THE HERMITE-MINKOWSKI DOMAIN OF REDUCTION OF POSITIVE DEFINITE QUADRATIC FORMS IN SIX VARIABLES

P. P. Tammela

1.

Let

$$\mathfrak{f} = \mathfrak{f}(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}\, x_i\, x_j$$

be a positive definite quadratic form with real coefficients a_{ij}, where $a_{ij} = a_{ji}$ $(i, j = 1, \ldots, n)$. The form $\mathfrak{f}$ is said to be Minkowski-reduced if any set of integers $\ell_1, \ldots, \ell_n$ with greatest common divisor $(\ell_1, \ldots, \ell_n) = 1$ satisfies

$$\mathfrak{f}(\ell_1, \ldots, \ell_n) \geq a_{ii}. \tag{1}$$

The general theory of Minkowski reduction of positive definite quadratic forms was developed in Minkowski's celebrated memoir [3] (see also the beautiful exposition [4]).

In the N-dimensional $N = \frac{n(n+1)}{2}$ space of coefficients $\{a_{11}, \ldots, a_{nn}, a_{12}, \ldots, a_{n-1,n}\}$, the Minkowski domain of reduction is a convex gonohedron with a finite number of faces. Minkowski [2] formulated a number of propositions which make it possible to determine the reduction gonohedron for $n \leq 6$. However, Minkowski published proofs of these propositions only for $n \leq 4$ [1, 3]. Ryshkov [5] has recently proved these propositions in the case $n = 5$. The purpose of the present note is to give a proof of Minkowski's propositions for $n = 6$. More precisely, we establish the following proposition (stated without proof by Minkowski [2]).

THEOREM. Let

$$\mathfrak{f} = \sum_{i,j=1} a_{ij}\, x_i\, x_j$$

be a positive definite quadratic form with real coefficients a_{ij} $(i, j = 1, \ldots, n)$ and suppose that $n \leq 6$. A necessary and sufficient condition for $\mathfrak{f}$ to be Minkowski-reduced is that it satisfies the conditions

$$a_{ii} \leq a_{i+1,\, i+1} \qquad (i = 1, \ldots, n-1) \tag{2}$$

and the conditions

$$\mathfrak{f}(\ell_1, \ldots, \ell_n) \geq a_{\kappa\kappa}, \tag{3}$$

where $\kappa = 1, \ldots, n$, and the values of ℓ_i are taken from the following table. Here $(\kappa, \kappa', \kappa'', \ldots, \kappa^{(n-1)})$ run through all permutations of the indices $(1, 2, \ldots, n)$. Only those rows of the table (4) with not more than n entries are taken, and zeros are inserted in the vacant positions.

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR, Vol. 33, pp. 72-89, 1973.

ℓ_{κ}	$\pm\ell_{\kappa'}$	$\pm\ell_{\kappa''}$	$\pm\ell_{\kappa'''}$	$\pm\ell_{\kappa^{IV}}$	$\pm\ell_{\kappa^{V}}$
I	I				
I	I	I			
I	I	I	I		
I	I	I	I	I	
I	I	I	I	2	
I	I	I	I	I	I
I	I	I	I	I	2
I	I	I	I	2	2
I	I	I	I	2	3

(4)

2.

Before proving the theorem, we state and prove a series of lemmas.

We introduce the following notation: 1) e_i is the vector which has its i-th coordinate equal to 1 and all its other coordinates equal to 0 $(i=1,\ldots,n)$; 2) if $i, j, \ldots, \kappa$ are indices (possibly identical) from the set $(1, 2, \ldots, n)$, then

$$e_{i,j,\ldots,\kappa} = e_i + e_j + \ldots + e_\kappa .$$

In this notation the inequalities (3) corresponding to the rows of the table (4) (other than the last row) can be written as follows:

$$\mathfrak{f}(e_{j_0,j_1,j_2,\ldots,j_t}) - \mathfrak{f}(e_{j_0}) \geq 0 \qquad (t = 1, 2, \ldots, n-1), \tag{5}$$

$$\mathfrak{f}(e_{j_0,j_1,\ldots,j_{t-1},j_t,j_t}) - \mathfrak{f}(e_{j_0}) \geq 0 \quad (t = 4, 5, \ldots, n-1), \tag{6}$$

$$\mathfrak{f}(e_{j_0,j_1,\ldots,j_{t-1},j_{t-1},j_t,j_t}) - \mathfrak{f}(e_{j_0}) \geq 0 \quad (t = 5, \ldots, n-1). \tag{7}$$

The inequalities (5), (6), and (7), respectively, are equivalent to the inequalities

$$2\mathfrak{f}(e_{j_0};\, e_{j_1,j_2,\ldots,j_t}) + \mathfrak{f}(e_{j_1,j_2,\ldots,j_t}) \geq 0 \quad (t = 1, 2, \ldots, n-1), \tag{8}$$

$$2\mathfrak{f}(e_{j_0};\, e_{j_1,j_2,\ldots,j_{t-1},j_t,j_t}) + \mathfrak{f}(e_{j_1,\ldots,j_{t-1},j_t,j_t}) \geq 0 \quad (t = 4, 5, \ldots, n-1), \tag{9}$$

$$2\mathfrak{f}(e_{j_0};\, e_{j_1,\ldots,j_{t-1},j_{t-1},j_t,j_t}) + \mathfrak{f}(e_{j_1,\ldots,j_{t-1},j_{t-1},j_t,j_t}) \geq 0 \quad (t = 5, \ldots, n-1), \tag{10}$$

where $\mathfrak{f}(x; y)$ is the bilinear form

$$\mathfrak{f}(x;\, y) = \sum_{i,j=1}^{n} a_{ij}\, x_i\, y_j,$$

corresponding to the quadratic form , and $j_0, j_1, \ldots, j_t$ are distinct indices chosen from the set $(1, 2, \ldots, n)$.

In addition, for brevity we use the notation $\ell = (\ell_1, \ldots, \ell_n)$ in the statements of Lemmas 1-3 for the point with integral coordinates $\ell_1, \ldots, \ell_n$ satisfying the inequalities

$$0 \leq \ell_1 \leq \ell_2 \leq \ldots \leq \ell_n . \tag{11}$$

LEMMA 1. Let κ be any index in the range $0 \leq \kappa \leq n-1$. Suppose that $\mathfrak{f}$ satisfies the inequalities* (5) for $t = 1, 2, \ldots, \kappa+1$, and suppose also that

$$2\ell_{n-\kappa} > \sum_{i=1}^{n-\kappa-1} \ell_i. \tag{12}$$

Finally, suppose that for $\kappa \geq 3$ we have the inequality

$$2\ell_{n-\kappa} > \sum_{i=1}^{n-\kappa-1} \ell_i + \sum_{i=0}^{\kappa-3} (i+1)(\kappa-i-2)(\ell_{n-i}-\ell_{n-i-1}) - (\ell_{n-\kappa+1}-\ell_{n-\kappa}). \tag{13}$$

Then

$$\mathfrak{f}(\ell) \geq \mathfrak{f}(\ell - e_{n-\kappa,\ldots,n}). \tag{14}$$

Proof. We have**

$$\mathfrak{f}(\ell) - \mathfrak{f}(\ell - e_{n-\kappa,\, n-\kappa+1,\ldots,n}) = 2\mathfrak{f}(\ell;\, e_{n-\kappa,\, n-\kappa+1,\ldots,n}) - \mathfrak{f}(e_{n-\kappa,\, n-\kappa+1,\ldots,n}).$$

Expanding the bilinear form $\mathfrak{f}(\ell;\, e_{n-\kappa,\, n-\kappa+1,\ldots,n})$ in its first argument according to the representation

$$\ell = \sum_{i=1}^{n-\kappa-1} \ell_i e_i + \ell_{n-\kappa} e_{n-\kappa,\, n-\kappa+1,\ldots,n} + \sum_{i=0}^{\kappa-1} (\ell_{n-i} - \ell_{n-i-1}) e_{n-i,\ldots,n},$$

we obtain

$$\begin{aligned}
\mathfrak{f}(\ell) - \mathfrak{f}(\ell - e_{n-\kappa,\ldots,n}) = {} & \sum_{i=1}^{n-\kappa-1} \ell_i \{2\mathfrak{f}(e_i;\, e_{n-\kappa,\ldots,n}) + \mathfrak{f}(e_{n-\kappa,\ldots,n})\} + \\
& + (\ell_{n-\kappa+1} - \ell_{n-\kappa})\{2\mathfrak{f}(e_{n-\kappa};\, e_{n-\kappa+1,\ldots,n}) + \mathfrak{f}(e_{n-\kappa+1,\ldots,n})\} + \\
& + (\ell_{n-\kappa+2} - \ell_{n-\kappa+1}) \sum_{j=n-\kappa}^{n-\kappa+1} \{2\mathfrak{f}(e_j;\, e_{n-\kappa+2,\ldots,n}) + \mathfrak{f}(e_{n-\kappa+2,\ldots,n})\} + \\
& + \sum_{i=0}^{\kappa-3} (\ell_{n-i} - \ell_{n-i-1}) \sum_{j=n-\kappa}^{n-\kappa+1} \{2\mathfrak{f}(e_j;\, e_{n-i,\ldots,n}) + \mathfrak{f}(e_{n-i,\ldots,n})\} + \\
& + \sum_{i=0}^{\kappa-3} (\ell_{n-i} - \ell_{n-i-1}) \sum_{j=n-\kappa+2}^{n-i-1} \sum_{h=n-i}^{n} \{2\mathfrak{f}(e_j;\, e_h) + \mathfrak{f}(e_j)\} + \\
& + \Bigl(2\ell_{n-\kappa} - \sum_{i=1}^{n-\kappa-1} \ell_i - 1\Bigr)\{\mathfrak{f}(e_{n-\kappa,\ldots,n}) - \max_{n-\kappa+2 \leq h \leq n} \mathfrak{f}(e_h)\} + \\
& + (\ell_{n-\kappa+1} - \ell_{n-\kappa})\{\mathfrak{f}(e_{n-\kappa+1,\ldots,n}) - \max_{n-\kappa+2 \leq h \leq n} \mathfrak{f}(e_h)\} + \\
& + \sum_{i=0}^{\kappa-3} (\ell_{n-i} - \ell_{n-i-1}) \sum_{j=n-\kappa+2}^{n-i-1} (i+1)\{\max_{n-\kappa+2 \leq h \leq n} \mathfrak{f}(e_h) - \mathfrak{f}(e_j)\} + \\
& + \Bigl(2\ell_{n-\kappa} - \sum_{i=1}^{n-\kappa-1} \ell_i - 1 + \ell_{n-\kappa+1} - \ell_{n-\kappa} - \sum_{i=0}^{\kappa-3} (\ell_{n-i} - \ell_{n-i-1})(\kappa-i-2)(i+1)\Bigr) \max_{n-\kappa+2 \leq h \leq n} \mathfrak{f}(e_h).
\end{aligned}$$

---

*And hence also the equivalent inequalities (8).

**We henceforth take the sum $\sum\limits_{j=u}^{v}$ to be equal to zero if $v < u$. We assume also that $\ell_m = \ell_n$ and

$$\max_{m \leq h \leq n} \mathfrak{f}(e_h) = 0 \quad \text{if} \quad m > n.$$

This establishes the inequality (14), since by (5) each of the terms in the curly brackets is nonnegative, so that by (11), (12), and (13) each of the terms on the right-hand side of the last equality is also nonnegative.

This proves Lemma 1.

Remark. For $\kappa = 3$ and $n = 6$ the inequality (13) takes the form

$$\ell_3 + \ell_4 + \ell_5 > \ell_1 + \ell_2 + \ell_6. \tag{13'}$$

LEMMA 2. Let $\kappa = 3$ or $\kappa = 4$. Suppose that $\mathfrak{f}$ satisfies the inequalities (5), (6), and (7) for $t = 1, 2, \ldots, \kappa + 1$ and also that

$$2\ell_{n-\kappa} > \sum_{i=1}^{n-\kappa-1} \ell_i, \qquad \ell_{n-\kappa} + \ell_n - \ell_{n-1} > \sum_{i=1}^{n-\kappa-1} \ell_i. \tag{15}$$

Finally, suppose that for $\kappa = 4$ we have the inequality

$$\ell_{n-4} + \ell_{n-3} + \ell_{n-2} > \sum_{i=1}^{n-5} \ell_i + \ell_{n-1}. \tag{16}$$

Then

$$\mathfrak{f}(\ell) \geq \mathfrak{f}(\ell - e_{n-\kappa,\ldots,n-1,n,n}). \tag{17}$$

Proof. Let

$$A = \min\{\ell_{n-\kappa};\ \ell_n - \ell_{n-1}\}, \qquad B = \min\{\ell_{n-\kappa} - A;\ \ell_{n-1} - \ell_{n-2}\},$$

$$C = \min\{\ell_{n-\kappa+1} - \ell_{n-\kappa},\ \ell_{n-1} - \ell_{n-2} - B\}.$$

We have

$$\mathfrak{f}(\ell) - \mathfrak{f}(\ell - e_{n-\kappa,\ldots,n-1,n,n}) = 2\mathfrak{f}(\ell;\ e_{n-\kappa,\ldots,n-1,n,n}) - \mathfrak{f}(e_{n-\kappa,\ldots,n-1,n,n}).$$

Expanding the bilinear form $\mathfrak{f}(\ell; e_{n-\kappa,\ldots,n-1,n,n})$ according to the representation

$$\ell = \sum_{i=1}^{n-\kappa-1} \ell_i e_i + A e_{n-\kappa,\ldots,n-1,n,n} + B e_{n-\kappa,\ldots,n-1,n-1,n,n} +$$

$$+ (\ell_{n-\kappa} - A - B) e_{n-\kappa,\ldots,n-1,n} + C e_{n-\kappa+1,\ldots,n-1,n-1,n,n} +$$

$$+ (\ell_{n-\kappa+1} - \ell_{n-\kappa} - C) e_{n-\kappa+1,n-1,n} + \sum_{i=2}^{\kappa-2} (\ell_{n-i} - \ell_{n-i-1}) e_{n-i,\ldots,n} +$$

$$+ (\ell_{n-1} - \ell_{n-2} - B - C) e_{n-1,n} + (\ell_n - \ell_{n-1} - A) e_n,$$

we obtain

$$\mathfrak{f}(\ell) - \mathfrak{f}(\ell - e_{n-\kappa,\ldots,n-1,n,n}) =$$

$$= \sum_{i=1}^{n-\kappa-1} \ell_i \{2\mathfrak{f}(e_i;\ e_{n-\kappa,\ldots,n-1,n,n}) + \mathfrak{f}(e_{n-\kappa,\ldots,n-1,n,n})\} +$$

$$+ (\ell_{n-\kappa} - A - B)\{\mathfrak{f}(e_{n-\kappa,\ldots,n-1,n}) - \mathfrak{f}(e_n)\} +$$

$$+ B\{\mathfrak{f}(e_{n-\kappa,\ldots,n-1,n}) - \mathfrak{f}(e_{n-1})\} +$$

$$+ (\ell_{n-\kappa+1} - \ell_{n-\kappa} - C)\{\mathfrak{f}(e_{n-\kappa+1,\ldots,n-1,n}) + 2\mathfrak{f}(e_{n-\kappa};\ e_{n-\kappa+1,\ldots,n-1,n})\} +$$

$$+ (\ell_{n-\kappa+1} - \ell_{n-\kappa} - C)\{\mathfrak{f}(e_{n-\kappa+1,\ldots,n-1,n}) - \mathfrak{f}(e_n)\} +$$

$$+C\{f(e_{n-\kappa+1,\ldots,n-1,n,n})-f(e_{n-1})\}+$$

$$+C\{2f(e_{n-\kappa};\ e_{n-\kappa+1,\ldots,n-1,n-1,n,n})+f(e_{n-\kappa+1,\ldots,n-1,n-1,n,n})\}+$$

$$+\sum_{i=2}^{\kappa-2}(\ell_{n-1}-\ell_{n-i-1})\sum_{j=n-2}^{n-1}\{2f(e_j;\ e_n)+f(e_n)\}+$$

$$+\sum_{i=2}^{\kappa-2}(\ell_{n-i}-\ell_{n-i-1})\sum_{j=n-\kappa}^{n-\kappa+1}\{2f(e_j;\ e_{n-i,\ldots,n-1,n})+f(e_{n-i,\ldots,n-1,n})\}+$$

$$+(\ell_{n-1}-\ell_{n-2}-B-C)\sum_{i=n-3}^{n-2}\{2f(e_i;\ e_{n-1,n})+f(e_{n-1,n})\}+$$

$$+(\ell_{n-1}-\ell_{n-2}-B-C)\{2f(e_{n-1};\ e_n)+f(e_n)\}+$$

$$+(\ell_n-\ell_{n-1}-A)\sum_{i=n-3}^{n-1}\{2f(e_i;\ e_n)+f(e_n)\}+$$

$$+B[f(e_{n-\kappa,\ldots,n-1,n-1,n,n})-f(e_{n-\kappa,\ldots,n-1,n})]+$$

$$+(\ell_n-\ell_{n-1}-A)[\sum_{i=n-\kappa}^{n-4}2f(e_i;\ e_n)+f(e_n)]+$$

$$+(\ell_{n-1}-\ell_{n-2}-B-C)[\sum_{i=n-\kappa}^{n-4}2f(e_i;\ e_n)+f(e_n)]+$$

$$+\{2A-1-\sum_{i=1}^{n-\kappa-1}\ell_i+\ell_{n-\kappa}-A-B+B\}\{f(e_{n-\kappa,\ldots,n-1,n,n})-f(e_{n-1})\}+$$

$$+(\ell_{n-\kappa+1}-\ell_{n-\kappa}-C)[f(e_{n-\kappa+1,\ldots,n-1,n,n})-f(e_{n-\kappa+1,\ldots,n-1,n})]+$$

$$+(\ell_{n-1}-\ell_{n-2}-B-C)\sum_{i=n-\kappa}^{n-4}\{2f(e_i;\ e_{n-1})+f(e_{n-1})\}+$$

$$+\{2A-1-\sum_{i=1}^{n-\kappa-1}\ell_i+\ell_{n-\kappa}-A-B+B-(\kappa-3)(\ell_{n-1}-\ell_{n-2}-B-C)\}f(e_{n-1}).$$

This establishes the inequality (17), since by (5), (6), and (7) each of the terms in the curly brackets is nonnegative, while for $\kappa=3$ or $\kappa=4$ Eq. (5) and Lemma 1 imply that the expressions in the curly brackets are nonnegative, so that by (11), (15), and (16) each of the terms on the right-hand side of the last equality is nonnegative.

This proves Lemma 2.

LEMMA 3. Let $n=6$. Suppose that f satisfies the inequalities (5), (6), and (7) for $t=1,\ldots,5$ and also that

$$1\le\ell_1=\ell_2=\ell_3,\qquad 2\ell_m\le\sum_{i=1}^{m-1}\ell_i\qquad(m=4,5,6). \tag{18}$$

Finally, suppose that either of the following conditions holds:

$$\text{I})\quad \ell_5=2\ell_4=2\ell_1, \tag{19}$$

$$2)\quad \ell_6=\ell_5>\ell_3. \tag{20}$$

Then

$$f(\ell)\ge f(\ell-e_{1,2,3,4,5,5,6,6}). \tag{21}$$

Proof. Let

$$A=min(\ell_1+\ell_4-\ell_5,\ \ell_4-\ell_3).$$

We have

$$\mathfrak{f}(\ell)-\mathfrak{f}(\ell-e_{1,2,3,4,5,5,6,6})=2\mathfrak{f}(\ell;\, e_{1,2,3,4,5,5,6,6})-\mathfrak{f}(e_{1,2,3,4,5,5,6,6}).$$

Expanding the bilinear form $\mathfrak{f}(\ell;\, e_{1,2,3,4,5,5,6,6})$ according to the representation

$$\ell=(\ell_5-\ell_4)e_{1,2,3,4,5,5,6,6}+Ae_{1,2,3,4,4,5,5,6,6}$$
$$+(\ell_1+\ell_4-\ell_5-A)e_{1,2,3,4,5,6}+(\ell_4-\ell_3-A)e_{4,5,6}+(\ell_6-\ell_5)e_6,$$

we obtain

$$\mathfrak{f}(\ell)-\mathfrak{f}(\ell-e_{1,2,3,4,5,5,6,6})=A\{\mathfrak{f}(e_{1,2,3,4,5,6})-\mathfrak{f}(e_4)\}+$$
$$+(\ell_1+\ell_4-\ell_5-A)\{\mathfrak{f}(e_{1,2,3,4,5,5,6})-\mathfrak{f}(e_5)\}+(\ell_1+\ell_4-\ell_5-A)\{\mathfrak{f}(e_{1,2,3,4,5,6,6})-\mathfrak{f}(e_5)\}$$
$$+(\ell_4-\ell_3-A)\sum_{i=2}^{3}\{2\mathfrak{f}(e_i;\, e_{4,5,6})+\mathfrak{f}(e_{4,5,6})\}+(\ell_4-\ell_3-A)\{2\mathfrak{f}(e_4;\, e_{5,6})+\mathfrak{f}(e_{5,6})\}+$$
$$+(\ell_4-\ell_3-A)\{2\mathfrak{f}(e_1;\, e_{5,6})+\mathfrak{f}(e_{5,6})\}+(\ell_4-\ell_3-A)\{2\mathfrak{f}(e_1;\, e_4)+\mathfrak{f}(e_1)\}+$$
$$+(\ell_6-\ell_5)\{\mathfrak{f}(e_{1,2,3,4,5,6,5})-\mathfrak{f}(e_5)\}+(\ell_6-\ell_5)\sum_{i=1}^{4}\{2\mathfrak{f}(e_i;\, e_{5,6})+\mathfrak{f}(e_{5,6})\}+$$
$$+(2(\ell_5-\ell_4)-1-(\ell_6-\ell_5)+A)\{\mathfrak{f}(e_{1,2,3,4,5,5,6,6})-\mathfrak{f}(e_1)\}+$$
$$+A[\mathfrak{f}(e_{1,2,3,4,4,5,5,6,6})-\mathfrak{f}(e_{1,2,3,4,5,6})]+\{2(\ell_5-\ell_4)-1-(\ell_6-\ell_5)+A-(\ell_4-\ell_3-A)\}\mathfrak{f}(e_1).$$

This establishes the inequality (21), since by (5), (6), and (7) each of the terms in the curly brackets is nonnegative, while by Eq. (5) and Lemma 1 the expression in the curly brackets is nonnegative, so that by (11), (18), (19), and (20) each of the terms on the right-hand side of the last equality is nonnegative.

This proves Lemma 3.

3.

We turn now to the proof of the theorem, assuming henceforth that $n \leq 6$. If a form is Minkowski-reduced, then it satisfies a set of inequalities which includes the inequalities (2) and (3).

We next prove the converse: if $\mathfrak{f}$ satisfies the inequalities (2) and (3), then it is Minkowski-reduced. Let $\ell=(\ell_1,\ldots,\ell_n)$ be a vector with integral coordinates having the greatest common divisor $(\ell_j,\ldots,\ell_n)=1$ for some j $(1\leq j\leq n)$. We must prove that

$$\mathfrak{f}(\ell_1,\ldots,\ell_n)\geq a_{jj}. \tag{22}$$

1°. Without loss of generality, we can assume that $\ell_i \geq 0$ for all $i=1,\ldots,n$; for otherwise the form $\mathfrak{f}$ could be replaced by the form $\mathfrak{f}_1=\mathfrak{f}(\pm x_1,\ldots,\pm x_n)$, where the signs are chosen so that

$$\mathfrak{f}_1(|\ell_1|,\ldots,|\ell_n|)=\mathfrak{f}(\ell_1,\ldots,\ell_n).$$

2°. Suppose that all the coordinates of the vector ℓ except the first coordinate are equal to zero, with the greatest common divisor $(\ell_j,\ldots,\ell_n)=1$. Then $\ell=e_i$ for some $i\geq j$ and

$$\mathfrak{f}(\ell)=\mathfrak{f}(e_i)=a_{ii}\geq a_{jj}$$

by (2). Thus the inequality (22) has been established in this case. We therefore assume in what follows that there are at least two nonzero coordinates.

3°. As a preliminary step, we prove the following lemma.

LEMMA 4. Suppose that $\mathfrak{f}$ satisfies the conditions (3). Suppose also that the integral-valued vector $\ell=(\ell_1,\ldots,\ell_n)$ is distinct from the vectors of the table (4), with $\ell_i \geq 0$ $(i=1,\ldots,n)$, and that there are precisely s nonzero coordinates $(2 \leq s \leq n \leq 6)$. Then there exists an integral-valued vector $\ell'=(\ell'_1,\ldots,\ell'_n)$ satisfying the conditions

1) $0 \leq \ell'_i \leq \ell_i \quad (i=1,\ldots,n)$;
2) $\ell'_i > 0 \quad$ if $\quad \ell_i > 0 \quad (i=1,\ldots,n)$;
3) $\ell'_1+\ldots+\ell'_n < \ell_1+\ldots+\ell_n$;
4) $\mathfrak{f}(\ell') \leq \mathfrak{f}(\ell)$.

Proof. Without loss of generality, we assume that the conditions (11) are satisfied for $\ell=(\ell_1,\ldots,\ell_n)$, since we could otherwise replace the form $\mathfrak{f}$ by a form $\mathfrak{f}'=\mathfrak{f}(x_{\kappa_1},\ldots,x_{\kappa_n})$ such that $\ell_{\kappa_1} \leq \ldots \leq \ell_{\kappa_n}$. In proving Lemma 4 for the form $\mathfrak{f}'$, we also prove it for the form $\mathfrak{f}$.

In the following table we give the construction of the vector ℓ' in terms of the vector ℓ (for various values of s and under various conditions on the coordinates of the vector ℓ). The remarks in the last column indicate why the new vector ℓ' satisfies the conditions 1-4 (the references are to various cases of Lemmas 1-3).

s	Condition on 1	Construction of the vector 1'	Remarks
2	$\ell_{n-1} < \ell_n$	$\ell' = \ell - e_n$	L. 1. for $\kappa=0$
	$\ell_{n-1} = \ell_n$	$\ell' = \frac{1}{\ell_n}\ell$	$\ell = \ell_{n-1} e_{n-1,n}$
3	$\ell_{n-2} < \ell_n$	$\ell' = \ell - e_n$	L. 1. for $\kappa=0$
	$\ell_{n-2} = \ell_n$	$\ell' = \frac{1}{\ell_n}\ell$	$\ell = \ell_n e_{n-2,n-1,n}$
4	$2\ell_n > \sum_{i=n-3}^{n-1} \ell_i$	$\ell' = \ell - e_n$	L. 1. for $\kappa=0$
	$2\ell_n \leq \sum_{i=n-3}^{n-1} \ell_i$ $2\ell_{n-1} > \sum_{i=n-3}^{n-2} \ell_i$	$\ell' = \ell - e_{n-1,n}$	L. 1. for $\kappa=1$
	$2\ell_m \leq \sum_{i=n-3}^{m-1} \ell_i \ (m=n-1,n)$ $\ell_{n-2} > 1$	$\ell' = \ell - e_{n-2,n-1,n}$	L. 1. for $\kappa=2$
5	$2\ell_n > \sum_{i=n-4}^{n-1} \ell_i$	$\ell' = \ell - e_n$	L. 1. for $\kappa=0$
	$2\ell_n \leq \sum_{i=n-4}^{n-1} \ell_i$ $2\ell_{n-1} > \sum_{i=n-4}^{n-2} \ell_i$	$\ell' = \ell - e_{n-1,n}$	L. 1. for $\kappa=1$
	$2\ell_m \leq \sum_{i=n-4}^{m-1} \ell_i \ (m=n-1,n)$ $2\ell_{n-2} > \sum_{i=n-4}^{n-3} \ell_i$	$\ell' = \ell - e_{n-2,n-1,n}$	L. 1. for $\kappa=2$

	$2\ell_m \leq \sum_{i=n-4}^{m-1} \ell_i \ (m=n-2,\ n-1,\ n),$ $\ell_{n-3}+\ell_{n-2}+\ell_{n-1} > \ell_{n-4}+\ell_n$ $\ell_{n-3} > 1$	$\ell'=\ell-e_{n-3,n-2,n-1,n}$	L. 1. for $\kappa=3$
	$2\ell_m \leq \sum_{i=n-4}^{m-1} \ell_i \ (m=n-2,n-1,n),$ $\ell_{n-3}+\ell_{n-2}+\ell_{n-1} \leq \ell_{n-4}+\ell_n$ $\ell_{n-3} > 1$	$\ell' = \frac{1}{\ell_{n-4}}\ell$	$\ell=\ell_{n-4}e_{n-4,n-3,n-2,n-1,n,n}$
	$2\ell_6 > \sum_{i=1}^{5} \ell_i$	$\ell'=\ell-e_6$	L. 1. for $\kappa=0$
	$2\ell_6 \leq \sum_{i=1}^{5} \ell_i$ $2\ell_5 > \sum_{i=1}^{4} \ell_i$	$\ell'=\ell-e_{5,6}$	L. 1. for $\kappa=1$
	$2\ell_m \leq \sum_{i=1}^{m-1} \ell_i \ (m=5,6),$ $2\ell_4 > \sum_{i=1}^{3} \ell_i$	$\ell'=\ell-e_{4,5,6}$	L. 1. for $\kappa=2$
$s=n=6$	$2\ell_m \leq \sum_{i=1}^{m-1} \ell_i \ (m=4,5,6),$ $\ell_3 > \ell_1$ $\ell_3+\ell_4+\ell_5 > \ell_1+\ell_2+\ell_6$	$\ell'=\ell-e_{3,4,5,6}$	L. 1. for $\kappa=3$
	$2\ell_m \leq \sum_{i=1}^{m-1} \ell_i \ (m=4,5,6),$ $\ell_3 > \ell_1$ $\ell_3+\ell_4+\ell_5 \leq \ell_1+\ell_2+\ell_6$ $\ell_2 = 1$	$\ell'=\ell-e_{3,4,5,6,6}$	$\ell=e_{1,2}+2e_{3,4,5,6,6}$ L. 2. for $\kappa=3$
	$2\ell_m \leq \sum_{i=1}^{m-1} \ell_i \ (m=4,5,6),$ $\ell_3 > \ell_1$ $\ell_3+\ell_4+\ell_5 \leq \ell_1+\ell_2+\ell_6$ $\ell_2 > 1$	$\ell'=\ell-e_{2,3,4,5,6,6}$	$\ell_6 > \ell_5,$ $\ell_2 \geqslant \ell_1$ $\ell_2+\ell_4 \geqslant \ell_5$ L. 2. for $\kappa=4$
	$2\ell_m \leq \sum_{i=1}^{m-1} \ell_i \ (m=4,5,6),$ $\ell_3 = \ell_2 = \ell_1 > 1$ $\ell_2+\ell_3+\ell_4 > \ell_1+\ell_5$ $\ell_2+\ell_6 > \ell_1+\ell_5$	$\ell'=\ell-e_{2,3,4,5,6,6}$	L. 2 for $\kappa=4$

$2\ell_m \le \sum_{i=1}^{m-1} \ell_i$ $(m=4,5,6)$, $\ell_3=\ell_2=\ell_1>1$ $\ell_2+\ell_3+\ell_4 \le \ell_1+\ell_5$	$\ell'=\ell-e_{1,2,3,4,5,5,6,6}$	$\ell_1=\ell_2=\ell_3=\ell_4$ $\ell_5=2\ell_1$ L. 3. (19)
$2\ell_m \le \sum_{i=1}^{m-1} \ell_i$ $(m=4,5,6)$ $\ell_3=\ell_2=\ell_1>1$ $\ell_2+\ell_6 \le \ell_1+\ell_5$ $\ell_5>\ell_3$	$\ell'=\ell-e_{1,2,3,4,5,5,6}$	$\ell_1=\ell_2=\ell_3$ $\ell_5=\ell_6$ $\ell_5>\ell_3$ L. 3. (20)
$2\ell_m \le \sum_{i=1}^{m-1} \ell_i$ $(m=4,5,6)$, $\ell_3=\ell_2=\ell_1>1$ $\ell_2+\ell_6 \le \ell_1+\ell_5$ $\ell_5=\ell_3$	$\ell'=\frac{1}{\ell_1}\ell$	$\ell=\ell_1 e_{1,2,3,4,5,6}$

It is straightforward to show that the conditions imposed on the coordinates ℓ_i of the vectors ℓ, exhaust all possible vectors which do not appear in the table (4). We note here only that if $\ell_{n-3}=1$, then it can be shown from the conditions

$$2\ell_{n-m} \le \sum_{i=1}^{n-m-1} \ell_i \qquad (m=0,1,2)$$

that ℓ is one of the vectors given in the table. It is also quite straightforward to verify that the conditions for the applicability of the lemmas are satisfied.

This proves Lemma 4.

In order to formulate the next proposition, we introduce the term "reductions of Lemma 4" for the transitions from ℓ to ℓ' given in the table which appears in the proof of Lemma 4.

Addendum to Lemma 4. The vector $\ell^{(1)}=e_{1,2,3,4,5,5,6,6}$ can be obtained by means of the reductions of Lemma 4 from only one of the following vectors $m^{(1)}=e_{1,2,3,4}+3e_{5,6}$; $m^{(2)}=e_{1,2}+2e_{3,4}+3e_{5,6}$, $m^{(3)}=e_1+2e_{2,3,4}+3e_5+4e_6$, $m^{(4)}=e_1+2e_{2,3}+3e_{4,5,6}$, $m^{(5)}=2e_{1,2,3,4,5,6,6}$, $m^{(6)}=2e_{1,2,3,4,5,5,6,6}$.

Proof. It is sufficient to consider the reductions of the table corresponding to $s=n=6$. These reductions are achieved by subtracting the vectors e_i; $e_{i,j}$; $e_{i,j,k}$; $e_{i,j,k,m}$, $e_{i,j,k,m,m}$, $e_{i,j,k,m,p,p}$, $e_{i,j,k,m,p,p,r,r}$ from the original vector ℓ. Using the fact that $\ell^{(1)}=e_{1,2,3,4,5,5,6,6}$ and checking directly that ℓ satisfies the necessary conditions for the reduction to be possible, we find the given set of vectors $m^{(i)}$.

LEMMA 5. If $\mathfrak{f}$ is any form in six variables which satisfies the conditions (3), then

$$\mathfrak{f}(m^{(i)}) \ge \max_{j=1,\dots,6} a_{jj} \qquad (i=1,\dots,6).$$

Proof. We have

$$\mathfrak{f}(m^{(1)}) = \mathfrak{f}(e_{1,2,3,4,5,6}) + 2\sum_{i=1}^{4}\{2\mathfrak{f}(e_i;\, e_{5,6}) + \mathfrak{f}(e_{5,6})\}$$

$$\mathfrak{f}(m^{(2)}) = \sum_{i=1}^{2}\{2\mathfrak{f}(e_i;\, e_{3,4,5,6}) + \mathfrak{f}(e_{3,4,5,6})\} +$$

$$+ \sum_{i=3}^{4}\{2\mathfrak{f}(e_i, e_{5,6}) + \mathfrak{f}(e_{5,6})\} + \mathfrak{f}(e_{1,2,3,4,5,5,6,6}) + \mathfrak{f}(e_{3,4,5,6})$$

$$\mathfrak{f}(m^{(3)}) = \{\mathfrak{f}(e_{1,2,3,4,5,5,6}) - \mathfrak{f}(e_1)\} + \{\mathfrak{f}(e_{2,3,4,5,6,6}) - \mathfrak{f}(e_5)\} +$$

$$+ \{\mathfrak{f}(e_{2,3,4,5,5,6,6}) - \mathfrak{f}(e_{2,3,4,5,6})\} + \mathfrak{f}(e_{1,2,3,4,5,5,6,6}) + \mathfrak{f}(e_{2,3,4,5,6})$$

$$\mathfrak{f}(m^{(4)}) = \{2\mathfrak{f}(e_1;\, e_{2,3,4,5,6}) + \mathfrak{f}(e_{2,3,4,5,6})\} + \sum_{i=1}^{3}\{2\mathfrak{f}(e_i;\, e_{4,5,6}) + \mathfrak{f}(e_{4,5,6})\} +$$

$$+ \sum_{i=2}^{3}\{2\mathfrak{f}(e_i;\, e_{4,5,6}) + \mathfrak{f}(e_{4,5,6})\} + \mathfrak{f}(e_{1,2,3,4,5,6}) + 2\mathfrak{f}(e_{2,3,4,5,6})$$

$$\mathfrak{f}(m^{(5)}) = \sum_{i=1}^{2}\{\mathfrak{f}(e_{1,2,3,4,5,6,6}) - \mathfrak{f}(e_i)\} + \sum_{i=1}^{2}\{2\mathfrak{f}(e_i;\, e_6) + \mathfrak{f}(e_i)\} +$$

$$+ \sum_{i=3}^{5}\{2\mathfrak{f}(e_i;\, e_6) + \mathfrak{f}(e_6)\} + \mathfrak{f}(e_{1,2,3,4,5,6}) + \mathfrak{f}(e_{1,2,3,4,5,6,6})$$

$$\mathfrak{f}(m^{(6)}) = 2\{\mathfrak{f}(e_{1,2,3,4,5,5,6,6}) - \mathfrak{f}(e_1)\} + \sum_{i=5}^{6}\{2\mathfrak{f}(e_1;\, e_i) + \mathfrak{f}(e_1)\} +$$

$$+ \sum_{i=2}^{4}\{2\mathfrak{f}(e_i;\, e_{5,6}) + \mathfrak{f}(e_{5,6})\} + \mathfrak{f}(e_{1,2,3,4,5,6}) + \mathfrak{f}(e_{1,2,3,4,5,5,6,6}).$$

By the inequalities (5), (6), and (7) the expressions in the curly brackets are nonnegative, and the sum of the remaining terms is not less than $\max\limits_{i=1,\dots,6} \mathfrak{f}(e_i)$.

This proves Lemma 5.

4°. Suppose that $\mathfrak{f}$ is a form which satisfies the conditions (2) and (3) of the theorem. By Lemma 4, if $\ell=(\ell_1,\dots,\ell_n)$ is any integral-valued nonnegative vector having s $(2\le s\le n\le 6)$ nonzero coordinates, then there exists a vector $\ell'=(\ell_1',\dots,\ell_n')$ of the table (satisfying $\ell_i'>0$ if $\ell_i>0$) such that

$$\mathfrak{f}(\ell)\ge\mathfrak{f}(\ell').$$

This vector also satisfies $\ell_i=1$ if $\ell_i'=1$. Thus if the last nonzero coordinate of ℓ is equal to unity, then the last nonzero coordinate of ℓ' is also equal to unity. By (3) we have

$$\mathfrak{f}(\ell)\ge\mathfrak{f}(\ell')\ge\mathfrak{f}(e_j)=a_{jj},$$

where $\ell_j=\ell_j'=1$ and $\ell_i=\ell_i'=0$ for $i>j$.

We therefore assume in what follows that the last nonzero coordinate of ℓ is greater than unity.

5°. If $s=2,3,4$ and ℓ is any integral-valued vector having s positive coordinates, Lemma 4 implies that $\ell_i'=1$ if $\ell_i=0$ and that $\ell_i'=0$ if $\ell_i=0$ $(i=1,\dots,n)$. By the conditions (3) we have

$$\mathfrak{f}(\ell)\ge\mathfrak{f}(\ell')\ge\max_{\ell_i\ne 0}\mathfrak{f}(e_i)$$

and the inequality (22) is valid.

6°. If $s=5$, Lemma 4 implies that $\mathfrak{f}(\ell)\ge\mathfrak{f}(\ell')$, where the vector ℓ' of the table has at least one of its last two positive coordinates equal to unity, and by (3) we have

$$\mathfrak{f}(\ell)\ge\mathfrak{f}(\ell')\ge\mathfrak{f}(e_k)=a_{kk},$$

where κ is the penultimate index of the positive coordinates of ℓ. Using the results of Sec. 3.5, this implies the inequality (22).

7°. Let $s=n=6$ and $\ell=(\ell_1, \ldots, \ell_n)$, and assume that we have the greatest common divisor $(\ell_j, \ldots, \ell_6)=1$ for $j=4$. By Lemma 4 there exists a vector $\ell'=(\ell'_1, \ldots, \ell'_6)$ of the table for which at least one of the last three coordinates is equal to unity. Then by (3) we have

$$f(\ell) \geqslant f(\ell') \geqslant a_{44}.$$

The case $\ell_6=1$ was considered in Sec. 3.4. Consequently, to prove (22) it is sufficient to study the case of the greatest common divisor $(\ell_5, \ell_6)=1$. We shall prove that in this case

$$f(\ell) \geqslant a_{55}. \tag{23}$$

8°. Let $s=n=6$ and assume that we have the greatest common divisor $(\ell_5, \ell_6)=1$. If by applying Lemma 4 we found a vector ℓ' of the table (4) for which the fifth or sixth coordinate were equal to unity, we would have

$$f(\ell) \geqslant f(\ell') \geqslant a_{55}.$$

To establish (23) it is therefore necessary to consider only those vectors ℓ' of the table (4) for which $\ell'_5>1$ and $\ell'_6>1$, i.e., the vectors

$$\ell^{(1)}=(1,1,1,1,2,2), \quad \ell^{(2)}=(1,1,1,1,2,3), \quad \ell^{(3)}=(1,1,1,1,3,2).$$

9°. We show here that

$$f(\ell^{(2)}) \geqslant a_{66} \geqslant a_{55}, \qquad f(\ell^{(3)}) \geqslant a_{55}. \tag{24}$$

We have

$$f(e_{i,j,\kappa,\ell,m,m,r,r,r}) = \{f(e_{i,j,\kappa,\ell,m,r,r}) - f(e_m)\} +$$
$$+ \sum_{t=i,j,\kappa,\ell} \{2f(e_t; e_{m,r}) + f(e_{m,r})\} + f(e_n) \geqslant f(e_r), \geqslant f(e_r),$$

where i, j, κ, ℓ, m, and r are distinct indices. By (5) and (6) the expressions in the curly brackets are nonnegative. This implies the inequalities (24) and hence also (23).

10°. To prove the theorem it remains only to note that, according to Lemma 5, all the vectors $m^{(i)}$ $(i=1,\ldots,6)$ from which the vector $\ell^{(1)}=(1,1,1,1,2,2)$ can be obtained by means of the reduction of Lemma 4 satisfy the condition

$$f(m^{(i)}) \geqslant a_{55} \qquad (i=1,\ldots,6),$$

so that we have established the inequality (23) and thus also (22).

This proves the theorem.

I thank A. V. Malyshev for suggesting the problem and for his interest in the work and S. S. Ryshkov for allowing me to become acquainted with his work [5] in manuscript form.

LITERATURE CITED

1. H. Minkowski, "Sur la reduction des formes quadratiques positives quaternaires," Gesamm. Abh., 1, 145-148 (1911).
2. H. Minkowski, Gesamm. Abh., 1, 217-218 (1911).
3. H. Minkowski, "Diskontinuitätsbereich für arithmetische Aequivalenz," Gesam. Abh. 2, 53-100 (1911).
4. B. L. van der Waerden, "Die Reduktionstheorie der positiven quadratischen Formen," Acta Mathem., 96, 265-309 (1956).
5. S. S. Ryshkov, "The theory of Hermite-Minkowski reduction of positive definite quadratic forms," J. of Soviet Math., 6, No. 6 pp. 659-679 (1976).