductivity $\lambda_{t0}^{(1)}$ and $\lambda_{t0}^{(2)}$ for the given temperature range. As is evident from the table, the greatest deviation of the temperature for the averaged values of the coefficients of thermal conductivity from their exact values is roughly 15% (≃48°C), and the largest deviation of the temperature for maximal values of the coefficients of thermal conductivity from the exact values of the temperature is roughly 10% (≃37°C).

The example of the solution of the thermal conductivity problem for a multilayer conduit illustrates the proposed method for solving the thermal conductivity problem for heat-sensitive bodies in contact. This method is also applicable to the solution of problems concerning the thermal conductivity of heat-sensitive bodies with convective heat-exchange. The effect of the temperature dependence of the coefficients of thermal conductivity for bodies in contact on the value of the temperature is evident in comparison with its values for constant coefficients of thermal conductivity.

LITERATURE CITED

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