BOUNDS FOR THE SPECTRAL RADIUS OF POSITIVE OPERATORS

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Beginning with the work of M. G. Krein and L. V. Kantorovich, methods of partially ordered spaces have been used to derive bounds for characteristic values and spectral radii of linear operators. We present below some new results in this direction.

1. Let K be a cone in the real Banach space E (for an explanation of our terminology see [1, 2]). An element $v_0 \in K$ is called a quasi-interior element of K if $l(v_0) > 0$ for any nonzero functional l of the conjugate semigroup K^*. For example in the cone of nonnegative functions in L_p, a quasi-interior function is a function taking positive values on a set of complete measure.

THEOREM 1. Let the positive, completely continuous, linear operator A satisfy the inequality

$$A^m v_0 \leqslant \alpha v_0, \tag{1}$$

where v_0 is a quasi-interior element of the cone K. Then

$$r(A) \leqslant \sqrt[m]{\alpha}, \tag{2}$$

where r(A) is the spectral radius of A.

THEOREM 2. Let K be a normal and reproducing cone. Let the positive linear operator A be unbounded above and satisfy the inequality (1). Then (2) holds.

These theorems supplement and generalize the results stated above concerning upper bounds of the spectral radius and also results concerning inconsistent inequalities [2-6].

2. Before proving the above theorems we state and prove a simple auxiliary result.

LEMMA 1. Let l_0 be a non-null positive linear functional such that

$$l_0(Ax) \geqslant r(A)\, l_0(x) \qquad (x \in K), \tag{3}$$

where A is a positive linear operator. Let (1) hold for some quasi-interior element v_0. Then (2) holds.

PROOF. First let $\alpha > 0$. We write

$$w_0 = \alpha^{1-\frac{1}{m}} v_0 + \alpha^{1-\frac{2}{m}} A v_0 + \cdots + A^{m-1} v_0. \tag{4}$$

Clearly w_0 is a quasi-interior element (since v_0 is a quasi-interior element). Since

$$A w_0 = \alpha^{1-\frac{1}{m}} A v_0 + \alpha^{1-\frac{2}{m}} A^2 v_0 + \cdots + A^m v_0,$$

it follows from (1) that

$$A w_0 \leqslant \alpha^{\frac{1}{m}} w_0. \tag{5}$$

Assume that (2) does not hold; we have

$$r(A)\, w_0 - A w_0 \geqslant \delta w_0,$$

where δ is positive. This implies that $r(A) w_0$ is a quasi-interior element and so

$$r(A)\, l_0(w_0) - l_0(A w_0) > 0$$

and (3) is contradicted.

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Now let $\alpha = 0$. For any $\varepsilon > 0$ we have

$$A^m v_0 \leqslant \varepsilon^m v_0$$

and what has already been proved implies $r(A) \leq \varepsilon$. It remains to note that ε is positive.

The lemma is proved.

We now prove Theorems 1 and 2.

In the proof of Theorem 1 we use a theorem due to M. G. Krein [1] stating that the spectral radius (if it is positive) of a positive completely continuous operator is a characteristic value of the operators A and $A,^*$ corresponding to characteristic vectors in the cones K and K^* if the linear span of K is dense E. Under the conditions of Theorem 1 the density of the linear span of the cone K in E follows from the existence of quasi-interior elements in the cone. Let l_0 be a positive functional satisfying the equation $A^* l_0 = r(A)\, l_0$. (Frein's theorem shows that such a functional exists). It follows from the definition of a conjugate operator that

$$l_0(Ax) = r(A)\, l_0(x) \qquad (x \in E). \tag{6}$$

It only remains to use Lemma 1.

We now turn to Theorem 2. We recall that the u_0-boundedness above of the operator A implies the existence of a nonzero element $u_0 \in K$ and a positive integer n such that

$$A^n x \leqslant \beta(x)\, u_0 \qquad (x \in K). \tag{7}$$

We write E_{u_0} for the set of $x \in E$, for which the so-called u_0-norm

$$\|x\|_{u_0} = \inf_{\{\beta:\, -\beta u_0 \leqslant x \leqslant \beta u_0\}} \beta$$

is finite. It is known that the normality of K implies the completeness of the space E_{u_0} in the u_0-norm [1, 2]. The intersection $K_{u_0} = K \;\; E_{u_0}$ is a solid and normal cone in E_{u_0} (it is also sharp in the sense of M. G. Krein). The operator $B = A^{nm}$ maps K_{u_0} into itself and so [1] B^* has in $E^*_{u_0}$ a characteristic vector l_0 of $K^*_{u_0}$ corresponding to a characteristic value equal to r(B):

$$B^* l_0 = r(B)\, l_0. \tag{8}$$

We are clearly interested only in the case in which $r(A) > 0$; in this case $r(B) > 0$. We write

$$l_0(x) = \frac{1}{r(B)}\, l_0(Bx) \qquad (x \in E). \tag{9}$$

This functional is positive and thus continuous since we are dealing with a reproducing cone [10]. The relation (9) can be considered to be equivalent to condition (3). The equality (1) implies that $Bv_0 \leq \alpha^n v_0$. Hence Lemma 1 yields the inequality

$$r(B) \leqslant \alpha^n, \tag{10}$$

which in turn yields (2).

Theorems 1 and 2 are proved.

Theorem 1 can also be proved by first establishing the inequality (10) without having recourse to the construction of the element (4); in our opinion, however, the derivation of (5) from (1) is of independent interest. We note that the essential part of the reasoning in the proof of Theorem 2 was used by one of the authors in an investigation of irresolvable linear operators [7].

3. As an example we consider the integral operator

$$Ax(t) = \int_\Omega K(t, s)\, x(s)\, ds, \tag{11}$$

where Ω is a bounded closed set of a finite-dimensional space. We assume that the kernel of this operator is finite. We also assume that A is completely continuous in some L_p space. Let $t \in \Omega$

$$\int_\Omega K(t, s)\, v_0(s)\, ds \leqslant \alpha v_0(t), \tag{12}$$

for almost all $t \in \Omega$, where $v_0(t) \in L_p$ and $v_0(t)$ is almost everywhere positive. Then Theorem 1 implies that $r(A) \leq \alpha$.

4. The natural question arises of whether (2) follows from (1) for arbitrary positive linear operators A. It turns out that (2) does not follow from (1), and we give a counter-example (we stress that the cone K is solid in this example).

Let E be the space of functions $f(z)$ analytic in the disc $|z| \leq 1$, continuous in $f(z)$, and taking real values for real z. This space can be considered as a real Banach space with norm

$$\|f\| = \max_{|z| \leqslant 1} |f(z)|. \tag{13}$$

Let K be the set of functions $f(z) \in E$, taking nonnegative values for real $z \in [-1, -1/2]$. We easily see that the cone K is solid (it does not possess the property of normality). F. Bonsall [8] considers this space from another point of view.

In E we define the linear operator

$$Af(z) = -(z + 1/2)\, f(z). \tag{14}$$

The spectrum of this operator coincides with the disc $|\lambda + 1/2| \leq 1$ and so

$$r(A) = 3/2.$$

But

$$A v_0 \leqslant v_0/2,$$

where $v_0(z) \equiv 1$. Hence (1) does not imply (2) for the operator (14).

This example supplies a negative answer to one of the questions posed in [6].

5. We now present another theorem concerning spectral radii; this theorem is similar to some assertions proved in [9] for spaces with mini-hedral cones.

Let E_1 and E_2 be Banach spaces with cones K_1 and K_2 respectively. Let K_1 be reproducing and let K_2 be normal. Let T be a mapping of K_1 into K_2 satisfying the inequality

$$\| Tx \| \geqslant c \| x \| \qquad (x \in K_1), \tag{15}$$

where c is a constant, $c > 0$ (we do not assume that T is linear).

THEOREM 3. Let A and B be positive linear operators with domains in E_1 and E_2 respectively. Let

$$TAx \leqslant BTx \qquad (x \in K_1). \tag{16}$$

Then we have

$$r(A) \leqslant r(B). \tag{17}$$

PROOF. Let f K_1, $f \neq 0$. From (16) we have

$$TA^n f \leqslant B^n T f \qquad (n = 1, 2, \ldots), \tag{18}$$

and, since K_2 is normal, we have

$$\| TA^n f \| \leqslant M \| B^n T f \| \qquad (n = 1, 2, \ldots), \tag{19}$$

where M is a positive number. Moreover (15) yields

$$\| A^n f \| \leqslant \frac{M}{c} \| B^n T f \| \qquad (n = 1, 2, \ldots). \tag{20}$$

Hence

$$\overline{\lim_{n\to\infty}} \| A^n f \|^{1/n} \leqslant \overline{\lim_{n\to\infty}} \| B^n \|^{1/n} \left(\frac{M}{c} \| Tf \| \right)^{1/n} = \overline{\lim_{n\to\infty}} \| B^n \|^{1/n}.$$

Using this result and the formula

$$r(B) = \lim_{n\to\infty} \| B^n \|^{1/n}$$

due to I. M. Gel'fand, we obtain

$$\overline{\lim_{n\to\infty}} \| A^n f \|^{1/n} \leqslant r(B) \qquad (f \in K_1). \tag{21}$$

Any element $f \in E_1$ can be expressed in the form $f = f_1 - f_2$, where $f_1, f_2 \in K_1$. Clearly,

$$\overline{\lim_{n\to\infty}} \| A^n (f_1 - f_2) \|^{1/n} \leqslant \overline{\lim_{n\to\infty}} (\| A^n f_1 \| + \| A^n f_2 \|)^{1/n} \leqslant \overline{\lim_{n\to\infty}} \{ 2^{1/n} \max [\| A^n f_1 \|^{1/n}, \| A^n f_2 \|^{1/n}] \}.$$

Hence (21) yields

$$\overline{\lim_{n\to\infty}} \| A^n f \|^{1/n} \leqslant r(B) \qquad (f \in E_1). \tag{22}$$

The inequality (22) implies that, for $|\lambda| > r(B)$, the equation

$$\lambda x = Ax + f \tag{23}$$

has the solution

$$x^* = \sum_{n=0}^{\infty} A^n f / \lambda^{n+1}, \tag{24}$$

the series on the right converging in norm.

Hence (17) holds and the theorem is proved.

We note that in the proof of Theorem 3 we have used the following useful corollary to Gel'fand's formula: the inequality

$$\overline{\lim_{n\to\infty}} \| A^n f \|^{1/n} \leqslant a \qquad (f \in E)$$

implies that $r(A) \leq a$.

6. THEOREM 4. Let the operator T act from the Banach space E_1 into the Banach space E_2, and let T be partially ordered by the reproducing cone E_2 and satisfy the conditions

1) $-Tx \quad K_2$ for every nonzero x in E_1.

2) $T(tx) = |t|\, Tx$.

Let the linear operator A with domain E_1 satisfy the following inequality analogues to (16):

$$TAx \leqslant BTx \qquad (x \in E_1),$$

where B is a positive linear operator acting in E_2.

Then every characteristic value λ of A satisfies the inequality $|\lambda| \leq (A)$.

PROOF. Let λ be a characteristic value of A corresponding to the characteristic vector x_0:

$$Ax_0 = \lambda x_0 \qquad (x_0 \in E_1).$$

Then

$$|\lambda|\, Tx_0 = T(\lambda x_0) = TAx_0 \leqslant BTx_0.$$

We write $y_0 = Tx_0$. Plainly $-y_0 \bar{\in} K_2$, $y_0 \neq \theta$, and $By_0 \geq |\lambda|\, y_0$. We will prove that $r(B) \geq |\lambda|$. Assume that this not so: $r(B) < |\lambda|$. We write f for the difference $By_0 - |\lambda|\, y_0$. Clearly $f \geq \theta$. Two cases are possible: 1) $f = \theta$, 2) $f \neq \theta$. In the first case $|\lambda|$ is a characteristic value of B and so $|\lambda| \leq r(B)$; this leads to a contradiction. Consider the second case: $f \neq \theta$. Our assumption implies that the equation $|\lambda|\, x = Bx - f$ has the single solution $x = y_0$ where

$$y_0 = -\sum_{n=0}^{\infty} B^n f.$$

Since $f \in K_2$, $f \neq \theta$, and $y_0 \in K_2$, $y_0 \neq \theta$. This contradicts the fact that $y_0 \in K_2$, and the theorem is proved.

The conditions of Theorem 4 are less restrictive than the conditions of Theorem 3. This is to be expected, since the spectral radius can be larger than the supremum of the absolute values of all characteristic values.

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