EXTENSIONS OF SYMMETRIC OPERATORS AND SYMMETRIC BINARY RELATIONS

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Various classes of extensions of symmetric operators with equal (finite or infinite) defect numbers are described in terms of abstract boundary conditions. The dual problem of the description of extensions of a symmetric binary relation is also considered.

1. Let A be a closed symmetric operator with dense domain of definition in a separable Hilbert space H and with equal defect numbers (n, n) ($n \leq \infty$). The problem of the description of all self-adjoint extensions of A was solved in the classical works of J. von Neumann. There is, however, interest in the question of the description of extensions of A in terms which, in the case of differential operators, immediately reduce to boundary conditions. A series of results in this direction (in the case $n < \infty$) were obtained in [1–3].*

In this paper we propose a method of description of various classes of extensions of A (among them self-adjoint, dissipative, accumulative), in terms of "abstract boundary conditions," which is applicable for arbitrary $n \leq \infty$.

2. Let us cite some results about binary relations in a Hilbert space H.

Let M_θ be an arbitrary closed set in $H \oplus H$. One says that an ordered pair of elements $x, x' \in H$ belongs to a binary relation $\theta = \theta_M$ and one writes $x\theta x'$, if $\{x, x'\} \in M_\theta$. A binary relation θ_1 is called an extension of $\theta (\theta_1 \supset \theta)$ if $x\theta_1 x'$ follows from $x\theta x'$.

Definition 1. A binary relation θ is called dissipative (accumulative, symmetric) if 1) the set M_θ is linear and 2) if $x\theta x'$, then $\text{Im}\,(x', x) \geq 0$ ($\text{Im}\,(x', x) \leq 0$, $\text{Im}\,(x', x) = 0$). A dissipative (accumulative symmetric) relation is called maximal dissipative (accumulative, symmetric) if it does not have a proper dissipative (accumulative, symmetric) extension. A symmetric relation is called Hermitian if it is simultaneously maximal dissipative and maximal accumulative.

The following theorem was proved in [4-5] (the assertion on Hermitian relations was proved in [6]).

THEOREM 1. For any contraction K in H (i.e., $\|K\| \leq 1$), the binary relations defined by the equations†

$$(K - E)\; x' + i\,(K + E)x = 0, \tag{1}$$

$$(K - E)x' - i\,(K + E)x = 0, \tag{2}$$

are maximal dissipative and maximal accumulative, respectively. Conversely, each maximal dissipative (accumulative) binary relation is representable in the form (1) [(2)], where the contraction K is uniquely defined by the relation. A maximal dissipative (accumulative) relation is maximal symmetric precisely when the operator K in (1) [(2)] is isometric. The general form of Hermitian operators is given by (1) or (2), where K is a unitary operator in H. The general form of any (in general, not maximal) dissipative (accumulative) binary relation in H is, respectively, given by the formulas

*The construction of A. V. Shtraus [3] is also suitable for an operator with a nondense domain of definition.
† E is the identity operator.

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$$K(x' + ix) = x' - ix, \quad x' + ix \in D(K), \tag{3}$$
$$K(x' - ix) = x' + ix, \quad x' - ix \in D(K), \tag{4}$$

where K is a linear operator for which $\|Kf\| \leq \|f\|$ for all $f \in D(K)$ [D(K) is the domain of definition of K]. A dissipative (accumulative) binary relation is symmetric if and only if the operator K in (3) [(4)] is isometric.

3. Let us introduce the concept of a space of boundary values of A.

Definition 2. The triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$, where $\mathcal{H}$ is a separable Hilbert space and Γ_1 and Γ_2 are linear transformations of $D(A^*)$ into $\mathcal{H}$, is called a space of boundary values of A if

1) for any $y, z \in D(A^*)$

$$(A^*y, z)_H - (y, A^*z)_H = (\Gamma_1 y, \Gamma_2 z)_{\mathcal{H}} - (\Gamma_2 y, \Gamma_1 z)_{\mathcal{H}};$$

2) for any $Y_1, Y_2 \in \mathcal{H}$ there exists a vector $y \in D(A^*)$ such that $\Gamma_1 y = y_1$, $\Gamma_2 y = Y_2$;

3) if $y \in D(A)$, then $\Gamma_1 y = \Gamma_2 y = 0$.

For various classes of differential operators, spaces of boundary values were constructed by many authors (cf., for example, [6-8]).

Let $(\mathcal{H}, \Gamma_1, \Gamma_2)$ be an arbitrarily selected space of boundary values of A. Recall that a linear operator B in H is called dissipative (accumulative) if for all $f \in D(B)$ $\operatorname{Im}(Bf, f)_H \geq 0$ ($\operatorname{Im}(Bf, f)_H \leq 0$). A dissipative (accumulative) operator is called maximal dissipative (accumulative) if it does not have a proper dissipative (accumulative) extension.

THEOREM 2. For any contraction K in $\mathcal{H}$ the restriction of A^* to the set of vectors $y \in D(A^*)$ satisfying the condition

$$(K - E)\Gamma_1 y + i(K + E)\Gamma_2 y = 0 \tag{5}$$

or

$$(K - E)\Gamma_1 y - i(K + E)\Gamma_2 y = 0, \tag{6}$$

is, respectively, a maximal dissipative (accumulative) extension of A. Conversely, each maximal dissipative (accumulative) extension of A is the restriction of A^* to the set of $y \in D(A^*)$ satisfying (5) [(6)], where the contraction K is uniquely defined by the extension. Maximal symmetric extensions of A in H are characterized by conditions (5) [(6)], in which K is an isometric operator. These conditions define self-adjoint extensions if K is unitary. The general form of dissipative (accumulative) extensions of A is given by the conditions

$$K(i\Gamma_1 y + i\Gamma_2 y) = \Gamma_1 y - i\Gamma_2 y, \Gamma_1 y + i\Gamma_2 y \in D(K), \tag{7}$$

$$K(\Gamma_1 y - i\Gamma_2 y) = \Gamma_1 y + i\Gamma_2 y, \quad \Gamma_1 y - i\Gamma_2 y \in D(K), \tag{8}$$

respectively, where K is a linear operator with $\|Kf\| \leq \|f\|$, $f \in D(K)$, and the general form of symmetric extensions by formulas (7) and (8), where K is an isometric operator.

We shall carry out the proof for maximal dissipative extensions (the remaining cases are considered in an analogous manner).

Let $\tilde{A}$ be a maximal dissipative extension of A. Then $\tilde{A} \subset A^*$ [9]. Let $M_\theta = \{\{\Gamma_1 y, \Gamma_2 y\} \in \mathcal{H} \oplus \mathcal{H} \mid y \in D(\tilde{A})\}$. Then θ is a dissipative binary relation in $\mathcal{H}$. If $\tilde{\theta} \supset \theta$ and $\tilde{\theta}$ is a dissipative binary relation, then the operator $\tilde{\tilde{A}}$, defined as the restriction of A^* to $D(\tilde{\tilde{A}}) = \{y \in D(A^*) \mid \{\Gamma_1 y, \Gamma_2 y\} \in M_{\tilde{\theta}}\}$, is a dissipative extension of $\tilde{A}$, i.e., $\tilde{A} = \tilde{\tilde{A}}$ and $\tilde{\theta} = \theta$. Consequently, θ is a maximal dissipative relation, and the required result follows from Theorem 1.

Let $\tilde{A}$ be the restriction of A^* to the set $D(\tilde{A})$ of vectors $y \in D(A^*)$ satisfying (5). One immediately verifies that $\tilde{A}$ is a dissipative extension of A. Let $\tilde{\tilde{A}}$ be a dissipative extension of $\tilde{A}$. Let us denote $M_\theta = \{\{\Gamma_1 y, \Gamma_2 y\} \mid y \in D(\tilde{A})\}$, $M_{\tilde{\theta}} = \{\{\Gamma_1 y, \Gamma_2 y\} \mid y \in D(\tilde{\tilde{A}})\}$. Obviously, $\tilde{\theta} \supset \theta$. But by Theorem 1, θ is a maximal dissipative relation, i.e., $\tilde{\theta} = \theta$, from which $\tilde{\tilde{A}} = \tilde{A}$. The theorem is proved.

Similar results are contained in [4–6] for operators for a special form (differential operators in a space of vector functions).

Let us note one particular case of the "boundary condition" (5):

$$\Gamma_1 y = B\Gamma_2 y, \tag{9}$$

where B is a bounded dissipative operator in $\mathcal{H}$. In the case $n < \infty$ N. A. Talyush [10] considered the extension $\tilde{A}_B$ corresponding to (9). In [10] it was assumed that for A there exists a space of boundary values $(\mathcal{H}, \Gamma_1, \Gamma_2)$ such that $\dim \mathcal{H} = n$. Below it will be shown that this is always valid. If $\dim \mathcal{H} = n < \infty$, then a $2n^2$-dimensional Lebesgue measure μ is induced in a natural way in the set of linear operators in $\mathcal{H}$. It was proved in [10] that for almost all B in (9) (with respect to the measure μ) the operator $\tilde{A}_B$ is of diagonal type (cf. [10]). Moreover, if A is a completely nonself-adjoint operator with singular spectrum, then for almost all B the operator $\tilde{A}_B$ has a complete system of eigenvectors. In reality these results are valid for extensions of the general form (5) for almost all K. For the proof one must note that for almost all K the operator (K−E) is invertible (cf. [11]). Further, the inverse image of a set of measure zero under the transformation

$$K \to -i(K-E)^{-1}(K+E) = -i[E + 2(K-E)^{-1}]$$

has measure zero. This follows from the unimodularity of the group GL (n, C) and the fact that Haar measure on this group is equivalent to μ (cf. [12]).

4. The following theorem gives an answer to the question of the existence of a space of boundary values.

THEOREM 3. For any symmetric operator A with defect indices (n, n) $(n \leq \infty)$ there exists a space of boundary values $(\mathcal{H}, \Gamma_1, \Gamma_2)$ with $\dim \mathcal{H} = n$.

Proof. As is known, $D(A^*) = D(A) \dotplus N_+ \dotplus N_-$, where N_+ and N_- are the eigensubspaces of A^* which correspond to the eigenvalues i and −i. Since $\dim N_+ = \dim N_- = n$ $(\leq \infty)$, there exists an isometric transformation from N_- to N_+, which we shall denote by U. Let P_+ and P_- be the projections of $D(A^*)$ onto N_+ and N_-, respectively. Let us put $\mathcal{H} = N_+$ (with the scalar product induced from H), $\Gamma_1 = P_+ + UP_-$, $\Gamma_2 = -iP_+ + iUP_-$.

The decomposition $y = y_0 + P_+y + P_-y$ $(y \in D(A^*),\ y_0 \in D(A))$ shows that for any $y, z \in D(A^*)$ $(A^*y, z)_H - (y, A^*z)_H = 2i[(P_+y, P_+z)_H - (P_-y, P_-z)_H]$. On the other hand, as a consequence of the isometricity of U,

$$(\Gamma_1 y, \Gamma_2 z)_{\mathcal{H}} - (\Gamma_2 y, \Gamma_1 z)_{\mathcal{H}} = 2i[(P_+y, P_+z)_H - (P_-y, P_-z)_H],$$

i.e., condition 1) of Definition 2 is satisfied.

If $Y_1, Y_2 \in \mathcal{H}$, let us select $y \in D(A^*)$ by putting $y = y_0 + y_+ + y_-$, where y_0 is an arbitrary vector from D(A), $y_+ = (1/2i)(iY_1 - Y_2) \in N_+$, $y_- = (1/2i)U^{-1}(iY_1 + Y_2) \in N_-$. One immediately verifies that $\Gamma_1 y = Y_1$ and $\Gamma_2 y = Y_2$. Finally, it is obvious that for $y \in D(A)$, $\Gamma_1 y = \Gamma_2 y = 0$. The theorem is proved.

5. Connections with the theory of spectral problems of the form $Su = \lambda Tu$ (S and T are symmetric differential operators). Bennewitz [13] and E. A. Coddington [14] posed the problem of the description of extensions of a symmetric binary relation. Symmetric and self-adjoint extensions are described in [13-14]; Theorem 1 formulated above also permits one to describe other classes of extensions.

Note that for symmetric relations representation (3) is equivalent to representation (4), in which K is replaced by K^{-1}.

THEOREM 4. Let a symmetric binary relation θ in a separable Hilbert space H be represented by Eq. (3). The general form of a symmetric extension $\tilde{\theta} \supset \theta$ is given by the equation

$$\tilde{K}(x' + ix) = x' - ix,\quad x' + ix \in D(\tilde{K}), \tag{10}$$

where $\tilde{K}$ is an isometric extension of K. The relation $\tilde{\theta}$ is maximal symmetric precisely when $D(\tilde{K}) = H$ or $R(\tilde{K}) = H$.* The relation $\tilde{\theta}$ is Hermitian precisely when $\tilde{K}$ is a unitary extension of K. The general form of a dissipative extension of θ is given by Eq. (10), in which $\tilde{K} \supset K$, $\|\tilde{K}f\| \leqslant \|f\|$ for all $f \in D(\tilde{K})$. This exten-

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*$R(\tilde{K})$ is the range of $\tilde{K}$.

sion is maximal dissipative if $D(\tilde{K}) = H$. The general form of an accumulative extension of θ is given by the equation

$$\widetilde{K^{-1}}\ (x' - ix) = x' + ix, \qquad x' - ix \in D\ (\widetilde{K^{-1}}),$$

where $\widetilde{K^{-1}} \supset K^{-1}$, $\|\widetilde{K^{-1}}f\| \leqslant \|f\|$, $f \in D\ (\widetilde{K^{-1}})$. This extension is maximal accumulative if $D(\widetilde{K^{-1}}) = H$.

Proof. Note that

$$D\ (K) = \{x' + ix \mid x\theta x'\}.$$

Indeed, if $y \in D(K)$, then, putting

$$x = \frac{1}{2i}(y - Ky),\ x' = \frac{1}{2}(y + Ky),$$

we see that $x\theta x'$ by virtue of (3). On the other hand,

$$y = x' + ix.$$

It is proved in an analogous manner that

$$D\ (K^{-1}) = \{x' - ix \mid x\ \theta x'\}.$$

Now all assertions of the theorem immediately follow from Theorem 1.

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