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SOLUTIONS OF THE YANG—BAXTER EQUATION

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We give the basic definitions connected with the Yang—Baxter equation (factorization condition for a multiparticle S-matrix) and formulate the problem of classifying its solutions. We list the known methods of solution of the Y—B equation, and also various applications of this equation to the theory of completely integrable quantum and classical systems. A generalization of the Y—B equation to the case of Z_2-graduation is obtained, a possible connection with the theory of representations is noted. The supplement contains about 20 explicit solutions.

0. By the Yang—Baxter equation [1, 2] is meant the following functional equation:

$$ {}_{\alpha\alpha'}R_{\gamma\gamma'}(u-v)\,{}_{\gamma\alpha''}R_{\beta\gamma''}(u)\,{}_{\gamma'\gamma''}R_{\beta'\beta''}(v) = {}_{\alpha'\alpha''}R_{\gamma'\gamma''}(v)\,{}_{\alpha\gamma''}R_{\gamma\beta''}(u)\,{}_{\gamma\gamma'}R_{\beta\beta'}(u-v) \tag{1} $$

for a collection of functions ${}_{\alpha\beta}R_{\gamma\delta}(u)$ of a complex parameter u, depending on four indices $\alpha, \beta, \gamma, \delta$, running through values from 1 to some natural number N. In (1) and later we understand summation over repeated indices.

Equation (1), which first appeared in [1, 2], has many applications to the theory of completely integrable quantum and classical systems and exactly solvable models of statistical physics. In recent years it has undergone intensive study. Here the profound connection of (1) with such areas of mathematics as group theory and algebraic geometry has become more and more apparent.

The present paper is an (apparently the first) attempt to give a systematic survey of the facts accumulated at the time it is written relating to the solutions of (1). The account is structured in the following way. In Sec. 1 we give the basic definitions and we

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formulate the problem of the classification of the solutions of (1). In Sec. 2 we list various applications of (1) to the theory of completely integrable systems. In Sec. 3 we discuss known methods of solution of (1) and we formulate some assertions about properties of its solutions. In Sec. 4 we consider generalizations of the Yang–Baxter equation, and finally, in the Supplement we give a summary of the known solutions of (1).

The authors hope that the present paper will be useful to specialists in the theory of completely integrable systems and the method of the inverse scattering problem, and also helps to attract attention of mathematicians who are specialists in group theory and algebraic geometry to a new promising object of investigation, the Yang–Baxter equation.

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1. Since the systematic study of the Yang–Baxter equation (Y–B) has only just begun, there is still no generally accepted terminology in this area. In this section we make an attempt to propose a system of terms and definitions for the theory of solutions of the Y–B equation. Practice will show how successful this attempt is.

First of all, we discuss a series of equivalent ways of writing the Y–B equation (1). For this we note that the four-indexed quantities ${}_{\alpha\beta}R_{\gamma\delta}(u)$ can constitute a linear operator $\mathbb{R}(u)$ in the tensor product of two N-dimensional complex spaces $V\otimes V$ $(V=\mathbb{C}^N)$. The action of this operator on the basis vector $e_\gamma\otimes e_\delta$ is given by the following formula:

$$\mathbb{R}(e_\gamma\otimes e_\delta)=(e_\alpha\otimes e_\beta)\,{}_{\alpha\beta}R_{\gamma\delta}\ . \tag{2}$$

The tensor ${}_{\alpha\beta}R_{\gamma\delta}$ can also constitute three operators $\mathbb{R}_{12}$, $\mathbb{R}_{13}$, $\mathbb{R}_{23}$ in the tensor product $V\otimes V\otimes V$, corresponding to the three ways of imbedding the space $V\otimes V$ in $V\otimes V\otimes V$:

$$\begin{aligned}
&\mathbb{R}_{12}(e_\gamma\otimes e_{\gamma'}\otimes e_{\gamma''})=(e_\alpha\otimes e_{\alpha'}\otimes e_{\gamma''})\,{}_{\alpha\alpha'}R_{\gamma\gamma'},\\
&\mathbb{R}_{13}(e_\gamma\otimes e_{\gamma'}\otimes e_{\gamma''})=(e_\alpha\otimes e_{\gamma'}\otimes e_{\alpha''})\,{}_{\alpha\alpha''}R_{\gamma\gamma''},\\
&\mathbb{R}_{23}(e_\gamma\otimes e_{\gamma'}\otimes e_{\gamma''})=(e_\gamma\otimes e_{\alpha'}\otimes e_{\alpha''})\,{}_{\alpha'\alpha''}R_{\gamma'\gamma''}.
\end{aligned} \tag{3}$$

Notation (3) we have introduced allows us to write the Yang–Baxter equation (1) as an operator equation:

$$\mathbb{R}_{12}(u-v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{23}(v)=\mathbb{R}_{23}(v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{12}(u-v)\ . \tag{4}$$

In order to avoid misunderstandings, it is necessary to note that there also exists another system of notation, used, e.g., in [3-5]. In these papers, instead of the operator $\mathbb{R}$, introduced above, there is used an operator $\check{\mathbb{R}}$, differing from $\mathbb{R}$ by multiplication by the permutation operator $\mathbb{P}$:

$$\check{\mathbb{R}}=\mathbb{P}\mathbb{R}\ , \tag{5}$$

where

$${}_{\alpha\beta}P_{\gamma\delta}=\delta_{\alpha\delta}\,\delta_{\beta\gamma}\ . \tag{6}$$

Here the Yang–Baxter equation assumes the form:

$$(\mathbb{I}\otimes\check{\mathbb{R}}(u-v))(\check{\mathbb{R}}(u)\otimes\mathbb{I})(\mathbb{I}\otimes\check{\mathbb{R}}(v))=(\check{\mathbb{R}}(v)\otimes\mathbb{I})(\mathbb{I}\otimes\check{\mathbb{R}}(u))(\check{\mathbb{R}}(u-v)\otimes\mathbb{I})\ . \tag{7}$$

This notation for the Y–B equation is interesting in that it preserves its form in the graduated case too (see Sec. 4).

Now we introduce a series of concepts which we need later. A solution $\mathbb{R}(u)$ of the Yang–Baxter equation (4) will be called a Yang–Baxter sheaf. The natural number N (dimension of the space V) will be called the dimension of the sheaf. The variable u, figuring in the Yang–Baxter equation (4), will be called the spectral parameter, in contrast with the other parameters $\xi, \eta, \zeta \ldots$, on which the sheaf $\mathbb{R}(u, \xi, \eta, \ldots)$ possibly depends, and which we shall call connection constants. We shall call the Yang–Baxter sheaf $\mathbb{R}(u)$ regular if for $u=0$ the operator $\mathbb{R}(u)$ is equal to the permutation operator $\mathbb{P}$ (6), which obviously satisfies (4):

$$\mathbb{P}_{12}\,\mathbb{P}_{13}\,\mathbb{P}_{23} = \mathbb{P}_{23}\,\mathbb{P}_{13}\,\mathbb{P}_{12}\,. \tag{8}$$

It is often useful to consider not the isolated sheaf $\mathbb{R}(u)$, but a family of Yang–Baxter sheaves $\mathbb{R}(u,\eta)$, depending on the connection constant η. We call the family $\mathbb{R}(u,\eta)$ quasiclassical if for some value of the parameter $\eta=\eta_0$ (one usually chooses the normalization $\eta_0=0$, which we shall also do in what follows) one has, identically in u,

$$\mathbb{R}(u,\eta)\big|_{\eta=\eta_0} = I\,, \tag{9}$$

where I is the identity operator in the space $V \otimes V$:

$${}_{\alpha\beta}I_{\gamma\delta} = \delta_{\alpha\gamma}\,\delta_{\beta\delta}\,. \tag{10}$$

If, moreover, for each η the sheaf $\mathbb{R}(u,\eta)$ is regular in the sense of the definition given above, such a family of sheaves will be called canonical.

Equation (4) admits a series of obvious transformations, leaving it invariant:

1) Multiplication of the solution $\mathbb{R}(u)$ by an arbitrary scalar function $f(u)$ again gives a solution of (4):

$$\mathbb{R}'(u) = f(u)\,\mathbb{R}(u)\,. \tag{11}$$

Sheaves $\mathbb{R}(u)$ and $\mathbb{R}'(u)$, connected by (11), will be called homothetic.

2) Similarity transformation. Let T be a nondegenerate operator in the space V. Then, as one verifies easily, the sheaf

$$\mathbb{R}'(u) = (T \otimes T)\,\mathbb{R}(u)\,(T \otimes T)^{-1} \tag{12}$$

satisfies the Yang–Baxter equation (4). Sheaves $\mathbb{R}(u)$ and $\mathbb{R}'(u)$, connected by (12), will be called similar. We note that a similarity transformation preserves the properties of regularity, quasiclassicism, and canonicity of a Y–B sheaf or family of sheaves. Two sheaves connected by a similarity transformation and homothetic will be called equivalent.

If in the space V the representation $T(g)$ of some group G acts, then we shall call the sheaf $\mathbb{R}(u)$ invariant with respect to the representation $T(g)$, if for any $g \in G$ one has

$$\mathbb{R}(u)(T(g) \otimes T(g)) = (T(g) \otimes T(g))\,\mathbb{R}(u)\,. \tag{13}$$

Let $\mathbb{R}^{(1)}(u)$ and $\mathbb{R}^{(2)}(u)$ be two solutions of (4) of dimensions N_1 and N_2, respectively. By the tensor product of the sheaves $\mathbb{R}^{(1)}(u)$ and $\mathbb{R}^{(2)}(u)$ we shall mean the sheaf $(\mathbb{R}^{(1)}\otimes\mathbb{R}^{(2)})(u)$ of dimension $N_1\times N_2$ defined by

$$(\mathbb{R}^{(1)}\otimes\mathbb{R}^{(2)})(u)=\mathbb{R}^{(1)}(u)\otimes\mathbb{R}^{(2)}(u)\ . \tag{14}$$

By the direct sum of the sheaves $\mathbb{R}^{(1)}(u)$ and $\mathbb{R}^{(2)}(u)$ we shall mean the sheaf $(\mathbb{R}^{(1)}+\mathbb{R}^{(2)})(u)$ of dimension N_1+N_2, defined in the following way. The operator $(\mathbb{R}^{(1)}+\mathbb{R}^{(2)})(u)$ acts on the basis vectors of the form $e^{(i)}_{\alpha_i}\otimes e^{(k)}_{\alpha_k}$ $(i,k=1,2;\ \alpha_i=1,2,\ldots,N_i;\ e^{(i)}_{\alpha_i}\in V_i)$ of the space $(V_1+V_2)\otimes(V_1+V_2)$ by:

$$\begin{aligned}
(\mathbb{R}^{(1)}+\mathbb{R}^{(2)})(u)(e^{(1)}_{\alpha_1}\otimes e^{(1)}_{\beta_1}) &= \mathbb{R}^{(1)}(u)(e^{(1)}_{\alpha_1}\otimes e^{(1)}_{\beta_1}),\\
(\mathbb{R}^{(1)}+\mathbb{R}^{(2)})(u)(e^{(1)}_{\alpha_1}\otimes e^{(2)}_{\beta_2}) &= e^{(1)}_{\alpha_1}\otimes e^{(2)}_{\beta_2},\\
(\mathbb{R}^{(1)}+\mathbb{R}^{(2)})(u)(e^{(2)}_{\alpha_2}\otimes e^{(1)}_{\beta_1}) &= e^{(2)}_{\alpha_2}\otimes e^{(1)}_{\beta_1},\\
(\mathbb{R}^{(1)}+\mathbb{R}^{(2)})(u)(e^{(2)}_{\alpha_2}\otimes e^{(2)}_{\beta_2}) &= \mathbb{R}^{(2)}(u)(e^{(2)}_{\alpha_2}\otimes e^{(2)}_{\beta_2}).
\end{aligned} \tag{15}$$

Direct verification shows that $(\mathbb{R}^{(1)}\otimes\mathbb{R}^{(2)})(u)$ and $(\mathbb{R}^{(1)}+\mathbb{R}^{(2)})(u)$ actually satisfy (4). It is also obvious that the operation of tensor multiplication of Y—B sheaves preserves the property of regularity of families of sheaves. Now the operation of addition, on the contrary, preserves only the property of quasiclassicism of sheaves. In addition, it follows from (15) that the direct sum of two sheaves is never a regular sheaf.

If the space V admits a decomposition into a direct sum of two subspaces V_1 and V_2, such that the action of the operator $\mathbb{R}(u)$ on the basis vectors of the form $e^{(i)}_{\alpha_i}\otimes e^{(k)}_{\beta_k}$ (notation is the same as in (15)) has the following property

$$\mathbb{R}(u)(e^{(i)}_{\alpha_i}\otimes e^{(k)}_{\alpha_k})\in V_i\otimes V_k\ ,\quad i,k=1,2\ ; \tag{16}$$

then the sheaf $\mathbb{R}(u)$ is called reducible. In particular, a reducible sheaf is always the direct sum of two sheaves. If no such decomposition exists, we shall call such a sheaf irreducible. It is easy to prove that for a reducible sheaf $\mathbb{R}(u)$ the operators $\mathbb{R}^{(1)}(u)$ and $\mathbb{R}^{(2)}(u)$, acting in the spaces $V_1\otimes V_1$ and $V_2\otimes V_2$, respectively, according to the formula

$$\mathbb{R}^{(i)}(u)(e^{(i)}_{\alpha_i}\otimes e^{(i)}_{\beta_i})=\mathbb{R}(u)(e^{(i)}_{\alpha_i}\otimes e^{(i)}_{\beta_i}),\quad i=1,2, \tag{17}$$

will also be Yang—Baxter sheaves. It is also obvious that a reducible sheaf cannot be regular.

As we shall see later, Y—B sheaves play a large role in the theory of quantum completely integrable systems. The analogous role in the theory of classical completely integrable systems is played by the classical Yang—Baxter sheaf, whose definition we shall now give. Let $\mathbb{R}(u,\eta)$ be a quasiclassical family of Yang—Baxter sheaves, depending smoothly on the parameter η. Then, differentiating (4) with respect to η and setting $\eta=0$, we get, keeping (9) in mind, for the quantities

$$r(u)=\frac{\partial}{\partial\eta}\,\mathbb{R}(u,\eta)\big|_{\eta=0} \tag{18}$$

the following equation

$$r_{12}(u-v)\, r_{13}(u)+r_{12}(u-v)\, r_{23}(v)+r_{13}(u)\, r_{23}(v) = r_{23}(v)\, r_{13}(u)+r_{23}(v)\, r_{12}(u-v)+r_{13}(u)\, r_{12}(u-v), \tag{19}$$

which can be rewritten in the following commutator form:

$$\left[r_{12}(u-v),\, r_{13}(u)+r_{23}(v)\right]+\left[r_{13}(u),\, r_{23}(v)\right] \doteq 0 \ . \tag{20}$$

By a classical Yang–Baxter sheaf will be meant any solution of the functional equation (20). The calculation given above shows that for a known quasiclassical family of Y–B sheaves one can always construct a classical Yang–Baxter sheaf. Whether the converse is valid, i.e., whether one can for any classical Y–B sheaf construct a corresponding quasiclassical family of Y–B sheaves, is still unknown. Using (18), one can transfer almost all concepts we have introduced for Y–B sheaves to the case of classical Y–B sheaves. In particular, instead of invariance with respect to homothety transformations (11), classical Y–B sheaves are invariant with respect to the translation transformation

$$r'(u) = r(u)+f(u)\, I \ . \tag{21}$$

The definitions of similarity transformation (12) and group invariance (13) carry over to classical Y–B sheaves without change. We shall call two classical Y–B sheaves equivalent if one of them can be turned into the other by a similarity transformation and a translation.

We shall call a classical Y–B sheaf $r(u)$ canonical if for it one has the following equation:

$$r(u) = -\mathbb{P}\, r(-u)\, \mathbb{P} \ . \tag{22}$$

The connection between the concepts of canonicity of classical and quantum Y–B sheaves is established by the following theorem.

<u>THEOREM.</u> Any canonical family of Y–B sheaves $\mathbb{R}(u,\eta)$ generates by (18) a classical Y–B sheaf $r(u)$, equivalent with a canonical one.

<u>Proof.</u> We differentiate (4) with respect to η and we set $u=0,\ \eta=0$. Multiplying the result obtained

$$r_{12}(-v)\,\mathbb{P}_{13}+\mathbb{P}_{13}\, r_{23}(v) = \mathbb{P}_{13}\, r_{12}(-v)+r_{23}(v)\,\mathbb{P}_{13} \tag{23}$$

on the right by $\mathbb{P}_{13}$ and using the obvious equations

$$\begin{aligned} \mathbb{P}_{13}\, r_{23}(v)\,\mathbb{P}_{13} &= r_{21}(v) = \mathbb{P}_{12}\, r_{12}(v)\,\mathbb{P}_{12} \ , \\ \mathbb{P}_{13}\, r_{12}(-v)\,\mathbb{P}_{13} &= r_{32}(-v) = \mathbb{P}_{23}\, r_{23}(-v)\,\mathbb{P}_{23} \ , \end{aligned} \tag{24}$$

we arrive at the equation

$$r_{12}(-v)+r_{21}(v) = r_{32}(-v)+r_{23}(v) \ . \tag{25}$$

By virtue of the obvious symmetry of the Y–B equation with respect to permutation of the spaces V_1, V_2, V_3 , one also has

$$r_{12}(-v)+r_{21}(v) = r_{13}(-v)+r_{31}(v) \ . \tag{26}$$

Comparing (25) and (26), we arrive at the inference that the operator $r_{12}(-v)\ +r_{21}(v)$ acts trivially on all three spaces $V_1, V_2\ , V_3$, i.e.,

$$r_{12}(-v)+r_{12}(v) = \varphi(v) I_1 \otimes I_2 \otimes I_3 ,$$

where the scalar function $\varphi(v)$ must be even $\varphi(v)=\varphi(-v)$. Redefining $r(v)\to r(v)+\frac{1}{2}\varphi(v)$, we get

$$r_{12}(-v)=-r_{21}(v) ,$$

equivalent with (22), which is what had to be proved.

The tensor product $(r^{(1)}\otimes r^{(2)})(u)$ of the classical Y—B sheaves $r^{(1)}(u)$ and $r^{(2)}(u)$ is given by:

$$(r^{(1)}\otimes r^{(2)})(u) = r^{(1)}(u)\otimes I^{(2)} + I^{(1)}\otimes r^{(2)}(u) , \tag{27}$$

and the direct sum by $(r^{(1)}+r^{(2)})(u)$

$$(r^{(1)}+r^{(2)})(u) = r^{(1)}(u)+r^{(2)}(u). \tag{28}$$

The definitions of reducible and irreducible sheaves are carried over to the case of classical Y—B sheaves unchanged.

We formulate to conclude this section a series of unsolved problems, standing in front of the theory of quantum and classical Yang—Baxter sheaves:

1) List all solutions of the Yang—Baxter equation (4) of a given dimension N up to equivalence. The analogous problem is of interest for regular, quasiclassical, and canonical sheaves, and families of Y—B sheaves, and also for Y—B sheaves having group invariance. The problem of listing all constant solutions of the Y—B equation, i.e., those independent of the spectral parameter, is also interesting:

$$\mathbb{R}_{12}\mathbb{R}_{13}\mathbb{R}_{23} = \mathbb{R}_{23}\mathbb{R}_{13}\mathbb{R}_{12} . \tag{29}$$

It is easy to see that (29) is satisfied, in particular, by the permutation operator $\mathbb{P}$ (6, 8) and the identity operator I (10). Setting in (4) $u=v=0$, we get that for any Y—B sheaf $\mathbb{R}(u)$, its value for $u=0$ also satisfies (29).

2) The same problems are naturally formulated also for classical Yang—Baxter sheaves. In addition to constant solutions, here there is also interest in solutions of the classical Yang—Baxter equation (20) of the form

$$r(u)=\frac{r}{u} , \tag{30}$$

where the operator r must by virtue of (20) satisfy

$$[r_{12} \cdot r_{13}+r_{23}] = 0, \quad [r_{12}+r_{13}, r_{23}] = 0 . \tag{31}$$

(For canonical sheaves $r=\mathbb{P}r\mathbb{P}$, and the equations in (31) are equivalent.)

It is easy to construct a wide class of solutions of (31). In fact, let $\mathfrak{g}$ be an arbitrary semisimple Lie algebra, J_α $(\alpha=1,2,\dots,N)$ be a basis of its generators in an arbitrary representation, $k^{\alpha\beta}$ be the matrix inverse to the matrix of the Killing form of the Lie algebra $\mathfrak{g}$ in the basis of generators J_α . Then, as is easy to verify, the operator r, defined by the formula

$$r = k^{\alpha\beta} J_\alpha \otimes J_\beta , \tag{32}$$

satisfies (31). This solution was also obtained in [6], where there is the assertion that (32) gives a complete description of the solutions of (31).

3) Can one associate with any classical Y–B sheaf $r(u)$ a classical family of Y–B sheaves such that (18) holds?

2. Now we discuss applications of the Yang–Baxter equation to the theory of quantum and classical completely integrable systems.

1) We shall show, first of all, that to any regular Yang–Baxter sheaf one can associate a quantum completely integrable system with locally mutually commuting integrals of motion. In fact, let $\mathbb{R}(u)$ be a regular Y–B sheaf of dimension N. We take as state space $\mathcal{H}$ of the quantum system sought the space $\mathcal{H}=V_1\otimes V_2\ldots\otimes V_M$ ($V_n\equiv\mathbb{C}^N$; $n=1,2,\ldots,M$), where M is an arbitrary natural number $\geqslant 2$, and we define in this space a Hamiltonian $\mathbb{H}$ by the formula:

$$\mathbb{H}=\sum_{n=1}^{M-1}\mathbb{H}_{n+1,n}+\mathbb{H}_{1,M}\quad, \tag{33}$$

where the local density of the Hamiltonian $\mathbb{H}_{n+1,n}$ is given by

$$\mathbb{H}_{n+1,n}=\left(\frac{d}{du}\mathbb{R}_{n+1,n}(u)\Big|_{u=0}\right)\mathbb{P}_{n+1,n}\ . \tag{34}$$

We explain the notation. The operator $\mathbb{R}_{n+1,n}(u)$ acts in the space $V_{n+1}\otimes V_n$, as the corresponding Yang–Baxter sheaf $\mathbb{R}(u)$, and on the remaining components of the tensor product $V_1\otimes\ldots\otimes V_M$ it acts as the identity operator. The same relates to the permutation operator $\mathbb{P}_{n+1,n}$. It is convenient to represent the quantum system we have constructed as a ring of M "atoms," each of which has N quantum states, where only the closest neighbors interact. We note that although the Hamiltonian $\mathbb{H}$, defined above, generally speaking need not be a self-adjoint operator (which, however, is not reflected in the following calculations), in practice, in the majority of cases it can be made self-adjoint by multiplying by a suitable constant.

A sequence of operators commuting with $\mathbb{H}$ is constructed in the following way. We extend our space $\mathcal{H}$ to the space $\hat{\mathcal{H}}=Q\otimes Q'\otimes\mathcal{H}$, introducing two auxiliary spaces Q and Q' isomorphic with $\mathbb{C}^N$. We define the transition operator $\mathbb{T}_1^M(u)$ by

$$\mathbb{T}_1^M(u)=\mathbb{L}_M(u)\mathbb{L}_{M-1}(u)\ldots\mathbb{L}_1(u)\,, \tag{35}$$

where $\mathbb{L}_n(u)=\mathbb{R}_{qn}(u)$ (the notation is the same as above, the index n relates to the space V_n, the index q to the space Q). Analogously, replacing Q by Q' one defines operators $\mathbb{L}'_n(u)$ and $\mathbb{T}_1'^M(u)$.

Using the notation introduced, one can rewrite (4) in the form

$$\mathbb{R}_{qq'}(u-v)\mathbb{L}_n(u)\mathbb{L}'_n(v)=\mathbb{L}'_n(v)\mathbb{L}_n(u)\mathbb{R}_{qq'}(u-v). \tag{36}$$

Starting from (36) one can prove [2–4] the following remarkable equation:

$$\mathbb{R}_{qq'}(u-v)\mathbb{T}_1^M(u)\mathbb{T}_1'^M(v)=\mathbb{T}_1'^M(v)\mathbb{T}_1^M(u)\,\mathbb{R}_{qq'}(u-v)\ . \tag{37}$$

The generating function $t(u)$ of the integrals of motion of the quantum system considered is defined as the trace of the transition operator $\mathbb{T}_1^M(u)$, taken with respect to the auxiliary space Q

$$t(u) = tr_q \mathbb{T}_1^M(u) \; . \tag{38}$$

It follows [2-4] from (37) that $t(u)$ is a family of mutually commuting operators in $\mathcal{H}$:

$$[t(u), t(v)] = 0 \, . \tag{39}$$

As shown in [7], $\ln(t^{-1}(0)t(u))$ is the generating functional of the local integrals of motion $\mathcal{J}_n$ for the Hamiltonian $\mathbb{H}$:

$$\mathcal{J}_n = \frac{d^n}{du^n} \ln(t^{-1}(0)t(u))\Big|_{u=0} \tag{40}$$

(locality means that we can represent the operator $\mathcal{J}_n$ as the sum of operators, each of which acts nontrivially at no more than $k+1$ neighboring nodes of the lattice). In particular, for $n=1$, (40) gives the Hamiltonian $\mathbb{H} = \mathcal{J}_1$.

The question of completeness of the system of integrals of motion $\mathcal{J}_n$ in the space $\mathcal{H}$ has only been weakly studied so far (completeness is strictly proved only for one of the simplest models — the Heisenberg ferromagnet [8]). The conjecture on the completeness of the integrals of motion (40) for the known Y—B sheaves of dimension $N=2$ is quite plausible. On the other hand, for dimensions $N > 2$, the completeness of the system $\mathcal{J}_n$ is automatically false as comparison with the corresponding classical completely integrable equations shows [9, 10]. The problem of constructing the missing integrals of motion in this case is still unsolved.

It is not excluded that there exist methods of constructing from a given Y—B sheaf other completely integrable quantum models too. For example, in the recent paper [11] there is constructed a relativistically invariant model of the quantum field theory, closely connected with the Y—B XYZ-sheaf (S8). Probably this result can be generalized to the case of an arbitrary Y—B sheaf.

2) The construction given above of the integrals of motion for a quantum model on a lattice was based on (37). This equation plays a most important role in the quantum method of the inverse problem [3, 4]. Besides the construction of integrals of motion, it allows one to find, in many cases, the eigenfunctions of the Hamiltonian $\mathbb{H}$ and its spectrum [3, 4]

If the Yang—Baxter sheaf $\mathbb{R}$, on which one constructs by the method described above a quantum completely integrable model on a lattice, depends also on additional parameters η, $\xi, \ldots$, then it often turns out to be possible to perform the passage to the limit, as a result of which one now gets a continuous completely integrable model of quantum field theory on the line. For example, from the XXZ model one gets in this way the nonlinear Schrödinger equation [12], from the XYZ-model one gets the Tirring model [7] and the quantum sin-Gordon equation.

The limit passage mentioned is usually realized in the following way. Depending on the character of the model, one chooses "critical" points of the spectral parameter and connection constant and performs a scaling limit passage with respect to the size of the lattice $a \to 0$, so that $an = x$ remains fixed (n is the number of nodes of the lattice). For the choice of "critical" values and scaling parameter, one considers the natural requirements of decomposability of $\mathbb{L}_n(u)$ from (36) in a series with respect to the scaling parameter and simplicity (diagonality or multiplicity 1) of the highest term

$$\mathbb{L}_n(u,\eta,\ldots) \sim \mathbb{L}^{(0)}(\alpha,\gamma,\ldots) + a\mathbb{L}^{(1)}(\alpha,\gamma,\ldots) + O(a^2) . \tag{41}$$

It is important to emphasize that the quantities $\mathbb{R}_{qq'}(\alpha,\gamma,\ldots)$ and $\mathbb{L}(x,a) = \mathbb{L}^{(0)} + a\mathbb{L}^{(1)}$ in (36) now get (after passage to the limit) a completely different interpretation in contrast with the lattice case, where $\mathbb{R}_{qq'}(u)$ and $\mathbb{L}_n(u)$ represent one and the same Yang—Baxter sheaf of dimension N. At the same time that $\mathbb{R}_{qq'}(\alpha,\gamma,\ldots)$ remains a numerical $N^2 \times N^2$ matrix, the operator $\mathbb{L}(x,a,\ldots)$ is interpreted as an $N \times N$ matrix, whose elements are operator-valued functions on the line, e.g., $\psi(x), \psi^{+}(x)$ for the nonlinear Schrödinger equation [12, 13] and $\pi(x)$, $\exp(\pm i\varphi(x))$ for the quantum sin-Gordon equation [14].

We choose the following parametrization of the Y—B sheaf (S8), connected with the XYZ-model [2],

$$\mathbb{R}(u,\eta,k) = \sum_{j=1}^{4} w_j(u,\eta,k)\, \sigma_j \otimes \sigma_j' ,$$

where $\sigma_j, \sigma_j', j = 1,2,3$ are Pauli matrices acting in $V = V' = \mathbb{C}^2$, and σ_4, σ_4' are identity matrices in these spaces,

$$w_1 = \frac{\operatorname{sn}\eta}{\operatorname{sn}u}, \quad w_2 = \frac{\operatorname{sn}\eta \operatorname{dn}u}{\operatorname{sn}u \operatorname{dn}\eta}, \quad w_3 = \frac{\operatorname{sn}\eta \operatorname{cn}u}{\operatorname{sn}u \operatorname{cn}\eta}, \quad w_4 = 1 .$$

The coefficients w_j can be expressed in terms of an analytic function of modulus k. Making the substitutions (K, K' are complete elliptic integrals of the first kind of moduli k and $k' = \sqrt{1-k^2}$)

$$u = i\alpha - iK' , \quad \eta = \gamma + \frac{\pi}{2} - K$$

and letting $k \to 0\ (k \sim a)$, we get the $\mathbb{L}$-operator of the quantum sin-Gordon equation [14]

$$\mathbb{L}(x,\alpha,\gamma,\ldots) \sim \begin{pmatrix} e^{i\gamma p} & 0 \\ 0 & e^{-i\gamma p} \end{pmatrix} + a \begin{pmatrix} 0 & e^{-\alpha}u^{-} - e^{\alpha}u^{+} \\ e^{\alpha}u^{-} - e^{-\alpha}u^{+} & 0 \end{pmatrix},$$

$$p = \int_x^{x+a} \pi(y)\,dy, \quad u^{\pm} = \exp\left(\pm i \frac{1}{a} \int_x^{x+a} \varphi(y)\,dy\right).$$

The linear problem for the quantum nonlinear Schrödinger equation [13] is obtained in the following way. First we carry out the degeneration of the Y—B sheaf (S8) in the sheaf of the XXZ-model on a lattice (S9) (the modulus of the elliptic functions $k = 0$). Then we proceed to the scaling limit in (S9) [12]

$$\eta = i\sqrt{2\gamma a}\ , \quad u = \frac{\pi}{2} + i(\alpha - \frac{\gamma}{2})\sqrt{\frac{a}{2\gamma}}\ ,$$

where γ is the connection constant of the nonlinear Schrödinger equation, α is the spectral parameter of the linear problem. As a result, we get $(\sigma_n^{\pm}/2\sqrt{a} \longrightarrow \psi^{\pm}(x))$

$$\mathbb{L}_n(u,\eta) \sim \begin{pmatrix} 1 + i\frac{\alpha}{2}a\ ; & i\sqrt{2\gamma}\int_x^{x+a}\psi^{-}(y)dy \\ i\sqrt{2\gamma}\int_x^{x+a}\psi^{+}(y)dy, & 1 - i\frac{\alpha}{2}a \end{pmatrix}.$$

Equation (36), in which $\mathbb{R}_{qq'}(u)$ is replaced by the limit of the matrices, and $\mathbb{L}_n$ is replaced by the approximate transition operator on the interval $(x, x+a)$, is satisfied only up to a. Now (37) remains valid after limit passage too, which allows one to apply for models on the line the quantum method of the inverse problem. The corresponding matrix $\mathbb{R}$, realizing the similarity of the tensor products of transition matrices of the quantum linear problems $\mathbb{T}(\lambda)\otimes\mathbb{T}(\mu)$ and $\mathbb{T}(\mu)\otimes\mathbb{T}(\lambda)$, will be called the quantum $\mathbb{R}$-matrix.

3) Most of the results of the first two parts of this section relating to quantum completely integrable systems can be carried over to the classical case too. The role of quantum Yang—Baxter sheaves will be played here by the classical sheaves. For example, analogously to the way for a quantum Yang—Baxter sheaf there was constructed a quantum completely integrable system, with any classical Yang—Baxter sheaf one can in a canonical way associate a classical completely integrable system of Heisenberg ferromagnet type [15]. Without describing this construction in detail, we note only that the quantum equations (36) and (37) correspond here to the classical equations

$$\{ L'(x,u), L''(y,v)\} = -i\left[r(u-v), L'(x,u) + L''(y,v)\right]\ , \tag{42}$$

$$\{ T_y'^{x}(u), T_y''^{x}(v)\} = -i\left[r(u-v), T_y'^{x}(u) + T_y''^{x}(v)\right]\ . \tag{43}$$

Equation (42) reproduces (20) with this difference that in the decomposition of $r_{13}(u)$ and $r_{23}(v)$ in terms of generators of some Lie algebra $\mathcal{J}_\alpha^{(i)}$, $i = 1,2,3$, the generators $\mathcal{J}_\alpha^{(3)}$ are replaced by functions $S_\alpha(x)$, of the Poisson brackets for which the commutation relations for the generators $\mathcal{J}_\alpha$ are reproduced and the commutator $[r_{13}(u), r_{23}(v)]$ is replaced by the Poisson brackets $L'(x,u)$ and $L''(y,v)$. Thus, $L(x,u)$ is an $N \times N$ matrix whose matrix elements are functions on the phase space of the dynamical system with Poisson bracket

$$\{ S_\alpha(x), S_\beta(y)\} = -c_{\alpha\beta}^{\gamma} S_\gamma(x)\,\delta(x-y),\ \left([\mathcal{J}_\alpha, \mathcal{J}_\beta] = i c_{\alpha\beta}^{\gamma}\mathcal{J}_\gamma\right).$$

The classical transition matrix is determined as a fundamental solution of the differential equation

$$\frac{\partial}{\partial x} T_y^x(u) = L(x,u) T_y^x(u), \quad T_x^x(u) = I\ , \tag{44}$$

and the matrices $L'(x,u)$ and $L''(y,v)$ are given by

$$L'(x,u) = L(x,u)\otimes I, \quad L''(y,v) = I\otimes L(y,v)\ . \tag{45}$$

In order to get (43) from (37), it is necessary to expand (37) in powers of the quasiclassical parameter η as $\eta \to 0$ and to use the relation $[\cdot,\cdot] \to -i\hbar\{\cdot,\cdot\}$ between the quantum commutator and the classical Poisson bracket, retaining in (37) terms of order η. The parameter η plays here the role of Planck's constant $\hbar$ [15].

As in the quantum case too, (43) allows one to calculate the Poisson brackets between matrix elements of the transition matrix and to construct commuting integrals of motion and action-angle variables [15]. The matrix figuring in (42), (43) will be called the classical r-matrix.

In the scheme described above there is contained a large number of completely integrable classical models, e.g., the nonlinear Schrödinger equation [13], the sin-Gordon equation, the Landau–Lifshits equation [15], Toda chain [3, 16], etc. [30]. It is essential here, however, that the Poisson brackets between dynamical variables are ultralocal in the terminology of [3], i.e., do not contain derivatives of the δ-function. There is interest in the problem of generalizing this scheme to equations with nonultralocal Poisson brackets, e.g., the Korteweg–de Vries equation. The first steps in this direction were taken by S. A. Tsyplyaev, who proved that for the sin-Gordon equation the classical r-matrices in the laboratory system $(\{\pi(x),\varphi(y)\}=\delta(x-y))$ and in the light cone system $(\{\varphi'(x),\varphi'(y)\}=\delta'(x-y))$ coincide.

4) To conclude this section, we note that there exists another important interpretation of the Yang–Baxter equation (1), as the condition for factorization of multiparticle S-matrices. The functions ${}_{\alpha\beta}R_{\gamma\delta}(\theta_1-\theta_2)$ are interpreted here as the scattering matrix of two particles of identical mass with relativistic speeds θ_1 and θ_2 and having N states ("polarizations") each, which are given by the indices $\alpha, \beta, \gamma, \delta$. Equation (1) here is the condition of reducibility of any multiparticle collision to a two-particle one (property of factorizability of multiparticle S-matrices). In this context (1) was first obtained by Yang [1], as a property of two-particle S-matrices of nonrelativistic one-dimensional Bose particles with exact interaction. Later, when the close connection was established between complete integrability of a model and the factorizability of its S-matrix [17], (1) was situated at the foundation of the method of calculation of factorized relativistic S-matrices (see [19] and the references in it). Here, in addition to (1) there are imposed on the S-matrix conditions of unitariness, analyticity, and crossing-symmetry. In the realms of this approach a large collection of S-matrices have been calculated. The fact is encouraging that the S-matrices found in a series of models of dynamics [20, 30] in the realms of the quantum Hamiltonian approach coincided with the answers obtained earlier by the method of factorization of S-matrices.

Using the S-matrix treatment of the Yang–Baxter equation one can give an intuitive interpretation of the operations on Y–B sheaves introduced in Sec. 1. The tensor product of sheaves describes the construction of the S-matrix for composite particles. The direct sum corresponds to the possibility of dividing the particles considered according to isotropic indices into two kinds such that particles of different kinds do not interact with one another. Reducibility means the possibility of dividing the particles into groups such that for particles from different groups the scattering is without reflections.

Besides the applications listed above, the Yang—Baxter equation is also used in the theory of exactly solvable models of statistical physics. The first paper in this area is Baxter [2]. Not having the possibility of discussing this interesting direction, we refer interested readers to [2, 4, 21].

3. In this section we list the basic methods of finding solutions of the Yang—Baxter equation, known from the literature.

1) The most straightforward method of solving (1) consists of choosing some more or less successful substitution for the matrix ${}_{\alpha\beta}R_{\gamma\delta}(u)$ (e.g., to impose some symmetry condition) and write down the so obtained cubical system of functional equations for the matrix elements ${}_{\alpha\beta}R_{\gamma\delta}(u)$. The system of equations obtained can either be solved directly, as was done by Baxter in [2], or one can differentiate it with respect to v and set $v=0$, getting a system of differential equations, which with some luck and skill can be solved. Although this method is rather awkward and success is not guaranteed, the overwhelming majority of known Yang—Baxter sheaves were obtained in precisely this way. In view of the large volume of calculations in verifying the Yang—Baxter equation for concrete sheaves, it can turn out to be helpful here to use a computer, in particular, programming languages allowing one to make analytic calculations [22].

2) An important improvement in the preceding method is connected with the algebra of Zamolodchikov [19]. We consider the algebra generated by elements $A_\alpha(\theta)$ and the commutation relations

$$A_\alpha(u)\,A_\beta(v) = {}_{\alpha\beta}R_{\gamma\delta}(u-v)A_\delta(v)A_\gamma(u). \tag{46}$$

The Yang—Baxter equation is the condition for associativity of this algebra under the assumption of the linear independence of the monomials of the third degree in A_α. The use of the Zamolodchikov algebra in practical calculations allows one easily to write down the matrix elements of the Yang—Baxter equation considering monomials of the form $A_\alpha(u)A_{\alpha'}(v)A_{\alpha''}(w)$ and performing commutations according to (46).

Cherednik constructed a realization of the operators $A_\alpha(u)$ for the XYZ- sheaf (S8) of dimension 2 [23] in the form of compositions of operators of multiplication and translation in the space of functions on an elliptic curves. Analogous relations of the Zamolodchikov algebra for $N>2$ were obtained in [5, 24]. Although Cherednik's method allows one to get new Yang—Baxter sheaves, for the corresponding realizations of the Zamolodchikov algebra for $N>2$ one does not have the independence of the third-degree monomials, and (1) must be verified independently each time.

3) And, finally, the third method of solving the Yang—Baxter equation consists of seeking a Y—B sheaf $\mathbb{R}(u)$ as an R-matrix $\mathbb{R}_{qq'}(u)$, involved in relations of type (36), (37) (or (42), (43) in the classical case) for some completely integrable model. The advantage of this method is that in the classical case the operator $L(u)$ is, as a rule, known in advance

from the classical method of the inverse scattering problem, and the problem reduces simply to the definition of $r(u)$ from the (it is true, redefined) system of linear equations (42). Knowledge of the classical $\mathbb{L}$-operator essentially facilitates the search for the quantum $\mathbb{L}$-operator for the corresponding quantum problem (which need not at all coincide with the classical $\mathbb{L}$-operator [14]).

Of course, as in the Cherednik realization, the Yang—Baxter equation (4) follows from (37) only under the condition of the independence of the monomials of third degree in $\mathbb{T}(u)$, which it is difficult to verify. Hence to get the sheaves $\mathbb{R}(u)$ it is necessary to verify (4) directly each time.

4. In this section we consider some generalizations of the Yang—Baxter equation.

One of the natural generalizations of the Yang—Baxter equation is connected with the introduction of a graduation [25]. In particular, the classical and quantum equations which can be solved by the method of the inverse problem and which along with the usual functions contain functions with anticommuting values, and in quantum theory, Fermi fields, lead to this generalization [26-28]. In what follows we shall speak only of $\mathbb{Z}_2$-graduation with the symbol $\mathbb{Z}_2$ frequently omitted.

A vector space V is called $\mathbb{Z}_2$-graded if it is decomposed into a direct sum of two subspaces $V_0 \oplus V_1$. Elements of V, having zero projection onto one of these subspaces, are called homogeneous. For the homogeneous elements x there is defined a function p(x) with values in the group $\mathbb{Z}_2$:

$p(x) = 0$, if $x \in V_0$ (even elements);
$p(x) = 1$, if $x \in V_1$ (odd elements).

If the dimensions of the spaces V_0 and V_1 are equal to n and m, respectively, then one writes the dimension of the graded space thus: $\dim V = (n, m)$.

An algebra $\mathcal{A}$ is called graded if it is graded as a vector space $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and for any homogeneous elements $a_\alpha \in \mathcal{A}_\alpha$ one has the property: $a_\alpha a_\beta \in \mathcal{A}_{\alpha+\beta}$, i.e., $p(a_\alpha a_\beta) = p(a_\alpha) + p(a_\beta)$ (addition of α and β reduced mod 2). $\mathbb{Z}_2$-graduations of an algebra are also called superalgebras. If for homogeneous elements of the graded algebra $\mathcal{A}$ one has the relation $ab = (-1)^{p(a)p(b)} ba$, then $\mathcal{A}$ is called a commutative superalgebra. An example of such an algebra is the Grassman algebra $\mathcal{G}$ [29].

We choose in the space $V = V_0 \oplus V_1$ a basis of homogeneous elements $e_1, \ldots, e_n \in V_0$ and $e_{n+1}, \ldots, e_{n+m} \in V_1$. The coefficients of the expansion of a vector $x \in V$ belong to the Grassman algebra $x = \sum_{i=1}^{n+m} e_i x_i$, $x_i \in \mathcal{G}$ (V is a right $\mathcal{G}$-module). Right linear operators in V can be represented in the chosen basis in the form of matrices

$$F(x) = F(e_i x_i) = F(e_i) x_i = e_j F_{ji} x_i \quad .$$

Such matrices F_{ij} are graded — to their rows and columns one can ascribe parity: $p_r(i) = p(e_i)$, $p_c(j) = p(e_j)$, $i, j = 1, \ldots, n+m$. A graduation is also introduced in the linear space of such matrices. To matrix F one ascribes a definite parity $p(F)$ if the expression

$$p(F) \equiv p(i) + p(j) + p(F_{ij})$$

is independent of i and j (the matrix elements $F_{ij} \in \mathcal{G}$ and $p(F_{ij})$ is its parity as an element of the Grassman algebra). In what follows we shall be interested only in matrices of parity zero, for which $p(F_{ij}) = p(i) + p(j)$.

We consider the graded Zamolodchikov algebra with generators $A_\alpha(u)$, some of which are even $p(A_\alpha(u)) \equiv p(\alpha) = 0$, and the others odd $p(A_\beta(u)) \equiv p(\beta) = 1$. The basic commutation relation (46) we rewrite in the form

$$A_\alpha(u) A_\beta(v) = (-1)^{p(\alpha)p(\beta)} {}_{\alpha\beta}R_{\gamma\delta}(u-v) A_\delta(v) A_\gamma(u), \tag{47}$$

assuming that all ${}_{\alpha\beta}R_{\gamma\delta}$ are even elements of $\mathcal{G}$. If ${}_{\alpha\beta}R_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}$, then the algebra introduced becomes a commutative superalgebra. Just as for an ordinary algebra (46) the condition for associativity under the assumption of the independence of monomials of third degree in the generators $A_\alpha(u)$, will be the relation

$$ {}_{\alpha\alpha'}R_{\gamma\gamma'}(u-v)\,{}_{\gamma\alpha''}R_{\beta\gamma''}(u)\,{}_{\gamma'\gamma''}R_{\beta'\beta''}(v)(-1)^{p(\gamma')(p(\alpha'')+p(\gamma''))} = {}_{\alpha'\alpha''}R_{\gamma'\gamma''}(v)\,{}_{\alpha\gamma''}R_{\gamma\beta''}(u)\,{}_{\gamma\gamma'}R_{\beta\beta'}(u-v)(-1)^{p(\gamma')(p(\gamma'')+p(\beta''))}. \tag{48}$$

Between the solutions of the Yang–Baxter equation (1) and its graded analog (48) there is a one-to-one correspondence. In fact, we overdetermine the coefficients in (48)

$$ {}_{\alpha\beta}\tilde{R}_{\gamma\delta}(u) = (-1)^{p(\alpha)p(\beta)} {}_{\alpha\beta}R_{\gamma\delta}(u). \tag{49}$$

Then for $\tilde{\mathbb{R}}(u)$ we get (1). Here it is essential that $p({}_{\alpha\beta}R_{\gamma\delta}) = 0$ for any nonzero ${}_{\alpha\beta}R_{\gamma\delta}$ and as a matrix $\mathbb{R}$ has null parity $p(\mathbb{R}) = 0$.

$\mathbb{R}(u)$ can be considered as a matrix in the tensor product of two graded spaces $V \otimes V$, $\dim V = (n, m)$. For compact notation for (48), we need the permutation operator in the tensor product of graded spaces and the operation of tensor product of graded matrices.

As a basis in the tensor product of two spaces $V \otimes W$ we take $v_i \otimes w_j$ (v_i, w_j are homogeneous elements). The components of the vector $x \otimes y$ in this basis are equal to $x_i y_j (-1)^{p(x_i)p(j)}$:

$$x \otimes y = (v_i x_i) \otimes (w_j y_j) = (v_i \otimes w_j) x_i y_j (-1)^{p(x_i)p(j)}.$$

We define the action of the (right) linear operator $F \otimes G$ in the space $V \otimes W$ as $(F \otimes G)(x \otimes y) = F(x) \otimes G(y)$ (we consider operators of null parity $p(F) = p(G) = 0$, otherwise there arises an additional factor $(-1)^{p(x)p(G)}$). As a result the matrix element of the tensor product of even matrices $\{F_{ij}\}$, $\{G_{ab}\}$ has the form

$$ {}_{ia}(F \otimes G)_{jb} = F_{ij} G_{ab} (-1)^{p(a)(p(i)+p(j))}. \tag{50}$$

The permutation operator $\mathbb{P}$ in $V \otimes V$, defined by its action on the product of homogeneous elements x, y has the form

$$\mathbb{P}(x \otimes y) = (-1)^{p(x)p(y)} y \otimes x, \quad {}_{ab}P_{cd} = \delta_{ad}\delta_{bc}(-1)^{p(c)p(d)}. \tag{51}$$

Just as the ordinary permutation operator satisfies (1), the operator (51) satisfies (48). Using this operator one can write one of the solutions of (48) for arbitrary dimension of the space V and graduation

$$\mathbb{R}(u)=a(u)+b(u)\mathbb{P}, \quad a(u)=1-b(u)=u/u+1 \ . \tag{52}$$

We note that the correspondence described above between $\mathbb{R}(u)$ and $\widetilde{\mathbb{R}}(u)$ (see (49)) preserves the property of regularity of Y—B sheaves, but not quasiclassicity.

We make use of the definition of the tensor product (50) and the graded permutation operator (51). Then for the operator

$$\check{\mathbb{R}}(u)=\mathbb{P}\mathbb{R}(u)$$

(48) assumes the same form as in Sec. 1 (7)

$$(\mathbb{I}\otimes\check{\mathbb{R}}(u-v))(\check{\mathbb{R}}(u)\otimes\mathbb{I})(\mathbb{I}\otimes\check{\mathbb{R}}(v))=(\check{\mathbb{R}}(v)\otimes\mathbb{I})(\mathbb{I}\otimes\check{\mathbb{R}}(u))(\check{\mathbb{R}}(u-v)\otimes\mathbb{I}) \ . \tag{53}$$

Just as in the nongraduated case, for a quasiclassical family of Y—B sheaves $(\mathbb{R}(u,\eta)|_{\eta=0}=\mathbb{I})$ one introduces a classical Y—B sheaf:

$$r(u)=\frac{d}{d\eta}\mathbb{R}(u,\eta)\Big|_{\eta=0} \ . \tag{54}$$

It is convenient to get the equation for $r(u)$ from the following form of (48):

$$\mathbb{R}_{12}(u-v)\overline{\mathbb{R}}_{13}(u)\mathbb{R}_{23}(v)=\mathbb{R}_{23}(v)\overline{\mathbb{R}}_{13}(u)\mathbb{R}_{23}(u-v) \ , \tag{55}$$

where the lower indices indicate in which of the three spaces $V_1\otimes V_2\otimes V_3$ the Y—B sheaf acts nontrivially, and

$$\overline{\mathbb{R}}_{13}(u)=\mathbb{P}_{23}\mathbb{R}_{12}(u)\mathbb{P}_{23}=\mathbb{P}_{12}\mathbb{R}_{23}(u)\mathbb{P}_{12} \ . \tag{56}$$

Differentiating (55) twice with respect to η and setting $\eta=0$, we get

$$[r_{12}(u-v),\bar{r}_{13}(u)+r_{23}(v)]+[\bar{r}_{13}(u),r_{23}(v)]=0 \ . \tag{57}$$

The arbitrariness connected with multiplication of $\mathbb{R}(u)$ by an arbitrary function leads to additive arbitrariness in r: if $r(u)$ is a solution of (57), then $r(u)+\varphi(u)\mathbb{I}$ is also a solution. One can make use of this arbitrariness and choose $\varphi(u)$ so that for $r(u)$ one has

$$r(u)=-\mathbb{P}r(-u)\mathbb{P} \ . \tag{58}$$

We shall say a few words about applications. Graduated Y—B sheaves, just like ordinary ones, about which we spoke in Sec. 2, arise if one applies the quantum method of the inverse problem to equations in which anticommuting quantum fields enter. Such are, e.g., the matrix nonlinear Schrödinger equation with Bose and Fermi fields, the massive model of Tirring with anticommuting fields [26], the supersymmetric sin-Gordon equation [27], etc. If $\mathbb{R}(u-v)$ interlaces the quantum transition matrices $\mathbb{T}(u)$, $\mathbb{T}(v)$, then $r(u-v)$ defines the Poisson brackets of the matricial elements of $\mathbb{T}(u),\mathbb{T}(v)$ in the classical theory. Equations (57) and (58) are the reflections, respectively, of the Jacobi identity and the antisymmetry property of the Poisson brackets.

Another possible generalization of the Yang—Baxter equation consists of considering (4) as a functional equation for three different operators $\mathbb{R}_{12}(u-v)$, $\mathbb{R}_{13}(u)$, and $\mathbb{R}_{23}(v)$ acting in the product of three different spaces $V_1\otimes V_2\otimes V_3$ with dimensions N_1, N_2, N_3, respectively, where the operator $\mathbb{R}_{\alpha\beta}(u)$ acts nontrivially only in $V_\alpha\otimes V_\beta$. In the language of the S-matrix interpretation, discussed at the end of Sec. 2, this means that we consider scattering of three kinds of particles, where N_α is the number of internal states of the α-th kind of particles $(\alpha = 1,2,3)$. The coefficient of reflection for the scattering of particles of different kinds here must be equal to zero.

One gets an especially simple such generalization for classical Y—B sheaves. In fact, any classical Y—B sheaf of dimension N can be decomposed as an operator in $C^N \otimes C^N$ with respect to basis elements of the form $\mathcal{J}_\alpha \otimes \mathcal{J}_\beta$ $(\alpha,\beta = 1,2,\ldots,N^2)$, where $\mathcal{J}_\alpha$ is any basis in the Lie algebra $gl(N,C)$:

$$r(u) = \sum_{\alpha,\beta=1}^{N^2} r_{\alpha\beta}(u)\, \mathcal{J}_\alpha \otimes \mathcal{J}_\beta \quad . \tag{59}$$

But since in the classical Yang—Baxter equation (20) only commutators appear, it can be considered as an equation on the Lie algebra $gl(N,C)$ and one can take as $\mathcal{J}_\alpha$ the generators of $gl(N,C)$ in any other representation besides the fundamental one. If the generators $\mathcal{J}_\alpha$, appearing in (59) with nonzero coefficients $r_{\alpha\beta}(u)$, form a subalgebra $\mathcal{A} \subset gl(N,C)$, then the same arguments work for the Lie algebra $\mathcal{A}$. Thus, for any classical Y—B sheaf one can construct an infinite series of classical Y—B sheaves of any dimensions, and also solutions of the generalized in the above sense Y—B equation.

If the conjecture discussed in Sec. 1 that to any classical Y—B sheaf corresponds a quantum quasiclassical Y—B sheaf is valid, then analogous results should be expected also in the quantum case. However, since the quantum Yang—Baxter equation (4) is not expressed in terms of commutators, the problem of extending a given Y—B sheaf to higher representations of $gl(N,C)$ (or other Lie algebras) becomes much more complicated than in the classical case. It is reasonable to assume that analogous to the way the classical Y—B equation (20) can be considered as an equation on a Lie algebra, the quantum Y—B equation (4) can be considered on the universal enveloping Lie algebra and one can get finite-dimensional Y—B sheaves by reducing a "universal" sheaf. The validity of this conjecture is verified by one of the authors (E.K.S.) for the simplest $gl(2,C)$ invariant sheaf (see Supplement, formula (S3)).* Using the generalized quantum linear problem for the sin-Gordon equation on higher representations with respect to an auxiliary space and applying the third method (Sec. 3) of finding solutions of the Y—B equation as quantum R-matrices, one can get a generalization of the XXZ-sheaf (S9) in terms of the universal enveloping algebra of $gl(2,C)$ [44].

SUPPLEMENT

In the Supplement we give a summary of known solutions of the quantum Yang—Baxter equation. Since in a series of cases the verification of (1) requires long and tiresome calculations, not all the sheaves given below will be reexamined. In the majority of examples, we indicate the authors to whom the assertion that the given sheaf satisfies (1) is due.

*As V. A. Fateev informed us, he, together with A. B. Zamolodchikov, obtained an analogous result.

It will be convenient for us to represent the tensor ${}_{\alpha\beta}R_{\gamma\delta}$ in the form of a square block-matrix, considering the index α as the number of the block-row, β as the number of the row in block α, γ as the number of the block-column, and δ as the number of the column in block γ.

The spectral parameter is denoted everywhere by u, the quasiclassical parameter (for quasiclassical sheaves) by η. The normalization of quasiclassical sheaves is chosen so that $\mathbb{R}(u,\eta)=\mathrm{I}$ for $\eta=0$. For quasiclassical sheaves we give the corresponding classical sheaf $r(u)=\frac{d}{d\eta}\mathbb{R}(u,\eta)|_{\eta=0}$ (18).

1) $GL(N,C)$-Invariant Sheaf (Yang [1])

$$\mathbb{R}(u)=\frac{u}{u+\eta}\mathrm{I}+\frac{\eta}{u+\eta}\mathbb{P},\quad \dim\mathbb{R}=N\geq 2\,. \tag{S1}$$

It is obvious that I and $\mathbb{P}$ are the unique operators in $C^N\otimes C^N$, invariant with respect to the group $GL(N,C)$ in the sense of (13). The sheaf (S1) is regular and quasiclassical, the corresponding classical sheaf $r(u)$ has the form

$$r(u)=\frac{1}{u}\mathbb{P}+\varphi(u)\mathrm{I}\;. \tag{S2}$$

The sheaf $\mathbb{R}(u)$ is widely used in the quantum method of the inverse problem [3]. For $N=2$ it arises in studying the quantum nonlinear Schrödinger equation [13], the XXX-model (Heisenberg ferromagnet) [3, 12], Toda chains [3, 16]. Sheaves with $N\geq 3$ are used in considering multicomponent analogs of the equations mentioned: vector and matrix nonlinear Schrödinger equations [10], generalized Heisenberg ferromagnet [30], non-Abelian Toda chains [16], systems of n-waves (Lie model) [31]. In addition, the sheaf (S1) is used as S-matrix for nonrelativistic particles with pointwise interaction [1].

As noted at the end of Sec. 4, for $N=2$ there are known analogs of the sheaf (S1) for any finite-dimensional irreducible representation of the group $GL(2,C)$. Let there act in the space $V=C^{\ell}$ an ℓ-dimensional irreducible representation $\mathcal{D}_\ell(g)$ of the group $GL(2,C)$. Then the generalized sheaf (S1) has the form:

$$\mathbb{R}(u)=\sum_{j=0}^{2\ell}\prod_{k=1}^{j}\frac{u-k\eta}{u+k\eta}P_j\;, \tag{S3}$$

where P_j is the projector onto the space of j-dimensional irreducible representations in the decomposition of the tensor product $\mathcal{D}_\ell\otimes\mathcal{D}_\ell$ into irreducible representations. The sheaf obtained is canonical. The proof of (S3) will be published separately.

2) $SO(N,C)$ Invariant Sheaves

In the space $C^N\otimes C^N$ there are in all three operators, invariant with respect to the action of the group $SO(N,C)$. We have already met this with the operators I and $\mathbb{P}$, and, in addition, the projector K:

$${}_{\alpha\beta}K_{\gamma\delta}=\delta_{\alpha\beta}\,\delta_{\gamma\delta}\;. \tag{S4}$$

An invariant Y–B sheaf corresponding to $SO(N,C)$ was found in [18] as S-matrix for the Gross–Neveu model:

$$\mathbb{R}(u)=\frac{u}{u+\eta}\mathbb{I}+\frac{\eta}{u+\eta}\mathbb{P}+\frac{u\eta}{(u+\eta)(\eta g-u)}\mathbb{K},\quad g=-\frac{N-2}{2}\ . \tag{S5}$$

The sheaf (S5) is canonical. The corresponding classical sheaf:

$$r(u)=\frac{1}{u}\mathbb{P}-\frac{1}{u}\mathbb{K}+\varphi(u)\mathbb{I}\ . \tag{S6}$$

It is interesting to note that if one takes a linear combination of only $\mathbb{I}$ and $\mathbb{P}$, we get a $GL(N,C)$-invariant solution of (S1), for combinations of $\mathbb{P}$ and $\mathbb{K}$ we get a new sheaf

$$\mathbb{R}(u)=\mathbb{P}-\frac{sh(u)}{sh(u+\gamma)}\mathbb{K},\quad e^{\gamma}=\frac{1}{2}(N-\sqrt{N^2-4}), \tag{S7}$$

which is regular, but not quasiclassical, and finally, for combinations of only $\mathbb{I}$ and $\mathbb{K}$ a solution does not exist.

3) XYZ-Sheaf (Baxter [2])

$$\mathbb{R}(u,\eta,k)=\begin{vmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{vmatrix}. \tag{S8}$$

TABLE 1

	R	$u=0$	$\eta=0$	$\partial/\partial\eta\cdot\vert_{\eta=0}$
a	1	1	1	0
b	$\dfrac{sn\,u}{sn(u+\eta)}$	0	1	$-\dfrac{cn\,u\,dn\,u}{sn\,u}$
c	$\dfrac{sn\,\eta}{sn(u+\eta)}$	1	0	$\dfrac{1}{sn\,u}$
d	$k\,sn\,u\,sn\,\eta$	0	0	$k\,sn\,u$

In the formulas of Table 1 all elliptic functions have modulus k. The XYZ-sheaf is canonical. To the quantum XYZ-sheaf corresponds a completely integrable lattice XYZ-model [4, 32]. The corresponding classical sheaf corresponds to the Landau–Lifshits equation [15]. The presence in the XYZ-sheaf of two parameters: η and k allows one to get different degenerate cases, e.g., XXZ- and XXX-sheaves (see below), and quantum models on the line (sin-Gordon equation, nonlinear Schrödinger equation).

4) XXZ-Sheaf

$$\mathbb{R}(u,\eta)=\begin{vmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{vmatrix}. \tag{S9}$$

The XXZ-sheaf is obtained from the XYZ-sheaf (Table 1) as the limit as $k\to 0$. Like the XYZ-sheaf, the XXZ-sheaf is canonical. To it corresponds a quantum lattice XXZ-model, which was considered in the realms of the quantum method of the inverse problem (QMIP) in

TABLE 2

	R	$u=0$	$\eta=0$	$\partial/\partial\eta \cdot \vert_{\eta=0}$
a	1	1	1	0
b	$\frac{\sin u}{\sin(u+\eta)}$	0	1	$-\mathrm{ctg}\, u$
c	$\frac{\sin\eta}{\sin(u+\eta)}$	1	0	$\frac{1}{\sin u}$

TABLE 3

	R		R
s_1	$\mathcal{J}(\mathrm{sn}\,u\,\mathrm{cn}\,2\eta + \frac{\mathrm{sn}\,\eta\,\mathrm{sn}\,2\eta}{\mathrm{sn}(u+\eta)}$	t	$\varepsilon_3 \mathcal{J}\,\mathrm{sn}\,u\,\mathrm{cn}\,2\eta$
s_2	$\mathcal{J}(\mathrm{sn}\,u(\mathrm{cn}2\eta+\mathrm{dn}2\eta-1)+\frac{\mathrm{sn}\,\eta\,\mathrm{sn}\,2\eta}{\mathrm{sn}(u+\eta)})$	T	$\varepsilon_4 \mathcal{J}\,\mathrm{sn}\,u$
s_3	$\mathcal{J}(\mathrm{sn}\,u\,\mathrm{dn}\,2\eta+\frac{\mathrm{sn}\,\eta\,\mathrm{sn}\,2\eta}{\mathrm{sn}(u+\eta)})$	R	$\mathcal{J}\,\mathrm{sn}\,2\eta$
α	$\varepsilon_1 \mathcal{J}\,\mathrm{sn}\,u\,\mathrm{cn}(u+\eta)\,\mathrm{sn}\,2\eta/\mathrm{sn}(u+\eta)$	a	$\varepsilon_5 \mathcal{J}\,\mathrm{sn}\,u\,\mathrm{dn}\,2\eta$
β	$\varepsilon_2 \mathcal{J}\,\mathrm{sn}\,u\,\mathrm{sn}\,2\eta/\mathrm{sn}(u+\eta)$	r	$\mathcal{J}\,\mathrm{cn}\,u\,\mathrm{sn}\,2\eta$
γ	$-\varepsilon_1\varepsilon_2 \mathcal{J}\,\mathrm{sn}\,u\,\mathrm{sn}\,2\eta/\mathrm{sn}(u+\eta)$	q	$\mathcal{J}\,\mathrm{dn}\,u\,\mathrm{sn}\,2\eta$

$$\mathcal{J} = 1/\mathrm{sn}(u+2\eta)\ , \qquad \varepsilon_i = \pm 1\ .$$

TABLE 4

	R	$u=0$	$\eta=0$	$\partial/\partial\eta \cdot \vert_{\eta=0}$
s	1	1	1	0
t	$\mathcal{J}\,\mathrm{sh}\,u$	0	1	$-2\,\mathrm{cth}\,u$
r	$\mathcal{J}\,\mathrm{sh}\,2\eta$	1	0	$\frac{2}{\mathrm{sh}\,u}$
a	$\varepsilon\mathcal{J}\frac{\mathrm{sh}\,u\,\mathrm{sh}\,2\eta}{\mathrm{sh}(u+\eta)}$	0	0	$\frac{2\varepsilon}{\mathrm{sh}\,u}$
R	$\mathcal{J}\frac{\mathrm{sh}\,\eta\,\mathrm{sh}\,2\eta}{\mathrm{sh}(u+\eta)}$	1	0	0
T	$\mathcal{J}\frac{\mathrm{sh}\,u\,\mathrm{sh}(u-\eta)}{\mathrm{sh}(u+\eta)}$	0	1	$-4\,\mathrm{cth}\,u$
σ	$t+R$	1	1	$-2\,\mathrm{cth}\,u$

$$\mathcal{J} = 1/\mathrm{sh}(u+2\eta)\ .$$

TABLE 5

	R	$u=0$	$\eta=0$	$\partial/\partial\eta \cdot \vert_{\eta=0}$
a	1	1	1	0
b	$g(\eta)\frac{\mathrm{sh}\,u}{\mathrm{sh}(u+\eta)}$	0	$g(0)$	$g'(0)-g(0)\,\mathrm{cth}\,u$
$\bar{b}$	$g^{-1}(\eta)\frac{\mathrm{sh}\,u}{\mathrm{sh}(u+\eta)}$	0	$g^{-1}(0)$	$\frac{-g'(0)}{g^2(0)} - g^{-1}(0)\,\mathrm{cth}\,u$
c	$e^{-\frac{1}{3}u}\frac{\mathrm{sh}\,\eta}{\mathrm{sh}(u+\eta)}$	1	0	$e^{\frac{1}{3}u}/\mathrm{sh}\,u$
$\bar{c}$	$e^{-\frac{1}{3}u}\frac{\mathrm{sh}\,\eta}{\mathrm{sh}(u+\eta)}$	1	0	$e^{\frac{1}{3}u}/\mathrm{sh}\,u$

[12]. This sheaf also arises in the quantization of the sin-Gordon equation by means of QMIP [14].

For the following degeneration $u=\varepsilon v$, $\eta=\varepsilon\gamma$, $\varepsilon\to 0$ the XXZ-sheaf degenerates into an XXX-sheaf corresponding with the sheaf (S1) for $N=2$.

5) XYZ-Sheaf for Spin 1 (Fateev [33])

$$\mathbb{R}(u,\eta,k)=\left|\begin{array}{ccc|ccc|ccc} s_1 & & & & \alpha & & & & \beta \\ & t & & r & & & & & \\ & & T & & & & R & & \\ \hline & r & & t & & & & & \\ \alpha & & & & s_2 & & & & \gamma \\ & & & & & a & & q & \\ \hline & & R & & & & T & & \\ & & & & & q & & a & \\ \beta & & & & \gamma & & & & s_3 \end{array}\right| \tag{S10}$$

As in Table 1, the elliptic functions in Table 3 have modulus k. The sheaf (S10) is canonical. The corresponding classical sheaf, which we shall not write down due to its complexity, is a sheaf (S8) rewritten in a basis of generators of a three-dimensional irreducible representation of $gl(2,\mathbb{C})$, which allows one to consider the sheaf (S10) as a generalization of the sheaf (S8) to a higher representation (see end of Sec. 4).

6) XXZ-Sheaf for Spin 1 (Zamolodchikov and Fateev [34])

$$\mathbb{R}(u,\eta)=\left|\begin{array}{ccc|ccc|ccc} s & & & & & & & & \\ & t & & r & & & & & \\ & & T & & a & & R & & \\ \hline & r & & t & & & & & \\ & & a & & \sigma & & a & & \\ & & & & & t & & r & \\ \hline & & R & & a & & T & & \\ & & & & & r & & t & \\ & & & & & & & & s \end{array}\right| \tag{S11}$$

In Table 4, $\varepsilon=\pm 1$. The sheaf (S11) for $\varepsilon=1$ is obtained from (S10) with $\varepsilon_i=1$ after passage to the limit as $k\to 0$ and a similarity transformation (12) with matrix

$$T=\frac{1}{\sqrt{2}}\left|\begin{array}{ccc} 1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ -i & 1 & 0 \end{array}\right|.$$

The property of canonicity of (S10) is also preserved for (S11).

7) Z_N-Invariant Sheaf (Cherednik [24])

For $N=3$

$$\mathbb{R}(u,\eta)=\left|\begin{array}{ccc|ccc|ccc} a & & & & & & & & \\ & b & & c & & & & & \\ & & \bar{b} & & & & \bar{c} & & \\ \hline & \bar{c} & & \bar{b} & & & & & \\ & & & & a & & & & \\ & & & & & b & & c & \\ \hline & & c & & & & b & & \\ & & & & & \bar{c} & & \bar{b} & \\ & & & & & & & & a \end{array}\right| \tag{S12}$$

TABLE 6

	R	$\partial/\partial\eta \cdot \vert_{\eta=0}$
a	$\mathcal{I}(\operatorname{sh}(u-3\eta)-\operatorname{sh}5\eta+\operatorname{sh}3\eta+\operatorname{sh}\eta)$	$X(\operatorname{ch}u-1)$
b	$\mathcal{I}(\operatorname{sh}(u-3\eta)+\operatorname{sh}3\eta)$	$X(\operatorname{ch}u+1)$
c	1	0
d	$\mathcal{I}(\operatorname{sh}(u-\eta)+\operatorname{sh}\eta)$	$2X\operatorname{ch}u$
e	$-2\mathcal{I}e^{-\frac{1}{2}u}\operatorname{sh}2\eta\operatorname{ch}(\frac{1}{2}u-3\eta)$	$-X(1+e^{-u})$
$\bar{e}$	$-2\mathcal{I}e^{\frac{1}{2}u}\operatorname{sh}2\eta\operatorname{ch}(\frac{1}{2}u-3\eta)$	$-X(1+e^{u})$
f	$-\mathcal{I}(2e^{-u}e^{2\eta}\operatorname{sh}\eta\operatorname{sh}2\eta+e^{-\eta}\operatorname{sh}4\eta)$	$-2X$
$\bar{f}$	$\mathcal{I}(2e^{u}e^{-2\eta}\operatorname{sh}\eta\operatorname{sh}2\eta-e^{\eta}\operatorname{sh}4\eta)$	$-2X$
g	$\mathcal{I}(e^{-\frac{1}{2}u}e^{2\eta}2\operatorname{sh}\frac{1}{2}u\operatorname{sh}2\eta)$	$X(1-e^{-u})$
$\bar{g}$	$-\mathcal{I}(e^{\frac{1}{2}u}e^{-2\eta}2\operatorname{sh}\frac{1}{2}u\operatorname{sh}2\eta)$	$-X(1-e^{u})$

$$\mathcal{I}=(\operatorname{sh}(u-5\eta)+\operatorname{sh}\eta)^{-1}, \qquad X=2/\operatorname{sh}u .$$

TABLE 7

	a_1	a_2	a_3	b_1	b_2	b_3	c_1	c_2	c_3	g
I	$\frac{u}{u+\eta}$	$\frac{\eta}{u+\eta}$	0	$\frac{u}{u+\eta}$	0	$\frac{u\eta}{(u+\eta)(\eta g-u)}$	0	0	0	$-\frac{N}{2}$
II	$\frac{u}{u+\eta}$	$\frac{\eta}{u+\eta}$	0	$\frac{u}{u+\eta}$	0	$\frac{u\eta}{(u+\eta)(\eta g-u)}$	0	$\frac{\eta}{u+\eta}$	$\frac{u\eta}{(u+\eta)(\eta g-u)}$	$-(N-1)$
III	$\frac{u}{u+\eta}$	$\frac{\eta}{u+\eta}$	0	$\frac{-u}{u+\eta}$	0	$\frac{u\eta}{(u+\eta)(\eta g+u)}$	0	$\frac{\eta}{u+\eta}$	$\frac{u\eta}{(u+\eta)(\eta g+u)}$	$1+N$
IV	0	1	0	0	0	$\frac{-\operatorname{sh}(u+\mu)}{\operatorname{sh}u}$	0	1	$\frac{-\operatorname{sh}(u+\mu)}{\operatorname{sh}u}$	$g=\operatorname{ch}\mu=N$
V	0	e^{u}	0	0	0	$-e^{-u-\mu}\frac{\operatorname{sh}(u+\mu)}{\operatorname{sh}u}$	0	1	$\frac{-\operatorname{sh}(u+\mu)}{\operatorname{sh}u}$	$g=e^{\mu}=N$

TABLE 8

	R	$u=0$	$\eta=0$	$\partial/\partial\eta \cdot \vert_{\eta=0}$
a	1	1	1	0
b	$\mathcal{I}\operatorname{sh}u\operatorname{ch}(u-\eta)/\operatorname{ch}(u+\eta)$	0	1	$-4\operatorname{cth}2u$
c	$\mathcal{I}\operatorname{sh}u$	0	1	$-2\operatorname{cth}u$
d	$\mathcal{I}(\operatorname{sh}u-\operatorname{sh}2\eta\operatorname{ch}\eta/\operatorname{ch}(u+\eta))$	-1	1	$-2\frac{\operatorname{ch}2u+3}{\operatorname{sh}2u}$
r	$\mathcal{I}\operatorname{sh}2\eta\operatorname{ch}\eta/\operatorname{ch}(u+\eta)$	1	0	$\frac{4}{\operatorname{sh}2u}$
x	$\mathcal{I}\operatorname{sh}2\eta$	1	0	$\frac{2}{\operatorname{sh}u}$
y	$\mathcal{I}\operatorname{sh}u\operatorname{sh}2\eta/\operatorname{ch}(u+\eta)$	0	0	$\frac{2}{\operatorname{ch}u}$

$$\mathcal{I}=1/\operatorname{sh}(u+2\eta)$$

The sheaf (S12) is regular, and if $g(0)=1$, then also quasiclassical. In addition, the sheaf is invariant with respect to a similarity transformation (12) with matrix T.

$$T=\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} .$$

TABLE 9

	R	$u=0$	$\eta=0$	$\partial/\partial\eta\cdot\mid_{\eta=0}$
a	$\mathcal{J}(\operatorname{sh} u+\operatorname{sh}\eta/\operatorname{ch} u)$	1	1	$-\operatorname{cth} u+2/\operatorname{sh} u$
$\bar{a}$	$\mathcal{J}(\operatorname{sh} u-\operatorname{sh}\eta/\operatorname{ch} u)$	-1	1	$-\operatorname{cth} u-2/\operatorname{sh} u$
b	$\mathcal{J}\operatorname{sh} u$	0	1	$-\operatorname{cth} u$
c	$\mathcal{J}\operatorname{sh}\eta$	1	0	$1/\operatorname{sh} u$
d	$\mathcal{J}\operatorname{sh} u\operatorname{sh}\eta/\operatorname{ch} u$	0	0	$1/\operatorname{ch} u$

$$\mathcal{J}=1/\operatorname{sh}(u+\eta).$$

TABLE 10

	R	$u=0$	$\eta=0$	$\partial/\partial\eta\cdot\mid_{\eta=0}$
a	$\mathcal{J}\operatorname{sh}(u-\eta)$	-1	1	$-2\operatorname{cth} u$
b	$\mathcal{J}\operatorname{sh} u$	0	1	$-\operatorname{cth} u$
c	$\mathcal{J}\operatorname{sh}\eta$	1	0	$1/\operatorname{sh} u$

$$\mathcal{J}=1/\operatorname{sh}(u+\eta).$$

In view of the identity $T^3=1$, the corresponding group of transformations is isomorphic with $\mathbb{Z}_3$. For $g(\eta)=1$ the sheaf (S12) arises upon quantization of the doubles Toda chain [36] with the help of QMIP (Reshetikhin).

Analogously, one constructs N-dimensional $\mathbb{Z}_N$-invariant sheaves for any $N\geqslant 3$ [24]:

$$\begin{aligned}
&{}_{\alpha\beta}R_{\gamma\delta}=0 \quad \text{for } \alpha+\beta\neq\gamma+\delta \pmod N;\ \alpha,\beta,\gamma,\delta=0,1,\ldots,N-1,\\
&{}_{\alpha\alpha}R_{\alpha\alpha}=1,\\
&{}_{\alpha\beta}R_{\beta\alpha}=\exp\Big(2u\frac{\alpha-\beta}{N}-\operatorname{sign}(\alpha-\beta)\Big)\frac{\operatorname{sh}\eta}{\operatorname{sh}(u+\eta)},\\
&{}_{\alpha\beta}R_{\alpha\beta}=\exp\Big(2\eta\frac{\beta-\alpha}{N}-\operatorname{sign}(\beta-\alpha)\Big)\frac{\operatorname{sh} u}{\operatorname{sh}(u+\eta)}.
\end{aligned} \tag{S13}$$

In algebraic terms a system of roots of the given sheaf, as we noted, is the quantum R-matrix for the relativistic field-theoretic model corresponding to the system of roots A_{N-1} [36]. It is not hard to calculate the classical r-matrix for the remaining root systems $(B_N, C_N,\ldots,E_8)$, and it is interesting to find the corresponding quantum R-matrices.

8) Sheaf of Dimension 3 (Izergin and Korepin [5])

$$\mathbb{R}(u,\eta)=\left|\begin{array}{ccc|ccc|ccc}
a & & & & & \bar{g} & & g & \\
 & b & & \bar{e} & & & & & \\
 & & b & & & & e & & \\
\hline
 & e & & b & & & & & \\
 & & & & c & & & & \\
g & & & & & d & & f & \\
\hline
 & & \bar{e} & & & & b & & \\
 & & & & & \bar{f} & & d & \\
\bar{g} & & & & & & & & c
\end{array}\right|. \tag{S14}$$

The sheaf (S14) is canonical — it is the $\mathbb{R}$-matrix for the quantum relativistic model of Mikhailov—Shabat [5].

9) Block $O(N)$-Invariant Sheaves [35]

$$\mathbb{R}(u) = \begin{vmatrix} A & & & \\ & B & C & \\ & C & B & \\ & & & A \end{vmatrix}, \tag{S15}$$

A, B, C are $N^2 \times N^2$ blocks of the form $A = a_1 I + a_2 P + a_3 K$ and analogously for B and C, where K is a projector (S4).

These sheaves were obtained in [35], as examples of factorized $SU(N)$-invariant S-matrices. This does not contradict our definition of them as $O(N)$-invariant, since the action of a group of transformations on an S-matrix is defined differently than we did it in (13).

The sheaf II is canonical, I is quasiclassical, III is regular.

Now we give some examples of Z_2-graded Yang—Baxter sheaves, satisfying (48).

10) $GL(n,m,C)$-Invariant Sheaf

$GL(n,m,C)$ is the analog of the group $GL(N,C)$ [25] in the graded space C^{n+m} with graduation (n,m). The sheaf has the form

$$\mathbb{R}(u) = \frac{u}{u+\eta} I + \frac{\eta}{u+\eta} P_{n,m}, \tag{S16}$$

where $P_{n,m}$ is the graded permutation operator (51). The sheaf given is a natural generalization to the graded case of the sheaf (S1). It is used in applying QMIP to equations containing Fermi fields [26, 27, 30].

11) Sheaf, Connected with the Tirring Massive Model for Fermi Fields

The auxiliary linear problem for the Tirring model [26] is defined by a (3 × 3) matrix differential operator of the first order. The graduation is equal to (2, 1). In the basis where $p(1) = p(2) = 0$, $p(3) = 1$, the $\mathbb{R}$-matrix, interlacing the monodromy operators for the auxiliary linear problem (37), has the form:

$$\mathbb{R}(u,\eta) = \left|\begin{array}{ccc|ccc|ccc} a & & & & & & & & y \\ & b & & r & & & & & \\ & & c & & & & x & & \\ \hline & r & & b & & & & & y \\ & & & & a & & & & \\ & & & & & c & & x & \\ \hline & & x & & & & c & & \\ & & & & & x & & c & \\ & y & & y & & & & & d \end{array}\right|. \tag{S17}$$

12) Graded Analog of XYZ-Sheaf (but in Hyperbolic Functions)

$$\mathbb{R}(u,\eta) = \begin{vmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & \bar{a} \end{vmatrix}. \tag{S18}$$

This sheaf arises as part of the interlacing $\mathbb{R}$-matrix for the quantum supersymmetric sin-Gordon equation [27], corresponding to a majorizing field.

13) Graded Analog of the XXZ-Sheaf

$$\mathbb{R}(u,\eta)=\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{vmatrix}. \tag{S19}$$

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