# QUANTUM VERSION OF THE METHOD OF INVERSE SCATTERING PROBLEM

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UDC 517.43+530.145

This paper is a developed and consecutive account of a quantum version of the method of the inverse scattering problem on the example of the nonlinear Schrödinger equation. The method of R-matrices developed by the author is given basic consideration. The generating functions of quantum integrals of motion and action-angle variables for the quantum nonlinear Schrödinger equation are obtained. There is also described a classical version of the method of R-matrices.

## Introduction

1. The last decade was marked by a sharp increase in interest in completely integrable systems of classical and quantum mechanics. Although exactly solvable problems are always of interest for physics and mathematics and models for the study of general laws of behavior of complicated nonlinear systems or as sources of "zeroth" approximations to nonintegrable equations, only comparatively recently were powerful methods for their study available, which allowed the essential extension of the class of completely integrable systems. We are concerned here, on the one hand, with the method of the inverse scattering problem [1], which has its origin in Gardner, Greene, Kruskal, and Miura [2], Lax [3], Zakharov and Faddeev [4] and allows one to investigate completely integrable models of classical mechanics. On the other hand, there exists a tradition of studying exactly solvable models of quantum mechanics and statistical physics, going back to Bethe [5] and Onsager [6] and achieving its highest development in Baxter [7-8]. (We intentionally schematize the situation here, leaving aside for example, the group-theoretic methods of investigation of classical and quantum completely integrable models.)

After the complete integrability of certain relativistically invariant models was proved: by the method of the inverse scattering problem sin-Gordon equations [9], chiral fields [10], etc. there arose the question, doesn't the corresponding quantum models also turn out to be completely integrable. A positive solution of this question would be of great interest for the quantum theory of fields, since it would give a nontrivial example of an exactly solvable quantum model. This circumstance gave rise to a series of attempts at the quasiclassical quantization of completely integrable models, e.g., [11, 12], which, however, were not completely satisfactory, since one could give only an approximate and not an exact answer.

Thus, there remains the actual problem of synthesis of the two approaches indicated above, the classical and the quantum, into one method of investigation of quantum fields of completely integrable systems, which one could call the "quantum method of the inverse scattering problem."

The first steps in this direction were undertaken in [13] by Faddeev and the author and in [14]. In [13] there was formulated a program for generalizing the method of the inverse scattering problem to the quantum case. In [14] this program was successfully realized for the nonlinear Schrödinger equation. However, the small size of [14] did not allow the inclusion in it of all the necessary proofs. The present paper, written on the basis of the author's dissertation [15], contains a developed and consecutive account of the quantum method of the inverse scattering problem for the nonlinear Schrödinger equation taking account of the achievements of the 2-yr period of development of this method [16-28]. In addition, we have included here a brief outline of the theory of the classical nonlinear Schrödinger equation, based on the method proposed by the author in [29] of the classical $\mathfrak{r}$-matrix.

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR, Vol. 95, pp. 55-128, 1980.

 0090-4104/82/1905-1546$07.50 

The author profoundly thanks Academician L. D. Faddeev for formulating the problem and constant interest in the work and also all the collaborators in the laboratory of mathematical problems of physics of the Leningrad Branch of the Mathematics Institute (LOMI) for many fruitful discussions.

2. We consider in more detail the basic lines of the approach to the quantum generalized method of the inverse scattering problem, proposed in [13, 14] and developed in the present paper. First of all we compare the formulation of the problem in classical and quantum mechanics. If in classical mechanics one is usually interested in the evolution of the initial data in time, and the classical method of the inverse scattering problem is usually aimed at finding the law of evolution in time of the scattering data of an auxiliary linear problem, then for quantum mechanics the "stationary" approach is more characteristic, in which the greatest interest is in the spectrum of the Hamiltonian and S-matrix. However, despite such an apparent dissimilarity in the formulation of the problem, there nevertheless exists an approach to the classical method of the inverse scattering problem, completely analogous to the "stationary" quantum mechanical approach. We are concerned here with the Hamiltonian interpretation of the method of the inverse scattering problem, developed in [4, 30]. The study of the nonlinear evolution equation with such an approach is aimed at constructing from the scattering data of the auxiliary linear problem variables of action-angle type for the equation considered and thus proving its complete integrability, determining in passing the spectrum of the elementary excitations of the system. Now the question of the time of evolution is from this point of view of secondary interest. Precisely this Hamiltonian approach was taken as the starting point for the quantum mechanical generalization of the method of the inverse scattering problem in [13, 14] and in the present paper.

The choice of the nonlinear Schrödinger equation (n.S.e.)

$$i\Psi_t = -\Psi_{xx} + 2\varkappa|\Psi|^2\Psi \tag{1}$$

as the object of study was dictated by the following considerations:

1) The quantum version of the nonlinear Schrödinger equation (with zero boundary conditions at infinity) describes a one-dimensional system of Bose particles with pointwise interaction. Thus, the problem reduces to the quantum mechanics of a finite number of particles and does not contain difficulties specific for the quantum field theory (nonfocal representations of commuting relations, divergence, etc.).

2) The nonlinear Schrödinger equation is studied in detail in the classical as well as the quantum case. The classical n.S.e. admits the application of the method of the inverse scattering problem [30-32], in the quantum case there is a complete description of the spectrum and eigenfunctions of the Hamiltonian [33, 34]. The latter circumstance is valuable in that it allows one to compare the results obtained by the new method with the exact quantum answer.

Now one can give a concrete formulation of the problem. An important role in the method of the inverse scattering problem is played by the transition matrix $T(\lambda)$ (see below Sec. 1.1)

$$T(\lambda) = \begin{pmatrix} a(\lambda), & \varkappa\bar{b}(\lambda) \\ b(\lambda), & \bar{a}(\lambda) \end{pmatrix}, \quad \lambda = \bar{\lambda}, \tag{2}$$

which is defined by means of the auxiliary linear problem, and whose matrix elements are functionals of the fields $\Psi(x)$, $\Psi(x)$.

As is known [30], (1) describes the dynamics of a Hamiltonian system with Hamiltonian

$$H = \int_{-\infty}^{\infty} dx\left(|\Psi_x|^2 + \varkappa|\Psi|^4\right)$$

and Poisson bracket, defined by the relations

$$\{\Psi(x), \Psi(y)\} = \{\overline{\Psi}(x), \overline{\Psi}(y)\} = 0 ,$$
$$\{\Psi(x), \overline{\Psi}(y)\} = i\delta(x-y) .$$

The method of the inverse scattering problem allows one to calculate the Poisson brackets between the matrix elements of the matrix $T(\lambda)$. In particular:

$$\{a(\lambda), a(\mu)\} = \{\bar{b}(\lambda), \bar{b}(\mu)\} = 0 , \tag{3}$$

$$\{a(\lambda), \bar{b}(\mu)\} = -\frac{\varkappa}{\lambda-\mu+i0}\, a(\lambda)\, \bar{b}(\mu) . \tag{4}$$

In turns out [31, 32], that $\ln a(\lambda)$ is the generating function of the local integrals of motion of (1), and from the quantities $b(\lambda)$ and $\bar{b}(\lambda)$ one can construct variables of action-angle type.

In [13] the problem of generalizing the method of the inverse problem to the quantum case was formulated as the problem of constructing quantum operators $A(\lambda)$ and $B^+(\lambda)$, which would have the following properties:

1) In the classical limit the operators $A(\lambda)$ and $B^+(\lambda)$ should go respectively into the quantities $a(\lambda)$ and $b(\lambda)$.

2) Between the operators $A(\lambda)$ and $B^+(\mu)$ there should be the following commutation relations:

$$[A(\lambda), A(\mu)] = [B^+(\lambda), B^+(\mu)] = 0 , \tag{5}$$

$$A(\lambda)\, B^+(\mu) = c(\lambda,\mu)\, B^+(\mu)\, A(\lambda) , \tag{6}$$

where $c(\lambda,\mu)$ is some numerical function of $\lambda$ and $\mu$.

We note that such a formulation makes sense not only for the n.S.e., but also for many other completely integrable equations [16].

The quantization of (1) is carried out in terms of the annihilation and birth operators $\Psi(x)$ and $\Psi^+(x)$, satisfying the canonical commutation relations

$$\begin{gathered}[\Psi(x), \Psi^+(y)] = [\Psi^+(x), \Psi^+(y)] = 0 , \\ [\Psi(x), \Psi^+(y)] = \delta(x-y) .\end{gathered} \tag{7}$$

Here the problem of constructing the operators $A(\lambda)$ and $B^+(\lambda)$ reduces, essentially, to the question of choosing a proper ordering of the operators $\Psi$ and $\Psi^+$, i.e., an ordering such that condition 2), formulated above, should be satisfied. As will be proved later, for the n.S.e. the normal (Wick) ordering is proper, i.e., the operators $A(\lambda)$ and $B^+(\lambda)$ are defined as the operators whose normal symbols are respectively the classical functionals $a(\lambda; \Psi, \overline{\Psi})$ and $\bar{b}(\lambda; \Psi, \overline{\Psi})$.

The basic technical difficulty of the approach considered consists in proving the commutation relations (5, 6) and calculating the coefficients $c(\lambda,\mu)$. This problem can be solved by the method of $R$-matrices proposed by the author in [14], which makes it possible to write down the commutation relations between the matrix elements of the quantum transition matrix in compact matrix form and to reduce their proof to the verification of simple infinitesimal relations. The idea of this method was suggested to the author by the papers of Baxter [7, 8].

Calculation of the coefficient $c(\lambda,\mu)$ by the method of $R$-matrices gives the following result:

$$c(\lambda,\mu)=\frac{\lambda-\mu+i\varkappa}{\lambda-\mu} \; . \tag{8}$$

We list the main results which will be proved in the basic text as consequences of the commutation relations (5, 6):

1) The operator-valued function $\ln A(\lambda)$, as in the classical case, is the generating function of locally pairwise commuting integrals of motion $\mathfrak{I}_m$ for the quantum n.S.e.

2) States $|k_1,\ldots,k_N\rangle_B$, obtained by the action in a vacuum of the operators $B^+(k_j)$ $(j=1,\ldots,N)$

$$|k_1,\ldots,k_N\rangle_B=B^+(k_1)\ldots B^+(k_N)|0\rangle \; , \tag{9}$$

are eigenvectors of the quantum Hamiltonian $\mathbb{H}$ and all integrals of motion $\mathfrak{I}_m$, where the corresponding eigenvalues are additive in the momenta $k_j$:

$$\mathbb{H}\,|k_1,\ldots,k_N\rangle_B=\sum_{j=1}^{N}k_j^2\,|k_1,\ldots,k_N\rangle_B \; . \tag{10}$$

The wave functions of the states $k_1,\ldots,k_N\rangle_B$ coincide with the wave functions obtained by the Bethe substitution method [33, 34].

3) The operators $\Phi(\lambda),\Phi^+(\lambda)$, defined by the formulas

$$\Phi^+(\lambda)=B^+(\lambda)\left(2\pi A^+(\lambda)A(\lambda)\right)^{-1/2},\ \Phi(\lambda)=\Phi^+(\lambda)^+, \tag{11}$$

satisfy the canonical commutation relations

$$\begin{gathered}\left[\Phi(\lambda),\Phi(\mu)\right]=\left[\Phi^+(\lambda),\Phi^+(\mu)\right]=0 \; ,\\ \left[\Phi(\lambda),\Phi^+(\mu)\right]=\delta(\lambda-\mu) \; .\end{gathered} \tag{12}$$

and allow (in the case $\varkappa>0$) the explicit diagonalization of the Hamiltonian

$$\mathbb{H}=\int_{-\infty}^{\infty}d\mu\,\mu^2\,\Phi^+(\mu)\,\Phi(\mu) \tag{13}$$

and all other integrals of motion $\mathfrak{I}_m$. On this basis the operators $\Phi(\lambda),\Phi^+(\lambda)$ can be called the quantum analogs of action-angle variables.

3. The basic text of the paper consists of two chapters, a Conclusion and a Supplement. In Chap. I we consider the classical nonlinear Schrödinger equation, in Chap. II the quantum one. Here the consideration of the classical case is carried out so that all the results obtained have direct analogs in the quantum case.

The composition of the paper is also subordinate to this idea: to each section of Chap. I corresponds an analogous section of Chap. II.

Chapter I consists of five sections. In Sec. 1.1 the basic concepts and notation are introduced. In Sec. 1.2 the Poisson brackets between matricial elements of the transition matrix $T_{x_1}^{x_2}(\lambda)$ on a finite interval are computed. In Sec. 1.3 the cases of semi-infinite and infinite intervals are considered. In Sec. 1.4, which has an auxiliary character, known results about the integrals of motion for the n.S.e. are collected and action-angle variables are constructed. In Sec. 1.5, with the help of the method of $R$-matrices, the generating functions of the $M$-operators for the n.S.e. are constructed.

Chapter II also consists of five sections. In Sec. 2.1 the known results for the quantum n.S.e. are listed, the quantum transition matrix $\mathbb{T}_{x_1}^{x_2}(\lambda)$ is introduced. In Sec. 2.2 the commutation relations between the matricial elements of the quantum transition matrix $\mathbb{T}_{x_1}^{x_2}(\lambda)$ are calculated. In Sec. 2.3 the cases of semi-infinite and infinite intervals are considered. In Sec. 2.4, the results obtained are summarized, the question of construction of quantum action-angle variables is considered. In Sec. 2.5 the question of the quantum M-operator is studied.

In the Conclusion the basic derivations and results of the paper are summarized, a brief survey is given of unsolved problems in the domain of quantum completely integrable systems, the future developments in this direction are discussed.

In the Supplement a summary is given of the classical and quantum commutation relations between the matricial elements of the transition matrix for finite, semi-infinite, and infinite intervals.

# CHAPTER I
# CLASSICAL NONLINEAR SCHRÖDINGER EQUATIONS

## 1.1. Transition Matrix

In the present section we introduce the basic notation and list some results, basically known in [30-32], for the classical nonlinear Schrödinger equation. We allow ourselves to deviate somewhat from the notation and formulations of the original papers [30-32], giving them a form more convenient for our goals.

The nonlinear Schrödinger equation, as was indicated in the Introduction, has the form

$$i\Psi_t = -\Psi_{xx} + 2\varkappa|\Psi|^2\Psi . \tag{1.1.1}$$

The complex-valued function $\Psi(x,t)$ will be assumed to be infinitely differentiable in both arguments and for any $t$, decreasing in $x$ faster than any power of $x$.

The study of (1.1.1) by the method of the inverse scattering problem reduces, as was shown in [31, 32], to the study of the spectral characteristics of the sheaf of linear differential operators $\frac{d}{dx} - L(x,\lambda)$ , where

$$L(x,\lambda) = \begin{pmatrix} -i\frac{\lambda}{2} & , i\varkappa\overline{\Psi}(x) \\ -i\Psi(x) , & i\frac{\lambda}{2} \end{pmatrix} = -i\frac{\lambda}{2}\sigma_3 + i\varkappa\overline{\Psi}(x)\sigma_+ - i\Psi(x)\sigma_- . \tag{1.1.2}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Starting here and up to Sec. 1.5 we shall consider the moment of time t fixed.

In contrast with [30-32], we have chosen the matrix L with nonsymmetric occurrence of the connection constant $\varkappa$, which allows us to consider in a uniform way both the case of repulsion $\varkappa > 0$ , and that of attraction $\varkappa < 0$ .

We introduce the transition matrix $T_{x_1}^{x_2}(\lambda)$ on the finite interval $[x_1, x_2]$ as the solution of the differential equation

$$\frac{\partial}{\partial x_2} T_{x_1}^{x_2}(\lambda) = L(x_2,\lambda)\, T_{x_1}^{x_2}(\lambda) \tag{1.1.3}$$

with the initial condition

$$T_x^x(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I. \tag{1.1.4}$$

We list some properties of the matrix $T_{x_1}^{x_2}(\lambda)$ :

1) $$\left(T_{x_1}^{x_2}(\lambda)\right)^{-1} = T_{x_2}^{x_1}(\lambda), \tag{1.1.5}$$

2) $$T_{x_2}^{x_3}(\lambda)\, T_{x_1}^{x_2}(\lambda) = T_{x_1}^{x_3}(\lambda) , \tag{1.1.6}$$

3) $$\frac{\partial}{\partial x_1} T_{x_1}^{x_2}(\lambda) = -T_{x_1}^{x_2}(\lambda)\, L(x_1,\lambda), \tag{1.1.7}$$

4) $$\overline{T_{x_1}^{x_2}(\lambda)} = K\, T_{x_1}^{x_2}(\bar{\lambda})\, K , \tag{1.1.8}$$

where

$$K = \begin{pmatrix} 0 & , \varkappa^{1/2} \\ \varkappa^{-1/2} , & 0 \end{pmatrix}, \quad K^2 = I ,$$

and the line over a matrix denotes elementwise complex conjugation.

5) $$\det T_{x_1}^{x_2}(\lambda) = 1 . \tag{1.1.9}$$

Properties 1)-3) follow directly from the definition of $T_{x_1}^{x_2}(\lambda)$. The symmetry property 4) follows from the analogous property of the L-operator

$$\overline{L(x,\lambda)} = K\, L(x,\bar{\lambda})\, K , \tag{1.1.10}$$

which can be verified directly. We note that (1.1.8) means that the matrix $T_{x_1}^{x_2}(\lambda)$ has the form

$$T_{x_1}^{x_2}(\lambda) = \begin{pmatrix} a_{x_1}^{x_2}(\lambda), & \varkappa \overline{b_{x_1}^{x_2}(\bar{\lambda})} \\ b_{x_1}^{x_2}(\lambda), & \overline{a_{x_1}^{x_2}(\bar{\lambda})} \end{pmatrix} \qquad (1.1.11)$$

Finally, property 5) follows from the equation $\operatorname{tr} L(x,\lambda) = 0$ .

Substituting (1.1.11) in (1.1.9) we get for real $\lambda$ the important relation of "unitariness"

$$\left|a_{x_1}^{x_2}(\lambda)\right|^2 - \varkappa \left|b_{x_1}^{x_2}(\lambda)\right|^2 = 1, \quad \lambda = \bar{\lambda}. \qquad (1.1.12)$$

Now we define the transition matrices $T_-(x,\lambda), T_+(x,\lambda), T(\lambda)$ for semi-infinite intervals $(-\infty, x]$ and $[x,\infty)$ and the infinite interval $(-\infty,\infty)$ , respectively, as the following limits:

$$T_-(x,\lambda) = \lim_{x_1 \to -\infty} T_{x_1}^{x}(\lambda)\, e(x_1,\lambda), \qquad (1.1.13)$$

$$T_+(x,\lambda) = \lim_{x_2 \to \infty} e(-x_2,\lambda)\, T_{x}^{x_2}(\lambda), \qquad (1.1.14)$$

$$T(\lambda) = \lim_{\substack{x_1 \to -\infty \\ x_2 \to +\infty}} e(-x_2,\lambda)\, T_{x_1}^{x_2}(\lambda)\, e(x_1,\lambda), \qquad (1.1.15)$$

where we have introduced the notation

$$e(x,\lambda) = \exp\left(-i \frac{\lambda}{2} \sigma_3 x\right) .$$

It follows from (1.1.13) that $T_-(x,\lambda)$ satisfies with respect to the variables $x$ the differential equation (1.1.3) with the boundary condition as $x \to -\infty$

$$T_-(x,\lambda) - e(x,\lambda) \xrightarrow[x \to -\infty]{} 0. \qquad (1.1.16)$$

Analogously, $T_+(x,\lambda)$ satisfies with respect to $x$ the differential equation (1.1.7) with the boundary condition as $x \to +\infty$

$$T_+(x,\lambda) - e(-x,\lambda) \xrightarrow[x \to +\infty]{} 0. \qquad (1.1.17)$$

It follows from (1.1.6) that for any $x$

$$T(\lambda) = T_+(x,\lambda)\, T_-(x,\lambda) . \qquad (1.1.18)$$

Analogously, to the case of a finite interval, for $T_\pm(x,\lambda)$ and $T(\lambda)$ one proves the symmetry property (1.1.8) and the "unitariness" property (1.1.12).

We list analytic properties of the matricial functions $T_{x_1}^{x_2}(\lambda)$, $T_\pm(x,\lambda)$, and $T(\lambda)$ with respect to the spectral parameter $\lambda$. $T_{x_1}^{x_2}(\lambda)$ is a holomorphic function on the entire complex plane $\lambda$. The matrix elements $a_-(x,\lambda), b_-(x,\lambda), a_+(x,\lambda), \overline{b_+(x,\bar{\lambda})}, a(\lambda)$ can be analytically extended to the half-plane $\operatorname{Im}\lambda > 0$ , then as matrix elements $\overline{a_-(x,\bar{\lambda})}, \overline{b_-(x,\bar{\lambda})}, b_+(x,\lambda), \overline{a_+(x,\bar{\lambda})}, \overline{a(\bar{\lambda})}$ can be analytically extended to the half-plane $\operatorname{Im}\lambda < 0$. The matrix elements $b(\lambda)$ and $\overline{b(\bar{\lambda})}$, in general, are defined only for real $\lambda$. (The notation for the matrix elements of the matrices $T_\pm(x,\lambda)$ and $T(\lambda)$ is given in the Supplement.)

The proof of the analytic properties listed above and also of the existence of the limits (1.1.13-17) is carried out in the standard way [30-32] using integral equations and integral representations for $T_{x_1}^{x_2}(\lambda)$, $T_{\pm}(x,\lambda)$, and $T(\lambda)$, and we now start in on its consideration.

The Cauchy problems (1.1.3-4) and (1.1.7)-(1.1.4) are equivalent respectively with the Volterra integral equations

$$T_{x_1}^{x_2}(\lambda) = I + \int_{x_1}^{x_2} dx\, L(x,\lambda) T_{x_1}^{x}(\lambda) \tag{1.1.19}$$

and

$$T_{x_1}^{x_2}(\lambda) = I + \int_{x_1}^{x_2} dx\, T_{x}^{x_2}(\lambda) L(x,\lambda) \ . \tag{1.1.20}$$

Extracting from $L(x,\lambda)$ the potential $V(x) = i\varkappa\overline{\Psi}(x)\sigma_+ - i\Psi(x)\sigma_-$, we get the following integral equations for $T_{x_1}^{x_2}(\lambda)$ :

$$T_{x_1}^{x_2}(\lambda) = e(x_2 - x_1,\lambda) + \int_{x_1}^{x_2} dx\, e(x_2 - x,\lambda) V(x) T_{x_1}^{x}(\lambda) \tag{1.1.21}$$

or

$$T_{x_1}^{x_2}(\lambda) = e(x_2 - x_1,\lambda) + \int_{x_1}^{x_2} dx\, T_{x}^{x_2}(\lambda) V(x) e(x - x_1,\lambda) \ . \tag{1.1.22}$$

From (1.1.13), (1.1.21), and (1.1.22) follow the integral equation

$$T_-(x,\lambda) = e(x,\lambda) + \int_{-\infty}^{x} d\xi\, e(x-\xi,\lambda) V(\xi) T_-(\xi,\lambda) \tag{1.1.23}$$

and the integral representation

$$T_-(x,\lambda) = e(x,\lambda) + \int_{-\infty}^{x} d\xi\, T_{\xi}^{x}(\lambda) V(\xi) e(\xi,\lambda) \tag{1.1.24}$$

for $T_-(x,\lambda)$. Analogously, from (1.1.14), (1.1.21), and (1.1.22) one derives the integral equation

$$T_+(x,\lambda) = e(-x,\lambda) + \int_{x}^{\infty} d\xi\, T_+(\xi,\lambda) V(\xi) e(\xi - x,\lambda) \tag{1.1.25}$$

and the integral representation

$$T_+(x,\lambda) = e(-x,\lambda) + \int_{x}^{\infty} d\xi\, e(-\xi,\lambda) V(\xi) T_{x}^{\xi}(\lambda) \tag{1.1.26}$$

for $T_+(x,\lambda)$. For $T(\lambda)$, from (1.1.15), (1.1.21), (1.1.22), we get the two integral representations:

$$T(\lambda) = I + \int_{-\infty}^{\infty} dx\, e(-x,\lambda) V(x) T_-(x,\lambda) \tag{1.1.27}$$

and

$$T(\lambda) = I + \int_{-\infty}^{\infty} dx\, T_+(x,\lambda) V(x) e(x,\lambda) \ . \tag{1.1.28}$$

Iterating the integral equation (1.1.21) or (1.1.22), we get the following expansions of the matricial elements of $T_{x_1}^{x_2}(\lambda)$ in power series in $\varkappa$:

$$a_{x_1}^{x_2}(\lambda)=e^{-i\frac{\lambda}{2}(x_2-x_1)}\left[1+\sum_{n=1}^{\infty}\varkappa^n\int\limits_{x_2>\xi_n>\eta_n>\xi_{n-1}>\ldots>\eta_1>x_1}\!\!\ldots\int d\xi_1\ldots d\xi_n\,d\eta_1\ldots d\eta_n\,e^{i\lambda(\xi_1+\ldots+\xi_n-\eta_1-\ldots-\eta_n)}\,\overline{\Psi}(\xi_1)\ldots\overline{\Psi}(\xi_n)\Psi(\eta_1)\ldots\Psi(\eta_n)\right], \tag{1.1.29}$$

$$b_{x_1}^{x_2}(\lambda)=-ie^{i\frac{\lambda}{2}(x_1+x_2)}\left[\sum_{n=0}^{\infty}\varkappa^n\int\limits_{x_2>\eta_{n+1}>\xi_n>\ldots>\eta_1>x_1}\!\!\ldots\int d\xi_1\ldots d\xi_n\,d\eta_1\ldots d\eta_{n+1}\times\right.$$

$$\left.\times e^{i\lambda(\xi_1+\ldots+\xi_n-\eta_1-\ldots-\eta_{n+1})}\,\overline{\Psi}(\xi_1)\ldots\overline{\Psi}(\xi_n)\Psi(\eta_1)\ldots\Psi(\eta_{n+1})\right]. \tag{1.1.30}$$

Expansions for $\overline{a_{x_1}^{x_2}(\bar{\lambda})}$ and $\overline{b_{x_1}^{x_2}(\bar{\lambda})}$ are obtained by complex conjugation. Analogous expansions for the matricial elements of $T_-(x,\lambda)$, $T_+(x,\lambda)$, and $T(\lambda)$ are obtained from (1.1.29) and (1.1.30) by cancellation, respectively, of $x_1, x_2$, or $x_1$ and $x_2$.

To conclude this section, some words on the discrete spectrum. As shown in [31], for $\varkappa>0$ the function $a(\lambda)$ has no zeros in $\mathrm{Im}\lambda>0$, but for $\varkappa<0$ can have in the upper half-plane a finite number of zeros:

$$a(\lambda_j)=0\,,\ \mathrm{Im}\lambda_j>0\,,\ j=1,\ldots,\mathcal{N}\,. \tag{1.1.31}$$

This property of the coefficient $a(\lambda)$ is closely connected with the existence for $\varkappa<0$ of soliton solutions of (1.1.1), and in the quantum case, as we shall see later, the connected states of the basic particles.

## 1.2. Poisson Brackets. $r$-Matrix

As is known [30], (1.1.1) describes the dynamics of a Hamiltonian system with Hamiltonian

$$H=\int_{-\infty}^{\infty}dx\left(|\Psi_x|^2+\varkappa|\Psi|^4\right) \tag{1.2.1}$$

and Poisson brackets

$$\begin{gathered}\{\Psi(x),\Psi(y)\}=\{\overline{\Psi}(x),\overline{\Psi}(y)\}=0\,,\\ \{\Psi(x),\overline{\Psi}(y)\}=i\delta(x-y)\,.\end{gathered} \tag{1.2.2}$$

In other words, (1.1.1) can be represented in the following form:

$$\Psi_t=\{H,\Psi\}\,. \tag{1.2.3}$$

In [30] from the matrix elements of the matrix $T(\lambda)$ (1.1.15) there were constructed action-angle variables for (1.1.1). The basis for this construction includes the calculation of the Poisson brackets between the matrix elements of $T(\lambda)$ as functionals of the fields $\Psi(x)$ and $\overline{\Psi}(x)$. Below we shall calculate these Poisson brackets by a new method, proposed by the author in [29]. This method is based on using the so-called $r$-matrix and has the advantage over the traditional methods [4, 30], that it admits direct generalization to the quantum case, and also allows one to appreciably simplify calculations.

In what follows it will be convenient for us to use the following notation. With each $(2 \times 2)$ matrix

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$

we associate two $(4 \times 4)$ matrices $\tilde{T}$ and $\tilde{\tilde{T}}$:

$$\tilde{T} = T \otimes I_2 = \begin{pmatrix} t_{11} & 0 & t_{12} & 0 \\ 0 & t_{11} & 0 & t_{12} \\ t_{21} & 0 & t_{22} & 0 \\ 0 & t_{21} & 0 & t_{22} \end{pmatrix}, \tag{1.2.4}$$

and

$$\tilde{\tilde{T}} = I_2 \otimes T = \begin{pmatrix} t_{11} & t_{12} & 0 & 0 \\ t_{21} & t_{22} & 0 & 0 \\ 0 & 0 & t_{11} & t_{12} \\ 0 & 0 & t_{21} & t_{22} \end{pmatrix}. \tag{1.2.5}$$

Thus, the matrix

$$\{\tilde{T}(\lambda), \tilde{\tilde{T}}(\mu)\} = \begin{pmatrix} \{a(\lambda), a(\mu)\}, \{a(\lambda), \varkappa\overline{b(\bar{\mu})}\}, \{\varkappa\overline{b(\bar{\lambda})}, a(\mu)\}, \{\varkappa\overline{b(\bar{\lambda})}, \varkappa\overline{b(\bar{\mu})}\} \\ \{a(\lambda), b(\mu)\}, \{a(\lambda), \overline{a(\bar{\mu})}\}, \{\varkappa\overline{b(\bar{\lambda})}, b(\mu)\}, \{\varkappa\overline{b(\bar{\lambda})}, \overline{a(\bar{\mu})}\} \\ \{b(\lambda), a(\mu)\}, \{b(\lambda), \varkappa\overline{b(\bar{\mu})}\}, \{\overline{a(\bar{\lambda})}, a(\mu)\}, \{\overline{a(\bar{\lambda})}, \varkappa\overline{b(\bar{\mu})}\} \\ \{b(\lambda), b(\mu)\}, \{b(\lambda), \overline{a(\bar{\mu})}\}, \{\overline{a(\bar{\lambda})}, b(\mu)\}, \{\overline{a(\bar{\lambda})}, \overline{a(\bar{\mu})}\} \end{pmatrix} \tag{1.2.6}$$

will contain all 16 possible Poisson brackets between the matrix elements of the matrices $T(\lambda)$ and $T(\mu)$. We note that the matrices $T(\lambda)$ and $T(\mu)$ commute with one another

$$\tilde{T}\tilde{\tilde{T}} = \tilde{\tilde{T}}\tilde{T}. \tag{1.2.7}$$

Later we shall also need the permutation matrix $\mathcal{P}$

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{P}^2 = I_4 \tag{1.2.8}$$

and the following easily verifiable relations

$$\mathcal{P}\tilde{T}\mathcal{P} = \tilde{\tilde{T}}, \quad \mathcal{P}\tilde{\tilde{T}}\mathcal{P} = \tilde{T}, \tag{1.2.9}$$

$$\mathcal{P}\tilde{T}\tilde{\tilde{T}}\mathcal{P} = \tilde{T}\tilde{\tilde{T}}. \tag{1.2.10}$$

The fundamental result of the present section is the following theorem.

THEOREM 1. The matrix of Poisson brackets $\{\tilde{T}_{x_1}^{x_2}(\lambda), \tilde{\tilde{T}}_{x_1}^{x_2}(\mu)\} = \mathcal{P}_{x_1}^{x_2}(\lambda,\mu)$ admits the following representation for $x_2 > x_1$

$$\mathcal{P}_{x_1}^{x_2}(\lambda,\mu) = \left[ r(\lambda-\mu), \tilde{T}_{x_1}^{x_2}(\lambda)\, \tilde{\tilde{T}}_{x_1}^{x_2}(\mu) \right], \tag{1.2.11}$$

where $[\ ,\ ]$ denotes the commutator of matrices, and the $4 \times 4$ matrix $r(\lambda-\mu)$ has the form

$$\tau(\lambda-\mu) = -\frac{\varkappa}{\lambda-\mu}\mathcal{P} . \tag{1.2.12}$$

For $x_2 < x_1$, as follows from (1.1.5), (1.2.11) goes into

$$\mathcal{P}_{x_1}^{x_2}(\lambda,\mu) = -\left[\tau(\lambda-\mu), \widetilde{T}_{x_1}^{x_2}(\lambda)\,\widetilde{\widetilde{T}}_{x_1}^{x_2}(\mu)\right].$$

Proof. For the proof it suffices to see that the right and left sides of (1.2.11) satisfy the same differential equation with the same initial conditions.

Differentiating the right side of (1.2.11) with respect to $x_2$ we get for $\mathcal{P}_{x_1}^{x_2}(\lambda,\mu)$ the following differential equation:

$$\frac{\partial}{\partial x_2}\mathcal{P}_{x_1}^{x_2}(\lambda,\mu) = \left[\tau(\lambda-\mu), \widetilde{L}(x_2,\lambda) + \widetilde{\widetilde{L}}(x_2,\mu)\right]\widetilde{T}_{x_1}^{x_2}(\lambda)\,\widetilde{\widetilde{T}}_{x_1}^{x_2}(\mu) + \left(\widetilde{L}(x_2,\lambda) + \widetilde{\widetilde{L}}(x_2,\mu)\right)\mathcal{P}_{x_1}^{x_2}(\lambda,\mu) \tag{1.2.13}$$

with the initial condition

$$\mathcal{P}_x^x(\lambda,\mu) = 0 . \tag{1.2.14}$$

In order to calculate the derivative with respect to $x_2$ of the left side of (1.2.11), we represent it in the form:

$$\frac{\partial}{\partial x_2}\left\{\widetilde{T}_{x_1}^{x_2}(\lambda), \widetilde{\widetilde{T}}_{x_1}^{x_2}(\mu)\right\} = \lim_{\delta\to 0}\left(\mathcal{P}_{x_1}^{x_2+\delta}(\lambda,\mu) - \mathcal{P}_{x_1}^{x_2}(\lambda,\mu)\right) =$$

$$= \lim_{\delta\to 0}\frac{1}{\delta}\left\{\widetilde{T}_{x_2}^{x_2+\delta}(\lambda), \widetilde{\widetilde{T}}_{x_2}^{x_2+\delta}(\mu)\right\}\widetilde{T}_{x_1}^{x_2}(\lambda)\,\widetilde{\widetilde{T}}_{x_1}^{x_2}(\mu) + \lim_{\delta\to 0}\frac{1}{\delta}\left(\widetilde{T}_{x_2}^{x_2+\delta}(\lambda)\,\widetilde{\widetilde{T}}_{x_2}^{x_2+\delta}(\mu) - I_4\right)\left\{\widetilde{T}_{x_1}^{x_2}(\lambda), \widetilde{\widetilde{T}}_{x_1}^{x_2}(\mu)\right\} . \tag{1.2.15}$$

We study the first summand of the expression obtained. Substituting in $\mathcal{P}_{x_2}^{x_2+\delta}(\lambda,\mu)$ the expression

$$T_{x_2}^{x_2+\delta} = I_2 + \int_{x_2}^{x_2+\delta} dx\, L(x) + O(\delta^2),$$

following from (1.1.19), we get

$$\left\{\widetilde{T}_{x_2}^{x_2+\delta}(\lambda), \widetilde{\widetilde{T}}_{x_2}^{x_2+\delta}(\mu)\right\} = \int_{x_2}^{x_2+\delta} dx \int_{x_2}^{x_2+\delta} dy \left\{\widetilde{L}(x,\lambda), \widetilde{\widetilde{L}}(y,\mu)\right\} + O(\delta^2) =$$

$$= \int_{x_2}^{x_2+\delta} dx \int_{x_2}^{x_2+\delta} dy \left\{\widetilde{L}(x,\lambda), \widetilde{\widetilde{L}}(x,\mu)\right\}' \delta(x-y) + O(\delta^2) = \int_{x_2}^{x_2+\delta} dx \left\{\widetilde{L}(x,\lambda), \widetilde{\widetilde{L}}(x,\mu)\right\}' + O(\delta^2).$$

Here $\{,\}'$ denotes the "local" Poisson bracket

$$\begin{gathered}\left\{\Psi(x), \Psi(x)\right\}' = \left\{\overline{\Psi}(x), \overline{\Psi}(x)\right\}' = 0 ,\\ \left\{\Psi(x), \overline{\Psi}(x)\right\}' = i .\end{gathered} \tag{1.2.16}$$

Finally, passing on (1.2.15) to the limit as $\delta\to 0$, we get

$$\frac{\partial}{\partial x_2}\mathcal{P}_{x_1}^{x_2}(\lambda,\mu) = \left\{\widetilde{L}(x_2,\lambda), \widetilde{\widetilde{L}}(x_2,\mu)\right\}'\widetilde{T}_{x_1}^{x_2}(\lambda)\,\widetilde{\widetilde{T}}_{x_1}^{x_2}(\mu) + \left(\widetilde{L}(x_2,\lambda) + \widetilde{\widetilde{L}}(x_2,\mu)\right)\mathcal{P}_{x_1}^{x_2}(\lambda,\mu) . \tag{1.2.17}$$

In order to identify (1.2.13) and (1.2.17), it suffices to note that one has

$$\left\{\widetilde{L}(x,\lambda), \widetilde{\widetilde{L}}(x,\mu)\right\}' = \left[\tau(\lambda-\mu), \widetilde{L}(x,\lambda) + \widetilde{\widetilde{L}}(x,\mu)\right] \tag{12.18}$$

or

$$\left\{\tilde{L}(x,\lambda),\tilde{\tilde{L}}(y,\mu)\right\}=\left[r(\lambda-\mu),\tilde{L}(x,\lambda)+\tilde{\tilde{L}}(y,\mu)\right]\delta(x-y), \tag{1.2.19}$$

which is easily verified directly, taking into account (1.1.2), (1.2.2), (1.2.12).

In order to complete the proof of Theorem 1, it remains to note that $\left\{\tilde{T}^{x_2}_{x_1}(\lambda),\tilde{\tilde{T}}^{x_2}_{x_1}(\mu)\right\}$ satisfies the initial condition (1.2.14).

A complete summary of all Poisson brackets between matrix elements of $T^{x_2}_{x_1}(\lambda)$ following from (1.2.11) is given in the Supplement (Eqs. (S1-6)).

We note that the domain of application of the method of the $r$-matrix is not restricted to (1.1.1). The existence of a representation of the Poisson bracket in the form (1.2.11) is based, as follows from the proof given above, only on (1.2.19), which also holds for the sin-Gordon equation and the Landau–Lifshits equations [29]. The matrix $r$ for these equations has a more complicated form than (2.2.12).

## 1.3. Passage to an Infinite Interval

To achieve our ultimate goal, the construction of action-angle variables, we need to calculate the Poisson brackets for the transition matrix on the interval $(-\infty,\infty)$. But first we concern ourselves with the calculation of the Poisson bracket $\left\{\tilde{T}_-(x,\lambda),\tilde{\tilde{T}}_-(x,\mu)\right\}$, which it is convenient to denote by $\mathcal{P}_-(x;\lambda,\mu)$.

Repeating word for word the argument of the preceding section, one can see that $\mathcal{P}_-(x;\lambda,\mu)$ satisfies with respect to the variable $x$, the differential equation (1.2.13). On the other hand, the same differential is satisfied by the expression $\left[r(\lambda-\mu),\ \tilde{T}_-(x,\lambda)\tilde{\tilde{T}}_-(x,\mu)\right]$. Consequently, their difference satisfies the corresponding homogeneous differential equation, whose general solution we can write as $\tilde{T}_-(x,\lambda)\tilde{\tilde{T}}_-(x,\mu)C_-(\lambda,\mu)$. Thus, we get the following representation for $\mathcal{P}_-(x;\lambda,\mu)$:

$$\mathcal{P}_-(x;\lambda,\mu)=\left[r(\lambda-\mu),\tilde{T}_-(x,\lambda)\tilde{\tilde{T}}_-(x,\mu)\right]-\tilde{T}_-(x,\lambda)\tilde{\tilde{T}}_-(x,\mu)C_-(\lambda,\mu). \tag{1.3.1}$$

The $(4\times4)$ matrix $C_-(\lambda,\mu)$ is determined from comparison with the asymptotics of (1.3.1) as $x\to-\infty$.

It is easy to get the asymptotic behavior of the right side of (1.3.1), using (1.1.16):

$$\mathcal{P}_-(x;\lambda,\mu)\underset{x\to-\infty}{\simeq}\left[r(\lambda-\mu),E(x;\lambda,\mu)\right]-E(x;\lambda,\mu)\,C_-(\lambda,\mu), \tag{1.3.2}$$

where

$$E(x;\lambda,\mu)=\tilde{e}(x,\lambda)\tilde{\tilde{e}}(x,\mu)=e^{-\frac{i}{2}\left(\lambda\tilde{\sigma}_3+\mu\tilde{\tilde{\sigma}}_3\right)x}. \tag{1.3.3}$$

To calculate the asymptotics of $\left\{\tilde{T}_-(x,\lambda),\tilde{\tilde{T}}_-(x,\mu)\right\}$ as $x\to-\infty$ we use the integral representation (1.1.24). We have:

$$\left\{\tilde{T}_-(x,\lambda),\tilde{\tilde{T}}_-(x,\mu)\right\}=\int_{-\infty}^{x}d\xi\int_{-\infty}^{x}d\eta\left\{\tilde{T}^{x}_{\xi}(\lambda)\tilde{V}(\xi),\tilde{\tilde{T}}^{x}_{\eta}(\mu)\tilde{\tilde{V}}(\eta)\right\}\tilde{e}(\xi,\lambda)\tilde{\tilde{e}}(\eta,\mu). \tag{1.3.4}$$

Calculating the Poisson bracket standing in (1.3.4) under the integral

$$\left\{\widetilde{T}^{x}_{\xi}(\lambda)\widetilde{V}(\xi),\widetilde{\widetilde{T}}^{x}_{\eta}(\mu)\widetilde{\widetilde{V}}(\eta)\right\}=\widetilde{T}^{x}_{\xi}(\lambda)\widetilde{\widetilde{T}}^{x}_{\eta}(\mu)\left\{\widetilde{V}(\xi),\widetilde{\widetilde{V}}(\eta)\right\}+\left\{\widetilde{T}^{x}_{\xi}(\lambda),\widetilde{\widetilde{T}}^{x}_{\eta}(\mu)\right\}\widetilde{V}(\xi)\widetilde{\widetilde{V}}(\eta)+$$
$$+\widetilde{\widetilde{T}}^{x}_{\eta}(\mu)\left\{\widetilde{T}^{x}_{\xi}(\lambda),\widetilde{\widetilde{V}}(\eta)\right\}\widetilde{V}(\xi)+\widetilde{T}^{x}_{\xi}(\lambda)\left\{\widetilde{V}(\xi),\widetilde{\widetilde{T}}^{x}_{\eta}(\mu)\right\}\widetilde{\widetilde{V}}(\eta), \tag{1.3.5}$$

we see that only the summand

$$\widetilde{T}^{x}_{\xi}(\lambda)\widetilde{\widetilde{T}}^{x}_{\eta}(\mu)\left\{\widetilde{V}(\xi),\widetilde{\widetilde{V}}(\eta)\right\}=\widetilde{T}^{x}_{\xi}(\lambda)\widetilde{\widetilde{T}}^{x}_{\eta}(\mu)\, i\varkappa\left(\widetilde{\sigma}_{-}\widetilde{\widetilde{\sigma}}_{+}-\widetilde{\sigma}_{+}\widetilde{\widetilde{\sigma}}_{-}\right)\delta(\xi-\eta) \tag{1.3.6}$$

gives a nondecreasing contribution as $x\to-\infty$ in (1.3.4). Substituting (1.3.5) and (1.3.6) in (1.3.4), and considering that as follows from (1.1.21),

$$T^{x}_{\xi}(\lambda)-e(x-\xi,\lambda)\longrightarrow 0,\quad x,\xi\longrightarrow-\infty,$$

we get, that as $x\to-\infty$,

$$\mathcal{P}_{-}(x;\lambda,\mu)\sim\int_{-\infty}^{x}E(x-\xi;\lambda,\mu)\,i\varkappa\left(\widetilde{\sigma}_{-}\widetilde{\widetilde{\sigma}}_{+}-\widetilde{\sigma}_{+}\widetilde{\widetilde{\sigma}}_{-}\right)E(\xi;\lambda,\mu)\,d\xi. \tag{1.3.7}$$

It is easy to calculate the integral in (1.3.7), and comparing the asymptotics of (1.3.7) with (1.3.2), we get

$$C_{-}(\lambda,\mu)=\varkappa\begin{pmatrix}0&0&0&0\\0&0&\frac{1}{\lambda-\mu-i0}&0\\0&\frac{1}{\lambda-\mu+i0}&0&0\\0&0&0&0\end{pmatrix}. \tag{1.3.8}$$

Completely analogously one calculates the Poisson bracket $\left\{\widetilde{T}_{+}(x,\lambda),\widetilde{\widetilde{T}}(x,\mu)\right\}=\mathcal{P}_{+}(x;\lambda,\mu)$ :

$$\mathcal{P}_{+}(x;\lambda,\mu)=\left[r(\lambda-\mu),\widetilde{T}_{+}(x,\lambda)\widetilde{\widetilde{T}}_{+}(x,\mu)\right]+C_{+}(\lambda,\mu)\widetilde{T}_{+}(x,\lambda)\widetilde{\widetilde{T}}_{+}(x,\mu), \tag{1.3.9}$$

where

$$C_{+}(\lambda,\mu)=\varkappa\begin{pmatrix}0&0&0&0\\0&0&\frac{1}{\lambda-\mu+i0}&0\\0&\frac{1}{\lambda-\mu-i0}&0&0\\0&0&0&0\end{pmatrix}. \tag{1.3.10}$$

It is interesting to note that although (1.3.1) and (1.3.9) should be understood in the sense of generalized functions, the summands in (1.3.1) and (1.3.9) containing $r(\lambda-\mu)$, are not needed in the regularization for $\lambda=\mu$, since the corresponding numerator $\left[P,\widetilde{T}_{\pm}(x,\lambda)\,\widetilde{\widetilde{T}}_{\pm}(x,\mu)\right]$ vanishes for $\lambda=\mu$ by virtue of (1.2.10).

However, the choice of a definite regularization of $r(\lambda-\mu)$, e.g., $r(\lambda-\mu)=-\varkappa P\,\mathrm{v.p.}\frac{1}{\lambda-\mu}$, allows one to write (1.3.1) and (1.3.9) in compact form:

$$\mathcal{P}_{-}(x;\lambda,\mu)=r(\lambda-\mu)\widetilde{T}_{-}(x,\lambda)\widetilde{\widetilde{T}}_{-}(x,\mu)-\widetilde{T}_{-}(x,\lambda)\widetilde{\widetilde{T}}_{-}(x,\mu)\,r_{-}(\lambda-\mu) \tag{1.3.11}$$

and

$$\mathcal{P}_+(x;\lambda,\mu) = r_+(\lambda-\mu)\widetilde{T}_+(x,\lambda)\widetilde{\widetilde{T}}_+(x,\mu) - \widetilde{T}_+(x,\lambda)\widetilde{\widetilde{T}}_+(x,\mu)\, r(\lambda-\mu) \ , \tag{1.3.12}$$

where

$$r_\pm(\lambda-\mu) = r(\lambda-\mu) + C_\pm(\lambda,\mu) = -\varkappa\begin{pmatrix} \text{v.p.}\frac{1}{\lambda-\mu}, & 0, & 0, & \\ 0, & 0, & \pm\pi i\delta(\lambda-\mu), & \\ 0, & \mp\pi i\delta(\lambda-\mu), & 0, & \\ 0, & 0, & 0, & \text{v.p.}\frac{1}{\lambda-\mu} \end{pmatrix}. \tag{1.3.13}$$

Now everything is ready to calculate the Poisson bracket $\{\widetilde{T}(\lambda),\widetilde{\widetilde{T}}(\mu)\} = \mathcal{P}(\lambda,\mu)$ . Keeping in mind (1.1.18), (1.3.11), and (1.3.12), we get

$$\mathcal{P}(\lambda,\mu) = r_+(\lambda-\mu)\widetilde{T}(\lambda)\widetilde{\widetilde{T}}(\mu) - \widetilde{T}(\lambda)\widetilde{\widetilde{T}}(\mu)\, r_-(\lambda-\mu)\,. \tag{1.3.14}$$

The results proved above allow us to formulate the following theorem.

THEOREM 2. The Poisson brackets between matrix elements of transition matrices for semi-infinite and infinite intervals are given by (1.3.11-14).

A complete summary of all Poisson brackets is given in the Supplement (Eqs. (S7-24)).

## 1.4. Integrals of Motion. Action-Angle Variables

In the present section, which has an auxiliary character, we list results basically known from [30-32] for the nonlinear Schrödinger equation, which will be useful later for comparison with the quantum case.

As shown in [31, 32], $\ln a(\lambda)$ is the generating function for the local integrals of motion $\mathcal{I}_m$ for (1.1.1), i.e., the coefficients $\mathcal{I}_m$ of the expansion of $\ln a(\lambda)$ in an asymptotic series in powers of $\lambda^{-1}$

$$\ln a(\lambda) = i\varkappa\sum_{m=1}^{\infty}\mathcal{I}_m\lambda^{-m} \tag{1.4.1}$$

are the integrals of the local densities with respect to $\Psi(x)$ and $\overline{\Psi}(x)$

$$\mathcal{I}_m = \int_{-\infty}^{\infty} dx\, \overline{\Psi}(x)\chi_m(x)\ , \tag{1.4.2}$$

where $\chi_m(x)$ are determined from the recursion relation

$$\chi_{m+1}(x) = -i\frac{d}{dx}\chi_m(x) + \varkappa\overline{\Psi}(x)\sum_{k=1}^{m-1}\chi_k(x)\chi_{m-k}(x) \tag{1.4.3}$$

with the initial condition

$$\chi_1(x) = \Psi(x)\ . \tag{1.4.4}$$

By virtue of (S19) the quantities $\mathcal{I}_m$ are in involution with respect to the Poisson bracket (1.2.2). We shall write down the first few of the integrals $\mathcal{I}_m$:

$$\mathcal{I}_1 = N = \int_{-\infty}^{\infty} dx\,|\Psi(x)|^2, \tag{1.4.5}$$

$$\mathcal{I}_2 = P = \frac{i}{2}\int_{-\infty}^{\infty} dx \left(\overline{\Psi}_x \Psi' - \overline{\Psi}\Psi_x\right), \tag{1.4.6}$$

$$\mathcal{I}_3 = H = \int_{-\infty}^{\infty} dx \left(|\Psi_x|^2 + \varkappa |\Psi|^4\right). \tag{1.4.7}$$

The quantities $N$, $P$, and $H$ are called, respectively, the number of particles, momentum, and energy. Since the Hamiltonian $H = \mathcal{I}_3$ is included among the $\mathcal{I}_m$, the quantities $\mathcal{I}_m$ are integrals of motion for (1.1.1).

We proceed now to the construction of action-angle variables for (1.1.1). The concept of "action-angle variables" we shall treat here broadly, calling such any canonical variables in which the Hamiltonian $H$ can be written as a quadratic form (and the equations of motion, correspondingly, become linear).

We introduce quantities $\varphi(\lambda)$ and $\overline{\varphi}(\lambda)$ by the formulas

$$\begin{aligned} \varphi(\lambda) &= \frac{b(\lambda)}{|b(\lambda)|}\sqrt{\frac{\ln|a(\lambda)|}{\pi\varkappa}}, \\ \overline{\varphi}(\lambda) &= \frac{\overline{b}(\lambda)}{|b(\lambda)|}\sqrt{\frac{\ln|a(\lambda)|}{\pi\varkappa}} \end{aligned} \tag{1.4.8}$$

In (1.4.8) it is necessary to take the positive value of the root. The expression under the radical sign here remains positive for any value of $\varkappa$, since by (1.1.12), $|a(\lambda)|>1$ for $\varkappa>0$ and $|a(\lambda)|<1$ for $\varkappa<0$, when $\lambda$ runs through the real axis.

The quantities $\varphi(\lambda)$ and $\overline{\varphi(\lambda)}$ satisfy all the requirements listed above for action-angle variables. In fact, using the Poisson brackets (Eqs. (S9-24)), it is easy to verify that $\varphi(\lambda)$ and $\overline{\varphi}(\lambda)$ are canonical conjugate variables:

$$\begin{aligned} \{\varphi(\lambda), \varphi(\mu)\} &= \{\overline{\varphi}(\lambda), \overline{\varphi}(\mu)\} = 0, \\ \{\varphi(\lambda), \overline{\varphi}(\mu)\} &= i\delta(\lambda-\mu). \end{aligned} \tag{1.4.9}$$

Further, for $\varkappa>0$, the generating function of the integrals of motion $\ln a(\lambda)$ has, as proved in [31, 32], the following integral representation:

$$\ln a(\lambda) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\mu\, \frac{\ln|a(\mu)|}{\lambda-\mu},$$

which can be rewritten, using (1.4.8), as

$$\ln a(\lambda) = i\varkappa\int_{-\infty}^{\infty} d\mu\, \frac{|\varphi(\mu)|^2}{\lambda-\mu}. \tag{1.4.10}$$

Decomposing (1.4.10) into powers of $\lambda^{-1}$, we get

$$\mathcal{I}_m = \int_{-\infty}^{\infty} d\mu\, \mu^{m-1}\, |\varphi(\mu)|^2. \tag{1.4.11}$$

In particular,

$$N = \int_{-\infty}^{\infty} d\mu\, |\varphi(\mu)|^2, \tag{1.4.12}$$

$$P=\int_{-\infty}^{\infty} d\mu\, \mu\, |\varphi(\mu)|^2, \tag{1.4.13}$$

$$H=\int_{-\infty}^{\infty} d\mu\, \mu^2\, |\varphi(\mu)|^2 . \tag{1.4.14}$$

Thus, all the integrals of motion $\mathcal{I}_m$ in the variables $\bar{\varphi}(\mu)$ are quadratic, and correspondingly, the equations of motion

$$\frac{d}{dt}\varphi(t;\mu)=\{\mathcal{I}_m, \varphi(t;\mu)\}=-i\mu^{m-1}\varphi(t;\mu) \tag{1.4.15}$$

are linear in $\varphi, \bar{\varphi}$, which allows us to call the variables $\varphi(\mu), \bar{\varphi}(\mu)$ variables of action-angle type for (1.1.1) in the sense of the definition given above. The action-angle variables introduced by us differ from the traditional ones [30], but have the advantage that they admit a quantum-mechanical generalization.

The case $\varkappa<0$ requires the calculation of the discrete spectrum. We shall not write down here the corresponding variables of action-angle type, since they, apparently, have no reasonable analogs in the quantum case, and we restrict ourselves to indicating how in this case one generalizes (1.4.10-14):

$$\ln a(\lambda)=\sum_{j=1}^{\mathcal{N}} \ln \frac{\lambda-\lambda_j}{\lambda-\bar{\lambda}_j} + i\varkappa\int_{-\infty}^{\infty} \frac{|\varphi(\mu)|^2}{\lambda-\mu}\, d\mu , \tag{1.4.10'}$$

$$\mathcal{I}_m=\frac{2}{m|\varkappa|}\sum_{j=1}^{\mathcal{N}} \mathrm{Im}(\lambda_j^m) + \int_{-\infty}^{\infty} d\mu\, \mu^{m-1}\, |\varphi(\mu)|^2 , \tag{1.4.11'}$$

$$\mathcal{N}=\frac{2}{|\varkappa|}\sum_{j=1}^{\mathcal{N}} \mathrm{Im}\,\lambda_j + \int_{-\infty}^{\infty} d\mu\, |\varphi(\mu)|^2 , \tag{1.4.12'}$$

$$P=\frac{2}{|\varkappa|}\sum_{j=1}^{\mathcal{N}} \mathrm{Re}\,\lambda_j\cdot \mathrm{Im}\,\lambda_j + \int_{-\infty}^{\infty} d\mu\, \mu\, |\varphi(\mu)|^2 , \tag{1.4.13'}$$

$$H=\frac{2}{3|\varkappa|}\sum \left(3\mathrm{Re}^2\lambda_j\cdot \mathrm{Im}\,\lambda_j - \mathrm{Im}^3\lambda_j\right) + \int_{-\infty}^{\infty} d\mu\, \mu^2\, |\varphi(\mu)|^2 . \tag{1.4.14'}$$

## 1.5. $M$-Operator

All the arguments of the preceding sections were based on the study of the operator $L(x,\lambda)$ at a fixed moment of time $t$. This did not prevent us from proving the complete integrability of (1.1.1) and finding the spectrum of its elementary excitations. For the traditional approach [31], however, the consideration of the time evolution from the very start is characteristic. The initial point here is the representation of the equation of motion (1.1.1) as the commutativity condition of two differential operators:

$$\left[\frac{\partial}{\partial x} - L(x,t;\lambda), \frac{\partial}{\partial t} - M(x,t;\lambda)\right]=0 , \tag{1.5.1}$$

or

$$L_t = M_x + [M, L] , \tag{1.5.2}$$

where

$$M(x,t;\lambda) = \begin{pmatrix} i\frac{\lambda^2}{2} + i\varkappa|\Psi(x,t)|^2, & \varkappa\overline{\Psi}_x(x,t) - i\varkappa\lambda\overline{\Psi}(x,t) \\ \Psi_x(x,t) + i\lambda\Psi(x,t), & -i\frac{\lambda^2}{2} - i\varkappa|\Psi(x,t)|^2 \end{pmatrix}. \tag{1.5.3}$$

Although, we stress again, the approach developed in the present paper is in principle not necessary in introducing the operator $M$ both in the classical and the quantum case, the study of the question of the $M$-operator is of definite systematic interest and allows us to demonstrate again the possibility of the method of the $\tau$-matrix.

We raise the question in the following way. Let the time evolution of the fields $\Psi(x,t)$ and $\overline{\Psi}(x,t)$ be given by the $m$-th local integral of motion:

$$\frac{\partial}{\partial t}\Psi(x,t) = \{\mathcal{J}_m, \Psi(x,t)\}. \tag{1.5.4}$$

How can we find under these conditions a matrix $M_m(x,t)$, satisfying (1.5.2)? Although the answer to this question is known [35], the method of the $\tau$-matrix allows us to express it in a simple and compact form.

We consider first the case of periodic boundary conditions on the interval $[x_1, x_2]$

$$\Psi(x+x_2-x_1) = \Psi(x) \tag{1.5.5}$$

and we formulate the following proposition:

Proposition 1.5.1. One has the equation

$$\{\operatorname{tr} T_{x_1}^{x_2}(\lambda), L(x,\mu)\} = \frac{\partial}{\partial x} M_{x_1}^{x_2}(x;\lambda,\mu) + [M_{x_1}^{x_2}(x;\lambda,\mu), L(x,\mu)], \tag{1.5.6}$$

where

$$M_{x_1}^{x_2}(x;\lambda,\mu) = \widetilde{\operatorname{tr}}\left(\widetilde{T}_{x}^{x_2}(\lambda)\,\tau(\lambda-\mu)\,\widetilde{T}_{x_1}^{x}(\lambda)\right). \tag{1.5.7}$$

The operation $\widetilde{\operatorname{tr}}$ introduced by us here is the convolution operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$ in the indices relating to the first factor. The result, thus, is a $(2\times 2)$ matrix. In particular,

$$\widetilde{\operatorname{tr}}(A\otimes B) = \widetilde{\operatorname{tr}}(\widetilde{A}\widetilde{\widetilde{B}}) = (\operatorname{tr}A)B; \quad A, B \in \mathrm{Mat}(2,2).$$

The proof of Proposition 1.5.1. is based on the following lemma:

LEMMA 1.5.1. For any functional $X(\psi,\overline{\psi})$ of the fields $\Psi(x)$ and $\overline{\Psi}(x)$ one has the following relation:

$$\{T_{x_1}^{x_2}(\lambda), X\} = \int_{x_1}^{x_2} dx\, T_{x}^{x_2}(\lambda)\{L(x,\lambda), X\} T_{x_1}^{x}(\lambda). \tag{1.5.8}$$

To prove Lemma 1.5.1 we introduce the notation $\mathcal{P}_{x_1}^{x_2}(\lambda,X) = \{T_{x_1}^{x_2}(\lambda), X\}$. Differentiating the left and right sides of (1.5.8) with respect to $x_2$, it is easy to see that they satisfy the same differential equation

$$\frac{\partial}{\partial x_2}\mathcal{P}_{x_1}^{x_2}(\lambda;X) = L(x_2,\lambda)\mathcal{P}_{x_1}^{x_2}(\lambda,X) + \{L(x_2,\lambda),X\}T_{x_1}^{x_2}(\lambda) \tag{1.5.9}$$

with the same initial condition

$$\mathcal{P}_{x_1}^{x_1}(\lambda; X) = 0 \ . \tag{1.5.10}$$

Reference to the corresponding uniqueness theorem completes the proof.

We begin now the proof of Proposition 1.5.1. First, using the notation $\widetilde{\mathrm{tr}}$, we have introduced, we transform the left side of (1.5.6):

$$\left\{ \mathrm{tr}\, T_{x_1}^{x_2}(\lambda), L(x,\mu) \right\} = \widetilde{\mathrm{tr}} \left\{ \widetilde{T}_{x_1}^{x_2}(\lambda), \widetilde{\widetilde{L}}(x,\mu) \right\} . \tag{1.5.11}$$

To calculate the right side of (1.5.11), we use Lemma 1.5.1, which we have just proved. Substituting in (1.5.8) $X = \widetilde{\widetilde{L}}(x,\mu)$, we get

$$\left\{ \widetilde{T}_{x_1}^{x_2}(\lambda), \widetilde{\widetilde{L}}(x,\mu) \right\} = \int_{x_1}^{x_2} d\xi\, \widetilde{T}_{\xi}^{x_2}(\lambda) \left\{ \widetilde{L}(\xi,\lambda), \widetilde{\widetilde{L}}(x,\mu) \right\} \widetilde{T}_{x_1}^{\xi}(\lambda) , \tag{1.5.12}$$

or, by virtue of (1.2.19),

$$\left\{ \widetilde{T}_{x_1}^{x_2}(\lambda), \widetilde{\widetilde{L}}(x,\mu) \right\} = \widetilde{T}_{x}^{x_2}(\lambda) \left[ r(\lambda-\mu), \widetilde{L}(x,\lambda) + \widetilde{\widetilde{L}}(x,\mu) \right] \widetilde{T}_{x_1}^{x}(\lambda) \ . \tag{1.5.13}$$

Substituting (1.5.13) in (1.5.11) and using (1.1.3) and (1.1.7), we get

$$\begin{aligned} \left\{ \mathrm{tr}\, T_{x_1}^{x_2}(\lambda), L(x,\mu) \right\} &= \widetilde{\mathrm{tr}}\, \widetilde{T}_{x}^{x_2}(\lambda) \left[ r(\lambda-\mu), \widetilde{L}(x,\lambda) \right] \widetilde{T}_{x_1}^{x}(\lambda) + \\ + \widetilde{\mathrm{tr}}\, \widetilde{T}_{x}^{x_2}(\lambda) \left[ r(\lambda-\mu), \widetilde{\widetilde{L}}(x,\lambda) \right] \widetilde{T}_{x_1}^{x}(\lambda) &= \frac{\partial}{\partial x} \widetilde{\mathrm{tr}}\, \widetilde{T}_{x}^{x_2}(\lambda)\, r(\lambda-\mu) \widetilde{T}_{x_1}^{x}(\lambda) + \\ + \left[ \widetilde{\mathrm{tr}}\, \widetilde{T}_{x}^{x_2}\, r(\lambda-\mu) \widetilde{T}_{x_1}^{x}(\lambda), L(x,\mu) \right] &= \frac{\partial}{\partial x} M_{x_1}^{x_2}(x;\lambda,\mu) + \left[ M_{x_1}^{x_2}(x;\lambda,\mu), L(x,\mu) \right] , \end{aligned} \tag{1.5.14}$$

where $M_{x_1}^{x_2}(x;\lambda,\mu)$ is given by (1.5.7), which is what had to be proved.

From (1.2.11) follows

$$\left\{ \mathrm{tr}\, T_{x_1}^{x_2}(\lambda), \mathrm{tr}\, T_{x_1}^{x_2}(\mu) \right\} = 0 \ , \tag{1.5.15}$$

which allows us to consider $\ln \mathrm{tr}\, T_{x_1}^{x_2}(\lambda)$ , as the generating function of the integrals of motion of (1.1.1) with periodic boundary conditions.

In order to find the analog of (1.5.6) for an infinite interval, it is necessary to divide both sides of (1.5.6) by $\mathrm{tr}\, T_{x_1}^{x_2}(\lambda)$ and pass to the limit as $x_1 \to -\infty$, $x_2 \to +\infty$. The answer depends, obviously, on the sign of $\mathrm{Im}\,\lambda$ and is given by the following formula:

$$\left\{ \ln a^{(\pm)}(\lambda), L(x,\mu) \right\} = M_x^{(\pm)}(x;\lambda,\mu) + \left[ M^{(\pm)}(x;\lambda,\mu), L(x,\mu) \right] , \tag{1.5.16}$$

where

$$\begin{aligned} &M^{(\pm)}(x;\lambda,\mu) = \\ &\widetilde{\mathrm{tr}} \left( \widetilde{P}^{(\pm)} \widetilde{T}_{+}(x,\lambda)\, r(\lambda-\mu) \widetilde{T}_{-}(x,\lambda) \widetilde{P}^{(\pm)} \right) \ . \end{aligned} \tag{1.5.17}$$

We explain the notation. The upper sign corresponds to $\mathrm{Im}\,\lambda > 0$, the lower to $\mathrm{Im}\,\lambda < 0$. $a^{(+)}(\lambda) = a(\lambda)$, $a^{(-)}(\lambda) = \overline{a(\bar{\lambda})}$. The projectors $P^{(\pm)}$ onto the eigenspaces $e(x,\lambda)$ have the form

$$P^{(+)}=\begin{pmatrix}1&0\\0&0\end{pmatrix} \quad , \quad P^{(-)}=\begin{pmatrix}0&0\\0&1\end{pmatrix} .$$

The proof of (1.5.16) and (1.5.17) becomes obvious if one notes that as $x_1 \to -\infty$, $x_2 \to +\infty$

$$\operatorname{tr} T_{x_1}^{x_2}(\lambda) \sim \operatorname{tr}\left(e(x_2,\lambda)T(\lambda)e(-x_1,\lambda)\right) \sim e^{\mp i\frac{\lambda}{2}(x_2-x_1)} \operatorname{tr} P^{(\pm)} T(\lambda) P^{(\pm)} = e^{\mp i\frac{\lambda}{2}(x_2-x_1)} a^{(\pm)}(\lambda) .$$

We consider the results obtained. Since $\ln a(\lambda)$ (or $\ln \overline{a(\bar\lambda)}$ ) is, as discussed in Sec. 1.4, the generating function of the local integrals of motion $\mathcal{J}_m$, (1.5.16) means that $M^{(\pm)}(x;\lambda,\mu)$ is the generating function of the corresponding $M$-operators:

$$M^{(\pm)}(x;\lambda,\mu) = \pm i\varkappa \sum_{m=1}^{\infty} \lambda^{-m} M_m(x,\mu) . \tag{1.5.18}$$

In particular,

$$M_1(x,\lambda) = \frac{i}{2}\sigma_3 , \tag{1.5.19}$$

$$M_2(x,\lambda) = -L(x,\lambda) = i\frac{\lambda}{2}\sigma_3 - i\varkappa\overline{\Psi}(x)\sigma_+ + i\Psi(x)\sigma_- , \tag{1.5.20}$$

$$M_3(x,\lambda) = \left(i\frac{\lambda^2}{2} + i\varkappa|\Psi(x)|^2\right)\sigma_3 + \varkappa\left(\overline{\Psi}_x(x) - i\lambda\overline{\Psi}(x)\right)\sigma_+ + \left(\Psi_x(x) + i\lambda\Psi(x)\right)\sigma_- . \tag{1.5.21}$$

We note that the specific structure of the $r$-matrix for the nonlinear Schrödinger equation (1.2.12) allows one to simplify the expression (1.5.17) for $M^{(\pm)}(x;\lambda,\mu)$:

$$M^{(+)}(x;\lambda,\mu) = -\frac{\varkappa}{\lambda-\mu}\frac{1}{a(\lambda)}\begin{pmatrix} a_-(x,\lambda) \\ b_-(x,\lambda)\end{pmatrix}\left(a_+(x,\lambda), \varkappa\overline{b_+(x,\bar\lambda)}\right) , \tag{1.5.22}$$

$$M^{(-)}(x;\lambda,\mu) = -\frac{\varkappa}{\lambda-\mu}\frac{1}{\overline{a(\bar\lambda)}}\begin{pmatrix} \varkappa\overline{b_-(x,\bar\lambda)} \\ \overline{a_-(x,\bar\lambda)}\end{pmatrix}\left(b_+(x,\lambda), \overline{a_+(x,\bar\lambda)}\right) , \tag{1.5.23}$$

thus reproducing a known result [35], stating that the generating function of the $M$-operators for the n.S.e. is proportional to the diagonal of the kernel of the resolvent operator $\frac{\partial}{\partial x} - L$.

We stress, however, that such a simplification makes essential use of the specifics of the nonlinear Schrödinger equation, at the same time that (1.5.17) carries a universal character and is suitable for any completely integrable models, whose $L$-operators have a $r$-matrix (e.g., the sin-Gordon equation, the Landau–Lifshits equation [29]).

To conclude this section, we introduce a series of formulas, describing the time evolution of the transition matrices $T_{x_1}^{x_2}, T_\pm$, and $T$.

<u>Proposition 1.5.2.</u> The transition matrices $T_{x_1}^{x_2}(t;\lambda)$, $T_\pm(x,t;\lambda)$ , and $T(t;\lambda)$ satisfy the following differential equations:

$$\frac{\partial}{\partial t} T_{x_1}^{x_2}(t;\lambda) = M(x_2,t;\lambda) T_{x_1}^{x_2}(t;\lambda) - T_{x_1}^{x_2}(t,\lambda) M(x_1;t,\lambda) , \tag{1.5.24}$$

$$\frac{\partial}{\partial t} T_-(x,t;\lambda) = M(x,t;\lambda) T_-(x,t;\lambda) - T_-(x,t;\lambda)\, i\frac{\lambda^2}{2}\sigma_3 , \tag{1.5.25}$$

$$\frac{\partial}{\partial t}T_+(x,t;\lambda)=i\frac{\lambda^2}{2}\sigma_3 T_+(x,t;\lambda)-T_+(x,t;\lambda)M(x,t;\lambda)\,, \tag{1.5.26}$$

$$\frac{\partial}{\partial t}T(t,\lambda)=\left[i\frac{\lambda^2}{2}\sigma_3\,,\,T(t,\lambda)\right]. \tag{1.5.27}$$

Proof. We introduce the notation $\frac{\partial}{\partial t}T_{x_1}^{x_2}(t,\lambda)=\mathcal{M}_{x_1}^{x_2}(t;\lambda)$ . Differentiating (1.5.24) with respect to $x_2$ and using (1.1.3) and (1.5.2), it is easy to see that both sides of (1.5.24) satisfy the same differential equation

$$\frac{\partial}{\partial x_2}\mathcal{M}_{x_1}^{x_2}(t;\lambda)=L(x_2,t;\lambda)\mathcal{M}_{x_1}^{x_2}(t;\lambda)+\Big(M_x(x_2,t;\lambda)+\big[L(x_2,t;\lambda),M(x_2,t;\lambda)\big]\Big)T_{x_1}^{x_2}(t;\lambda) \tag{1.5.28}$$

with the same initial condition

$$\mathcal{M}_{x_1}^{x_1}(t,\lambda)=0\;, \tag{1.5.29}$$

which proves (1.5.24). Equations (1.5.25-27) are now obtained from (1.5.24) by passage to the limit according to (1.1.13-15), taking into account

$$\lim_{|x|\to\infty}M(x,\lambda)=i\frac{\lambda^2}{2}\sigma_3\,, \tag{1.5.30}$$

which follows directly from (1.5.3) and the boundary condition $\Psi(x)\xrightarrow[|x|\to\infty]{}0$ .

We note that (1.5.27) has the obvious general solution:

$$T(t;\lambda)=e^{i\frac{\lambda^2}{2}\sigma_3 t}\,T(0,\lambda)\,e^{-i\frac{\lambda^2}{2}\sigma_3 t}\,. \tag{1.5.31}$$

Writing out the matrix elements of (1.5.31), we get the well-known result [31]:

$$a(t,\lambda)=a(0,\lambda)\,, \tag{1.5.32}$$

$$\bar{a}(t,\lambda)=\bar{a}(0,\lambda), \tag{1.5.33}$$

$$b(t,\lambda)=e^{-i\lambda^2 t}\,b(0,\lambda) \tag{1.5.34}$$

$$\bar{b}(t,\lambda)=e^{i\lambda^2 t}\,\bar{b}(0,\lambda)\,. \tag{1.5.35}$$

# CHAPTER II

# QUANTUM NONLINEAR SCHRÖDINGER EQUATION

## 2.1. Quantization

In this section we enter upon the study of the quantum version of the nonlinear Schrödinger equation, which constitutes the basic object of study of the present paper. As already noted in the Introduction, the nonlinear Schrödinger equation admits detailed description in the classical as well as the quantum case, which stipulated its choice as the object of the first application of the quantum method of the inverse problem.

We shall briefly describe the quantum system corresponding to the classical equation (1.1.1). The Hilbert space of states of the system is the Focke space $\mathcal{F}$ for Bose particles in one dimension [36, 37].

The elements of the space $\mathcal{F}$ are columns of the form

$$f=\begin{pmatrix} f_0 \\ f_1(x_1) \\ \cdots\cdots \\ f_N(x_1,\ldots,x_N) \\ \cdots\cdots\cdots \end{pmatrix}, \tag{2.1.1}$$

where $f_0 \in \mathbb{C}$, and $f_N(x_1,\dots,x_N)$ is a complex-valued symmetric square-integrable function of $N$ real variables. We define the scalar product in the space $\mathbb{F}$ by the formula

$$\langle f,g\rangle = \bar{f}_0 g_0 + \sum_{N=1}^{\infty} \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} dx_1\dots dx_N \bar{f}_N(x_1,\dots,x_N)\, g_N(x_1,\dots,x_N).$$

The Focke space $\mathbb{F}$ splits into the orthogonal sum of $N$ partial subspaces $\mathbb{F}_N$:

$$\mathbb{F} = \sum_{N=0}^{\infty}{}^{\oplus} \mathbb{F}_N \quad ; \quad f_N(x_1,\dots,x_N) \in \mathbb{F}_N .$$

A vector of the form

$$\begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \\ \dots \end{pmatrix} = |0\rangle \quad , \quad \langle 0|0\rangle = 1,$$

will be called a vacuum and will be denoted by $|0\rangle$.

Let the generalized operator-valued functions $\Psi(x)$ and $\Psi^+(x)$ define the standard [36, 37] representation of the canonical commutation relations

$$[\Psi(x),\Psi(y)] = [\Psi^+(x),\Psi^+(y)] = 0 ,$$
$$[\Psi(x),\Psi^+(y)] = \delta(x-y) \tag{2.1.2}$$

in the space $\mathbb{F}$ (we set Planck's constant $\hbar$ equal to 1), having the property

$$\Psi(x)|0\rangle = 0 . \tag{2.1.3}$$

Here any element $f$ [Eq. (2.1.1)] of the space $\mathbb{F}$ can be represented in the form

$$f = f_0|0\rangle + \sum_{N=1}^{\infty}\frac{1}{\sqrt{N!}}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} dx_1\dots dx_N\, f_N(x_1,\dots,x_N)\Psi^+(x_1)\dots\Psi^+(x_N)|0\rangle . \tag{2.1.4}$$

In what follows we shall use as representative both (2.1.1) and (2.1.4).

The classical Hamiltonian $H$ (1.2.1) corresponds in the quantum case to the self-adjoint operator $\mathbb{H}$ in the space $\mathbb{F}$, defined by the expression

$$\mathbb{H} = \int_{-\infty}^{\infty} dx \left(\Psi_x^+\Psi_x + \varkappa\Psi^+\Psi^+\Psi\Psi\right). \tag{2.1.5}$$

The Heisenberg equation of motion for the operator $\Psi(x,t)$, generated by the Hamiltonian $\mathbb{H}$, has the form

$$i\Psi_t = [\Psi,\mathbb{H}] = -\Psi_{xx} + 2\varkappa\Psi^+\Psi\Psi . \tag{2.1.6}$$

Using the commutation relations (2.1.2), it is easy to verify that the Hamiltonian $\mathbb{H}$ commutes with the operators of the number of particles

$$\mathbb{N} = \int_{-\infty}^{\infty} dx\, \Psi^+\Psi \tag{2.1.7}$$

and momentum

$$\mathbb{P} = \frac{i}{2}\int_{-\infty}^{\infty} dx\left(\Psi_x^+\Psi - \Psi^+\Psi_x\right). \tag{2.1.8}$$

On vectors $f_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}})$ from the $\mathcal{N}$-partial subspace $\mathbb{F}_{\mathcal{N}}$, the Hamiltonian $\mathbb{H}$ acts as a differential operator [33, 34, 38]:

$$\mathbb{H} f_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}}) = \left(-\sum_{j=1}^{\mathcal{N}} \frac{\partial}{\partial x^2} + 2\varkappa \sum_{i<j} \delta(x_i - x_j)\right) f_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}}), \tag{2.1.9}$$

having the form of a multipartial Schrödinger operator with twin pointwise interaction. It is easy to give strict meaning to the singular potential $\delta(x_i - x_j)$ in (2.1.9), replacing it by the boundary condition [33, 34, 38]

$$\frac{\partial}{\partial x_i} f_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}})\Big|_{x_i = x_j - 0}^{x_i = x_j + 0} = \varkappa f_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}})\Big|_{x_i = x_j}, \quad i \neq j. \tag{2.1.10}$$

The eigenfunctions of the operator $\mathbb{H}$ admit a simple description [33, 34]. Namely, the eigenfunction $f_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}} | k_1,\dots,k_{\mathcal{N}})$, corresponding to the eigenvalue $\sum_{j=1}^{\mathcal{N}} k_j^2$ and describing the state of scattering of $\mathcal{N}$ particles with momenta $k_1,\dots,k_{\mathcal{N}}$ $(\operatorname{Im} k_j = 0;\ j = 1,\dots,\mathcal{N})$, has for $x_1 < x_2 < \dots < x_{\mathcal{N}}$ the following form

$$f_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}} | k_1,\dots,k_{\mathcal{N}}) = \frac{(i)^{\mathcal{N}}}{(2\pi)^{\mathcal{N}/2}\sqrt{\mathcal{N}!}} \sum C_{\ell_1\dots\ell_{\mathcal{N}}}\, e^{i(k_{\ell_1}x_1 + \dots + k_{\ell_{\mathcal{N}}}x_{\mathcal{N}})} \tag{2.1.11}$$

(to other values of $x$ the function $f_{\mathcal{N}}$ is extended by symmetry). A substitution of the form (2.1.11) for an eigenfunction is usually called a Bethe substitution in honor of Bethe, who in [5] first proposed such a substitution, studying a lattice model of a ferromagnet.

The summation in (2.1.11) is taken over all permutations $(\ell_1,\dots,\ell_{\mathcal{N}})$ of $(1,\dots,\mathcal{N})$, and the coefficients $C_{\ell_1\dots\ell_{\mathcal{N}}}$ must satisfy the condition

$$\frac{C_{\ell_1\dots\ell_r\dots\ell_s\dots\ell_{\mathcal{N}}}}{C_{\ell_1\dots\ell_s\dots\ell_r\dots\ell_{\mathcal{N}}}} = \frac{k_{\ell_r} - k_{\ell_s} + i\varkappa}{k_{\ell_r} - k_{\ell_s} - i\varkappa}. \tag{2.1.12}$$

Following [38], we choose a solution of (2.1.12) in the form

$$C^{(\mathrm{norm})}_{\ell_1\dots\ell_{\mathcal{N}}} = \prod_{r<s} \sqrt{\frac{k_{\ell_r} - k_{\ell_s} + i\varkappa}{k_{\ell_r} - k_{\ell_s} - i\varkappa}}. \tag{2.1.13}$$

For such a choice of $C^{(\mathrm{norm})}_{\ell_1\dots\ell_{\mathcal{N}}}$ the system of functions $f^{(\mathrm{norm})}_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}} | k_1,\dots,k_{\mathcal{N}})$ is orthonormalized:

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} dx_1\dots dx_{\mathcal{N}}\, \overline{f^{(\mathrm{norm})}_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}} | k_1,\dots,k_{\mathcal{N}})}\, f^{(\mathrm{norm})}_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}} | p_1,\dots,p_{\mathcal{N}}) = \sum_{(\ell_1,\dots,\ell_{\mathcal{N}})} \prod_{j=1}^{\mathcal{N}} \delta(k_j - p_{\ell_j}). \tag{2.1.14}$$

In the case of repulsion $\varkappa > 0$, the system of functions $f^{(\mathrm{norm})}_{\mathcal{N}}(x_1,\dots,x_{\mathcal{N}} | k_1,\dots,k_{\mathcal{N}})$ $(\mathcal{N} = 0, 1, 2\dots)$, moreover, is complete in $\mathbb{F}$. In the case of attraction $\varkappa < 0$, we must consider connected states. It turns out that for each $\mathcal{N} = 2, 3, \dots$ there is only one $\mathcal{N}$-particle connected state obtained from (2.1.11) by analytic continuation with respect to the momenta $k_j$:

$$k_j = \frac{P}{\mathcal{N}} + i|\varkappa|\left(j - \frac{\mathcal{N}+1}{2}\right); \quad j = 1, 2, \dots, \mathcal{N}, \tag{2.1.15}$$

where $P$ is the total momentum of the connected state. The corresponding normalized wave functions are given in [38].

The proofs of all the assertions given above are contained in [33, 34, 38].

Concluding the list of results obtained for the quantum n.S.e. by the method of Bethe substitution, we can enter upon the consecutive account of the quantum version of the method of the inverse problem. We note right away that in contrast with the method of Bethe substitution, our method allows us to find the spectrum of the Hamiltonian and other integrals of motion of (2.1.6), without using the explicit form of the eigenfunctions (2.1.11).

The main object of our investigation will be, as noted in the Introduction, the quantum analog of the fundamental solution $T_{x_1}^{x_2}(\lambda)$ of the auxiliary linear problem (1.1.3). Generally speaking, there exist many methods of associating with a given functional of classical canonical variables $\Psi(x)$ and $\overline{\Psi}(x)$ a quantum operator (for example, Wick, anti-Wick, Weyl quantization [39]). We dwell here on Wick (normal) quantization. The advantage of such a choice is indicated, for example, by the result of [38] in which it is proved that the integrals of motion $\mathfrak{I}_m$ (1.4.2) for the classical equation (1.1.1) after Wick quantization go into quantum integrals of motion for (2.1.6).

Thus, we define the quantum transition matrix $\mathbb{T}_{x_1}^{x_2}(\lambda)$ by

$$\mathbb{T}_{x_1}^{x_2}(\lambda) = :T_{x_1}^{x_2}(\lambda): \quad . \tag{2.1.16}$$

The colons :: in (2.1.16) denote Wick quantization. In other words, the matrix elements of the matrix $\mathbb{T}_{x_1}^{x_2}(\lambda)$ are defined as quantum operators whose Wick symbols are the corresponding elements of $T_{x_1}^{x_2}(\lambda)$.

There immediately arises the question of the propriety of such a definition, i.e., of the existence of such quantum operators in the space $\mathrm{F}$. The clarification of this question we postpone to the end of the section, and meanwhile we list the properties of the quantum transition matrix, formally following from the definition (2.1.16).

$$1)\quad \mathbb{T}_{x_1}^{x_3}(\lambda) = \mathbb{T}_{x_2}^{x_3}(\lambda)\,\mathbb{T}_{x_1}^{x_2}(\lambda) \quad \text{for} \quad x_1 < x_2 < x_3 \quad \text{or} \quad x_1 > x_2 > x_3 \tag{2.1.17}$$

2) $\mathbb{T}_{x_1}^{x_2}(\lambda)$ has the form

$$\mathbb{T}_{x_1}^{x_2}(\lambda) = \begin{pmatrix} A_{x_1}^{x_2}(\lambda), & \varkappa B^{+}{}_{x_1}^{x_2}(\lambda) \\ B_{x_1}^{x_2}(\lambda), & A^{+}{}_{x_1}^{x_2}(\lambda) \end{pmatrix}, \tag{2.1.18}$$

where the superscript + denotes Hermitian conjugation, and $A_{x_1}^{x_2}(\lambda) = :a_{x_1}^{x_2}(\lambda):$, $B_{x_1}^{x_2}(\lambda) = :b_{x_1}^{x_2}(\lambda):$, $A^{+}{}_{x_1}^{x_2}(\lambda) = \left(A_{x_1}^{x_2}(\bar\lambda)\right)^{+}$, $B^{+}{}_{x_1}^{x_2}(\lambda) = \left(B_{x_1}^{x_2}(\bar\lambda)\right)^{+}$.

3) $\mathbb{T}_{x_1}^{x_2}(\lambda)$ satisfies the differential equations

$$\frac{\partial}{\partial x_2}\mathbb{T}_{x_1}^{x_2}(\lambda) = :L(x_2,\lambda)\,T_{x_1}^{x_2}(\lambda): = \left(-i\frac{\lambda}{2}\sigma_3 + i\varkappa\Psi^{+}(x_2)\sigma_{+}\right)\mathbb{T}_{x_1}^{x_2}(\lambda) - i\sigma_{-}\mathbb{T}_{x_1}^{x_2}(\lambda)\,\Psi(x_2) \tag{2.1.19}$$

and

$$\frac{\partial}{\partial x_1}\mathbb{T}_{x_1}^{x_2}(\lambda) = -:T_{x_1}^{x_2}(\lambda)\,L(x_1,\lambda): = -i\varkappa\Psi^{+}(x_1)\,\mathbb{T}_{x_1}^{x_2}(\lambda)\sigma_{+} - \mathbb{T}_{x_1}^{x_2}(\lambda)\left(-i\frac{\lambda}{2}\sigma_3 - i\sigma_{-}\Psi(x_1)\right) \tag{2.1.20}$$

with the initial condition

$$\mathbb{T}_x^x(\lambda) = \mathrm{I} . \tag{2.1.21}$$

Property 1) follows from the analogous property (1.1.6) for the classical transition matrix and the commutativity of the operators $\Psi(x)$ and $\Psi^+(x)$ on disjoint intervals. We stress that in contrast with the classical case, in the quantum case the condition $x_1<x_2<x_3$ or $x_1>x_2>x_3$ is essential.

Property 2) follows from (1.1.11) for the classical transition matrix and the obvious property of Wick quantization that the complex conjugate of the Wick symbol corresponds to the Hermitian conjugate operator.

In order to prove property 3), we formulate the following simple assertion. Let $\chi(\psi,\bar\psi)$ be a functional of the fields $\psi(x)$ and $\bar\psi(x)$. Then one has

$$\begin{aligned} &:\bar\psi\chi: = \Psi^+ :\chi: , \\ &:\chi\psi: = :\chi: \Psi . \end{aligned} \tag{2.1.22}$$

The proof is obvious.

Now, differentiating $\mathbb{T}_{x_1}^{x_2}(\lambda)$ with respect to $x_2$ or $x_1$, and using (2.1.16), (1.1.3), (1.1.7), and (2.1.22), we get (2.1.19-21).

In order to write (2.1.19) and (2.1.20) more compactly, we introduce the sign of normal arrangement of operator factors $\vdots\,\vdots$. The sign $\vdots\,\vdots$ should not be confused with the sign for Wick quantization $::$, applied here only to classical functionals, the sign $\vdots\,\vdots$, applied to the product of several operator factors (including $\Psi$ and $\Psi^+$), guarantees the arrangement of all $\Psi^+$ on the left, and all $\Psi$ on the right, without altering the order of the remaining factors. For example,

$$\vdots\, X\,\Psi\,\Psi^+ Y \,\vdots = \Psi^+ X Y \Psi . \tag{2.1.23}$$

Using the notation introduced, (2.1.19) and (2.1.20) can be rewritten, respectively, in the form

$$\frac{\partial}{\partial x_2}\mathbb{T}_{x_1}^{x_2}(\lambda) = \vdots\, \mathbb{L}(x_2,\lambda)\,\mathbb{T}_{x_1}^{x_2}(\lambda)\,\vdots \tag{2.1.24}$$

$$\frac{\partial}{\partial x_1}\mathbb{T}_{x_1}^{x_2}(\lambda) = -\vdots\, \mathbb{T}_{x_1}^{x_2}(\lambda)\,\mathbb{L}(x_1,\lambda)\,\vdots\ , \tag{2.1.25}$$

where $\mathbb{L}(x,\lambda)$ is the quantum $\mathbb{L}$-operator

$$\mathbb{L}(x,\lambda) = :L(x,\lambda): \ = -i\frac{\lambda}{2}\sigma_3 - i\sigma_-\Psi(x) + i\varkappa\sigma_+\Psi^+(x) = \begin{pmatrix} -i\frac{\lambda}{2}, & i\varkappa\Psi^+(x) \\ -i\Psi(x), & i\frac{\lambda}{2} \end{pmatrix} . \tag{2.1.26}$$

As also in the classical case, the differential equations (2.1.24) and (2.1.25) with initial condition (2.1.21) are equivalent with the Volterra integral equations

$$\mathbb{T}_{x_1}^{x_2}(\lambda) = \mathrm{I} + \int_{x_1}^{x_2} dx \,\vdots\, \mathbb{L}(x,\lambda)\,\mathbb{T}_{x_1}^{x}(\lambda)\,\vdots \tag{2.1.27}$$

and

$$\mathbb{T}_{x_1}^{x_2}(\lambda) = \mathrm{I} + \int_{x_1}^{x_2} dx \,\vdots\, \mathbb{T}_{x}^{x_2}(\lambda)\, \mathbb{L}(x,\lambda) \,\vdots \tag{2.1.28}$$

or

$$\mathbb{T}_{x_1}^{x_2}(\lambda) = e(x_2 - x_1, \lambda) + \int_{x_1}^{x_2} dx\, e(x_2 - x, \lambda) \,\vdots\, \mathbb{V}(x)\, \mathbb{T}_{x_1}^{x}(\lambda) \,\vdots \tag{2.1.29}$$

and

$$\mathbb{T}_{x_1}^{x_2}(\lambda) = e(x_2 - x_1, \lambda) + \int_{x_1}^{x_2} dx \,\vdots\, \mathbb{T}_{x}^{x_2}(\lambda)\, \mathbb{V}(x) \,\vdots\, e(x - x_1, \lambda), \tag{2.1.30}$$

where

$$\mathbb{V}(x) = \,:\mathbb{V}(x): = i\varkappa \sigma_+ \Psi^+(x) - i\sigma_- \Psi(x) . \tag{2.1.31}$$

Iterating (2.1.29) and (2.1.30), we get for the matrix elements of $\mathbb{T}_{x_1}^{x_2}(\lambda)$ [Eq. (2.1.18)] the following expansions:

$$A_{x_1}^{x_2}(\lambda) = e^{-i\frac{\lambda}{2}(x_2 - x_1)} \Big[ 1 + \sum_{n=1}^{\infty} \varkappa^n \int \ldots \int_{x_2 > \xi_n > \eta_n > \ldots > \eta_1 > x_1} d\xi_1 \ldots d\xi_n\, d\eta_1 \ldots d\eta_n \; e^{i\lambda(\xi_1 + \ldots + \xi_n - \eta_1 - \ldots - \eta_n)}\, \Psi^+(\xi_1) \ldots \Psi^+(\xi_n)\, \Psi(\eta_1) \ldots \Psi(\eta_n) \Big], \tag{2.1.32}$$

$$A_{x_1}^{+x_2}(\lambda) = e^{i\lambda(x_2 - x_1)} \Big[ 1 + \sum_{n=1}^{\infty} \varkappa^n \int \ldots \int_{x_2 > \eta_n > \xi_n > \ldots > \xi_1 > x_1} d\xi_1 \ldots d\xi_n\, d\eta_1 \ldots d\eta_n \; e^{i\lambda(\xi_1 + \ldots + \xi_n - \eta_1 - \ldots - \eta_n)}\, \Psi^+(\xi_1) \ldots \Psi^+(\xi_n)\, \Psi(\eta_1) \ldots \Psi(\eta_n) \Big], \tag{2.1.33}$$

$$B_{x_1}^{x_2}(\lambda) = -ie^{i\frac{\lambda}{2}(x_1 + x_2)} \sum_{n=0}^{\infty} \varkappa^n \int \ldots \int_{x_2 > \eta_{n+1} > \xi_n > \ldots > \eta_1 > x_1} d\xi_1 \ldots d\xi_n\, d\eta_1 \ldots d\eta_{n+1} \; e^{i\lambda(\xi_1 + \ldots + \xi_n - \eta_1 - \ldots - \eta_{n+1})}\, \Psi^+(\xi_1) \ldots \Psi^+(\xi_n)\, \Psi(\eta_1) \ldots \Psi(\eta_{n+1}), \tag{2.1.34}$$

$$B_{x_1}^{+x_2}(\lambda) = ie^{-i\frac{\lambda}{2}(x_1 + x_2)} \sum_{n=0}^{\infty} \varkappa^n \int \ldots \int_{x_2 > \xi_{n+1} > \eta_n > \ldots > \xi_1 > x_1} d\xi_1 \ldots d\xi_{n+1}\, d\eta_1 \ldots d\eta_n \; e^{i\lambda(\xi_1 + \ldots + \xi_{n+1} - \eta_1 - \ldots - \eta_{n+1})}\, \Psi^+(\xi_1) \ldots \Psi^+(\xi_{n+1})\, \Psi(\eta_1) \ldots \Psi(\eta_n), \tag{2.1.35}$$

analogous to the classical expansions (1.1.29) and (1.1.30).

To conclude this section we turn to the question of the propriety of the definition (2.1.16) of the quantum transition matrix $\mathbb{T}_{x_1}^{x_2}(\lambda)$. Unfortunately, the general theorems contained, e.g., in [40], do not give an answer to this question, since too restrictive conditions are imposed on the Wick symbol of the operator (of the type of decreasing as $|\Psi| \to \infty$ or summability). However, the specific construction of $\mathbb{T}_{x_1}^{x_2}(\lambda)$ as a functional of $\Psi(x)$ and $\bar{\Psi}(x)$ essentially simplifies the situation.

We take as a basis the expansions (2.1.32-35). Analysis of (2.1.32-35) shows that the operators $A_{x_1}^{x_2}(\lambda)$ and $A^{+}{}_{x_1}^{x_2}(\lambda)$ do not change the number of particles, $B^{+}{}_{x_1}^{x_2}(\lambda)$ increases it by 1, and lowers it by 1 (annihilating vacuum). Here, in order to define the action of any of these four operators on an $\mathcal{N}$-particle state, it suffices to know only a finite number of first terms of the series (2.1.32-35) (up to terms containing $\mathcal{N}$ annihilating operators, inclusive). Thus, on functions from $F_{\mathcal{N}}$ the matrix elements of $\mathbb{T}_{x_1}^{x_2}(\lambda)$ act as certain integral operators.

We shall not discuss here such properties of these operators as the domain of definition, range of values, etc. For our purposes (i.e., for calculating commutation relations between matrix elements $T^{x_2}_{x_1}(\lambda)$) it will suffice to consider the operators $A^{x_2}_{x_1}(\lambda)$, $A^{+x_2}_{x_1}(\lambda)$, $B^{x_2}_{x_1}(\lambda)$, and $B^{+x_2}_{x_1}(\lambda)$, as formal series (2.1.32-35). The integral equations (2.1.27-30) we shall consider here as compact notation for the series (2.1.32-35).

Some additional information on the properties of the matrix elements of $T^{x_2}_{x_1}(\lambda)$ as operators in the Focke space will be given in Sec. 2.4.

## 2.2. $R$-Matrix

As in the classical case too, our final goal is the definition of the quantum transition matrix $T(\lambda)$ for an infinite interval and the calculation of the commutation relations between its elements. An important intermediate stage here is the calculation of the commutation relations between the matrix elements of $T^{x_2}_{x_1}(\lambda)$. Analogously to Sec. 1.2, it is convenient to introduce the matrices $\tilde{T}^{x_2}_{x_1}(\lambda)$ and $\tilde{\tilde{T}}^{x_2}_{x_1}(\mu)$ by (1.2.4-5). The basic result of the present section is the following

THEOREM 3. The commutation relations $T^{x_2}_{x_1}(\lambda)$ and $T^{x_2}_{x_1}(\mu)$ can be written compactly in the form

$$R(\lambda-\mu)\tilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)\tilde{\tilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)=\tilde{\tilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)\tilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)R(\lambda-\mu), \tag{2.2.1}$$

where

$$R(\lambda)=I_4+ir(\lambda)=I_4-\frac{i\varkappa}{\lambda}\mathcal{P}. \tag{2.2.2}$$

As also in Sec. 1.2, the proof of Theorem 3 is based on the verification of (2.2.1) in infinitesimal form. Here the following lemma will be useful.

LEMMA 2.2.1. The products $\tilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)\tilde{\tilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)$ and $\tilde{\tilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)\tilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)$ satisfy the following differential equations:

$$\frac{\partial}{\partial x_2}\left(\tilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)\tilde{\tilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)\right)=\vdots\,\mathcal{L}(x_2;\lambda,\mu)\tilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)\tilde{\tilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)\,\vdots \tag{2.2.3}$$

and

$$\frac{\partial}{\partial x_2}\left(\tilde{\tilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)\tilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)\right)=\vdots\,\mathcal{L}'(x_2;\lambda,\mu)\tilde{\tilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)\tilde{\mathbb{T}}^{x_2}_{x_1}\,\vdots \tag{2.2.4}$$

and initial condition

$$\tilde{\mathbb{T}}^{x}_{x}(\lambda)\tilde{\tilde{\mathbb{T}}}^{x}_{x}(\mu)=\tilde{\tilde{\mathbb{T}}}^{x}_{x}(\mu)\tilde{\mathbb{T}}^{x}_{x}(\lambda)=I. \tag{2.2.5}$$

The operators $\mathcal{L}(x;\lambda,\mu)$ and $\mathcal{L}'(x;\lambda,\mu)$ in (2.2.3) and (2.2.4) have the following form

$$\mathcal{L}(x;\lambda,\mu)=\tilde{\mathbb{L}}(x,\lambda)+\tilde{\tilde{\mathbb{L}}}(x,\mu)+\varkappa\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+=\begin{pmatrix} -i\frac{\lambda+\mu}{2}, & i\varkappa\Psi^+(x), & i\varkappa\Psi^+(x), & 0 \\ -i\Psi(x), & i\frac{\mu-\lambda}{2}, & 0, & i\varkappa\Psi^+(x) \\ -i\Psi(x), & \varkappa, & i\frac{\lambda-\mu}{2}, & i\varkappa\Psi^+(x) \\ 0, & -i\Psi(x), & -i\Psi(x), & i\frac{\lambda+\mu}{2} \end{pmatrix}, \tag{2.2.6}$$

$$\mathcal{L}'(x;\lambda,\mu) = \mathbb{L}(x,\lambda) + \tilde{\tilde{\mathbb{L}}}(x,\mu) + \varkappa\,\tilde{\sigma}_+\tilde{\tilde{\sigma}}_- = \begin{pmatrix} -i\frac{\lambda+\mu}{2}, & i\varkappa\Psi^+(x), & i\varkappa\Psi^+(x), & 0 \\ -i\Psi(x), & i\frac{\mu-\lambda}{2}, & \varkappa, & i\varkappa\Psi^+(x) \\ -i\Psi(x), & 0, & i\frac{\lambda-\mu}{2}, & i\varkappa\Psi^+(x) \\ 0 & ,-i\Psi(x) & ,-i\Psi(x) & , i\frac{\lambda+\mu}{2} \end{pmatrix}. \tag{2.2.7}$$

We give two proofs of Lemma 2.2.1.

Proof 1. Let $\varepsilon$ be an arbitrary positive number. We differentiate the product $\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)$ with respect to $x_2$. Using (2.1.19), we get

$$\frac{\partial}{\partial x_2}\left(\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\right) = \left(-i\frac{\lambda}{2}\tilde{\sigma}_3 - i\frac{\mu}{2}\tilde{\tilde{\sigma}}_3 + i\varkappa\sigma_+\Psi^+(x_2+\varepsilon)\right)\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu) - i\tilde{\tilde{\sigma}}_-\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\Psi(x_2) +$$
$$+ i\varkappa\tilde{\tilde{\sigma}}_+\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda)\Psi^+(x_2)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu) - i\tilde{\sigma}_-\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda)\Psi(x_2+\varepsilon)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu) = \vdots\left(\tilde{\mathbb{L}}(x_2+\varepsilon,\lambda) + \tilde{\tilde{\mathbb{L}}}(x_2,\mu)\right)\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\vdots\, -$$
$$- i\tilde{\sigma}_-\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda)\left[\Psi(x_2+\varepsilon), \tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\right] + i\varkappa\tilde{\tilde{\sigma}}_+\left[\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda), \Psi^+(x_2)\right]\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu). \tag{2.2.8}$$

In order to get the final answer (2.2.3), we must calculate the commutators in the last two terms on the right side of (2.2.8) and pass to the limit as $\varepsilon \to 0$. The commutator $\left[\Psi(x_2+\varepsilon), \tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\right]$ obviously vanishes, since $\Psi(x_2+\varepsilon)$ commutes with all operators $\Psi(x), \Psi^+(x)$ for $x \in [x_1, x_2]$ since $\varepsilon > 0$. To calculate the second commutator in (2.2.8), we use Lemma 1.5.1 and the following easily verifiable equation,

$$\left[:\chi(\psi,\bar{\psi}):, \Psi^+(x)\right] = :\frac{1}{i}\left\{\chi(\psi,\bar{\psi}), \bar{\psi}(x)\right\}:, \tag{2.2.9}$$

which is valid for any functional $\chi(\psi,\bar{\psi})$ of the fields $\psi(x)$, $\bar{\psi}(x)$. As a result we get

$$\left[\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda), \Psi^+(x_2)\right] = :\int_{x_1}^{x_2+\varepsilon} dx\, \tilde{\mathbb{T}}_x^{x_2+\varepsilon}(\lambda)\frac{1}{i}\left\{\tilde{\mathbb{L}}(x,\lambda), \bar{\Psi}(x_2)\right\}\tilde{\mathbb{T}}_{x_1}^{x}(\lambda):\ \tilde{\mathbb{T}}_{x_2}^{x_2+\varepsilon}(\lambda)\,\varkappa\tilde{\sigma}_-\,\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda). \tag{2.2.10}$$

Substituting (2.2.10) in (2.2.8) and letting $\varepsilon \to 0$, we get the answer required. We note that the result is independent of the sign of $\varepsilon$. For $\varepsilon < 0$ the second commutator in (2.2.8) vanishes, and the first gives the needed summand in (2.2.3). One proves (2.2.4) analogously.

The method of "extension" used in the proof given above we borrow from [30].* This method allows us to avoid consideration of indeterminate expressions of the form $\left[\mathbb{T}_{x_1}^{x_2}(\lambda), \Psi^+(x_2)\right]$ (containing indeterminacies of the type of the product of a function and a discontinuous one, as is easy to see, using, for example, the expansions (1.1.29-30)). However, here one uses implicitly an unproved, generally speaking, proposition about the continuous dependence of $\frac{\partial}{\partial x_2}\left(\tilde{\mathbb{T}}_{x_1}^{x_2+\varepsilon}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\right)$ on $\varepsilon$. Hence we give a second proof of Lemma 2.2.1, more straightforward, although also more complicated.

Proof 2. We substitute into the product $\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)$ the integral equations for $\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)$ and $\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)$ of the form (2.1.27). As a result, we get:

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*The author thanks S. V. Manakov for indicating the possibility of using the method of "extension" in the quantum case.

$$\widetilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)=\mathrm{I}+\int_{x_1}^{x_2}d\xi\Big[\Big(-i\tfrac{\lambda}{2}\tilde{\sigma}_3+i\varkappa\Psi^+(\xi)\tilde{\sigma}_+\Big)\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)-i\tilde{\sigma}_-\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\Psi(\xi)\Big]+$$

$$+\int_{x_1}^{x_2}d\eta\Big[\Big(-i\tfrac{\mu}{2}\tilde{\tilde{\sigma}}_3+i\varkappa\Psi^+(\eta)\tilde{\tilde{\sigma}}_+\Big)\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)-i\tilde{\tilde{\sigma}}_-\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)\Psi(\eta)\Big]+$$

$$+\int_{x_1}^{x_2}d\xi\int_{x_1}^{x_2}d\eta\Big[\Big(-i\tfrac{\lambda}{2}\tilde{\sigma}_3+i\varkappa\Psi^+(\xi)\tilde{\sigma}_+\Big)\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)-i\tilde{\sigma}_-\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\Psi(\xi)\Big]\times$$

$$\times\Big[\Big(-i\tfrac{\mu}{2}\tilde{\tilde{\sigma}}_3+i\varkappa\Psi^+(\eta)\tilde{\tilde{\sigma}}_+\Big)\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)-i\tilde{\tilde{\sigma}}_-\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)\Psi(\eta)\Big]. \tag{2.2.11}$$

Subsequently, complicated, but in terms of ideas entirely transparent, calculations produce as their ultimate goal the reduction, using the commutation relations (2.1.2), of equation (2.2.11) to a Volterra integral equation, equivalent with the differential equation (2.2.3) with initial condition (2.2.5).

We transform the fourth summand in (2.2.11), opening the brackets in the integrand and using the commutation relation (2.1.2), after which it assumes the form

$$\int_{x_1}^{x_2}d\xi\int_{x_1}^{x_2}d\eta\Big[\Big(-i\tfrac{\lambda}{2}\tilde{\sigma}_3+i\varkappa\Psi^+(\xi)\tilde{\sigma}_+\Big)\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\Big(-i\tfrac{\mu}{2}\tilde{\tilde{\sigma}}_3+i\varkappa\Psi^+(\eta)\tilde{\tilde{\sigma}}_+\Big)\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)+$$

$$+\Big(-i\tfrac{\lambda}{2}\tilde{\sigma}_3+i\varkappa\Psi^+(\xi)\tilde{\sigma}_+\Big)\Big(-i\tilde{\tilde{\sigma}}_-\Big)\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)\Psi(\eta)-\tilde{\sigma}_-\tilde{\tilde{\sigma}}_-\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\Psi(\xi)\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)\Psi(\eta)+$$

$$-i\tilde{\sigma}_-\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\Big(-i\tfrac{\mu}{2}\tilde{\tilde{\sigma}}_3+i\varkappa\Psi^+(\eta)\tilde{\tilde{\sigma}}_+\Big)\Psi(\xi)\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)+\varkappa\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\delta(\xi-\eta)\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)\Big]. \tag{2.2.12}$$

We transform (2.2.12) in the following way. Firstly, we integrate the $\delta$-function in the fifth summand. Secondly, we divide the domain of integration in the remaining summands into two parts: $x_1<\xi<\eta<x_2$ and $x_1<\eta<\xi<x_2$. Then, for $\xi<\eta$, in the first and fourth summands we transfer $\Psi^+(\eta)$ to the left, using the fact that $\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)$ commutes with $\Psi^+(\eta)$ for the indicated relation of $\xi$ and $\eta$, and for $\xi>\eta$ analogously we transfer $\Psi(\xi)$ to the right in the third and fourth summands. Then using (2.1.19), we can rewrite (2.2.12) in the form

$$\int_{x_1}^{x_2}d\eta\int_{x_1}^{\eta}d\xi\Big[\Big(-i\tfrac{\mu}{2}\tilde{\tilde{\sigma}}_3+i\varkappa\Psi^+(\eta)\tilde{\tilde{\sigma}}_+\Big)\Big(\tfrac{\partial}{\partial\xi}\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\Big)\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)-i\tilde{\tilde{\sigma}}_-\Big(\tfrac{\partial}{\partial\xi}\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\Big)\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)\Psi(\eta)\Big]+$$

$$+\int_{x_1}^{x_2}d\xi\int_{x_1}^{\xi}d\eta\Big[\Big(-i\tfrac{\lambda}{2}\tilde{\sigma}_3+i\varkappa\Psi^+(\xi)\tilde{\sigma}_+\Big)\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\Big(\tfrac{\partial}{\partial\eta}\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)\Big)-$$

$$-i\tilde{\sigma}_-\widetilde{\mathbb{T}}^{\xi}_{x_1}(\lambda)\Big(\tfrac{\partial}{\partial\eta}\widetilde{\widetilde{\mathbb{T}}}^{\eta}_{x_1}(\mu)\Big)\Psi(\xi)\Big]+\varkappa\int_{x_1}^{x_2}dx\,\tilde{\tilde{\sigma}}_-\tilde{\tilde{\sigma}}_+\widetilde{\mathbb{T}}^{x}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}^{x}_{x_1}(\mu). \tag{2.2.13}$$

Carrying out integration of the total derivatives in (2.2.13) and substituting the result in (2.2.11), we get for $\widetilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)$ the following integral equation

$$\widetilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)=\mathrm{I}+\int_{x_1}^{x_2}dx\,:\Big(\widetilde{\mathbb{L}}(x,\lambda)+\widetilde{\widetilde{\mathbb{L}}}(x,\mu)+\varkappa\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\Big)\widetilde{\mathbb{T}}^{x}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}^{x}_{x_1}(\mu):, \tag{2.2.14}$$

which, obviously, is equivalent with the Cauchy problem (2.2.5) for (2.2.3). The investigation of the product $\widetilde{\widetilde{\mathbb{T}}}^{x_2}_{x_1}(\mu)\widetilde{\mathbb{T}}^{x_2}_{x_1}(\lambda)$ is carried out analogously. Thus, Lemma 2.2.1 is proved.

The proved lemma allows the reduction of the proof of Theorem 3 to the verification of the equation

$$R(\lambda-\mu)\mathcal{L}(x;\lambda,\mu)=\mathcal{L}'(x;\lambda,\mu)R(\lambda-\mu) \ . \tag{2.2.15}$$

In fact, by virtue of (2.2.3-5) and (2.2.15), the quantities $\widetilde{\widetilde{\mathbb{T}}}{}^{x_2}_{x_1}(\mu)\widetilde{\mathbb{T}}{}^{x_2}_{x_1}(\lambda)$ and $R(\lambda-\mu)\widetilde{\mathbb{T}}{}^{x_2}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}{}^{x_2}_{x_1}(\mu)\cdot R^{-1}(\lambda-\mu)$ satisfy the same differential equation with the same initial condition. We note that in establishing this fact it is extraordinarily important that $R$ is a numerical matrix, whose matrix elements commute with the matrix elements of $\mathbb{T}^{x_2}_{x_1}$.

Equation (2.2.15), to which the proof of Theorem 3 is reduced, is easily verified directly.

We discuss in the conclusion of this section the connection of the formula (2.2.1) which we have obtained with the result (1.2.11) of Theorem 1. For this it is convenient to introduce in the commutation relations (2.1.2) the Planck constant $\hbar$:

$$\left[\Psi(x),\Psi^{+}(y)\right]=\hbar\,\delta(x-y) \ . \tag{2.2.16}$$

Then the $R$-matrix assumes the form

$$R(\lambda)=I-\frac{i\varkappa\hbar}{\lambda}\mathcal{P}=I+i\hbar\, r(\lambda) \ . \tag{2.2.17}$$

We shall show that in the quasiclassical limit $\hbar\to 0$ (2.2.1) goes into (1.2.7). In fact, in view of (2.2.17), (1.2.1) can be written in the form

$$\left[\widetilde{\mathbb{T}}{}^{x_2}_{x_1}(\lambda),\widetilde{\widetilde{\mathbb{T}}}{}^{x_2}_{x_1}(\mu)\right]=-i\hbar\left(r(\lambda-\mu)\widetilde{\mathbb{T}}{}^{x_2}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}{}^{x_2}_{x_1}(\mu)-\widetilde{\widetilde{\mathbb{T}}}{}^{x_2}_{x_1}(\mu)\widetilde{\mathbb{T}}{}^{x_2}_{x_1}(\lambda)r(\lambda-\mu)\right). \tag{2.2.18}$$

Using the fact that as $\hbar\to 0$, $\mathbb{T}^{x_2}_{x_1}(\lambda)$ goes into the classical transition matrix $T^{x_2}_{x_1}(\lambda)$, and the commutator goes into the Poisson bracket

$$[\ ,\ ]\longrightarrow -i\hbar\{\ ,\ \} \tag{2.2.19}$$

and retaining in (2.2.18) terms of order $\hbar$, we arrive at (1.2.11).

We note that this result is also valid for $R$-matrices of more general form, for which (2.2.17) is false (see [29]). In the general case it is replaced by the relation

$$R(\lambda)=I+i\hbar r(\lambda)+O(\hbar^2) \tag{2.2.20}$$

or

$$r(\lambda)=-i\frac{\partial}{\partial\hbar}\Big|_{\hbar=0}R(\lambda,\hbar). \tag{2.2.21}$$

## 2.3. Passage to an Infinite Interval

This section is devoted to the derivation of the most important result of the present paper, the commutation relations between the matrix elements of the quantum transition matrix for an infinite interval.

Analogously to the way the quantum transition matrix $\mathbb{T}^{x_2}_{x_1}(\lambda)$ was introduced in Sec. 2.1 for a finite interval, we define quantum transition matrices $\mathbb{T}_{-}(x,\lambda)$, $\mathbb{T}_{+}(x,\lambda)$ for the semi-infinite and $\mathbb{T}(\lambda)$ for the infinite intervals by the formulas

$$\mathbb{T}_{\pm}(x,\lambda)=:\mathbb{T}_{\pm}(x,\lambda): \ , \tag{2.3.1}$$
$$\mathbb{T}(\lambda)=:\mathbb{T}(\lambda): \ .$$

The properties of the matrices $\mathbb{T}_{\pm}(x,\lambda)$ and $\mathbb{T}(\lambda)$ follow directly from (2.3.1), completely analogously to the corresponding properties of the matrix $\mathbb{T}_{x_1}^{x_2}(\lambda)$, established in Sec. 2.1. The matrices $\mathbb{T}_{\pm}(x,\lambda)$ and $\mathbb{T}(\lambda)$ have the same symmetry as $\mathbb{T}_{x_1}^{x_2}(\lambda)$ (for the notation for the matrix elements $\mathbb{T}_{\pm}(x,\lambda)$ and $\mathbb{T}(\lambda)$, see Points 6-8 in the Supplement) $\mathbb{T}_{-}(x,\lambda)$ satisfies the differential equation

$$\frac{\partial}{\partial x}\mathbb{T}_{-}(x,\lambda)=\vdots\ \mathbb{L}(x,\lambda)\,\mathbb{T}_{-}(x,\lambda)\ \vdots\ , \tag{2.3.2}$$

and $\mathbb{T}_{+}(x,\lambda)$ the equation

$$\frac{\partial}{\partial x}\mathbb{T}_{+}(x,\lambda)=-\vdots\ \mathbb{T}_{+}(x,\lambda)\,\mathbb{L}(x,\lambda)\ \vdots\ . \tag{2.3.3}$$

For $\mathbb{T}_{\pm}(x,\lambda)$ and $\mathbb{T}(\lambda)$ the quantum analogs of the integral equations (1.1.23-28) are valid, which we shall write down later insofar as they are needed.

We recall again that for now we are considering the matrix elements of $\mathbb{T}_{\pm}(x,\lambda)$ and $\mathbb{T}(\lambda)$ which are formal series of the form (2.1.32-35) (we shall not write these series down here, since they are obtained from (2.1.32-35) by eliminating $x_1$ and/or $x_2$).

Now we can formulate the basic result of the present paper, Theorem 4.

THEOREM 4. The commutation relations between the matrix elements of the quantum transition matrices $\mathbb{T}_{\pm}(x,\lambda)$ and $\mathbb{T}(\lambda)$ can be written for real $\lambda$ and $\mu$ in the following form:

$$R(\lambda-\mu)\tilde{\mathbb{T}}_{-}(x,\lambda)\tilde{\tilde{\mathbb{T}}}_{-}(x,\mu)\left(1-\frac{i\varkappa}{\lambda-\mu+i0}\,\tilde{\sigma}_{-}\,\tilde{\tilde{\sigma}}_{+}\right)=\tilde{\tilde{\mathbb{T}}}_{-}(x,\mu)\tilde{\mathbb{T}}_{-}(x,\lambda)\left(1+\frac{i\varkappa}{\lambda-\mu-i0}\,\tilde{\sigma}_{+}\,\tilde{\tilde{\sigma}}_{-}\right)R(\lambda-\mu), \tag{2.3.4}$$

$$R(\lambda-\mu)\left(1+\frac{i\varkappa}{\lambda-\mu-i0}\,\tilde{\sigma}_{-}\,\tilde{\tilde{\sigma}}_{+}\right)\tilde{\mathbb{T}}_{+}(x,\lambda)\tilde{\tilde{\mathbb{T}}}_{+}(x,\mu)=\left(1-\frac{i\varkappa}{\lambda-\mu+i0}\,\tilde{\sigma}_{+}\,\tilde{\tilde{\sigma}}_{-}\right)\tilde{\tilde{\mathbb{T}}}_{+}(x,\mu)\tilde{\mathbb{T}}_{+}(x,\lambda)R(\lambda-\mu), \tag{2.3.5}$$

$$\begin{aligned}&R(\lambda-\mu)\left(1+\frac{i\varkappa}{\lambda-\mu-i0}\,\tilde{\sigma}_{-}\,\tilde{\tilde{\sigma}}_{+}\right)\tilde{\mathbb{T}}(\lambda)\tilde{\tilde{\mathbb{T}}}(\mu)\left(1-\frac{i\varkappa}{\lambda-\mu+i0}\,\tilde{\sigma}_{-}\,\tilde{\tilde{\sigma}}_{+}\right)=\\&=\left(1-\frac{i\varkappa}{\lambda-\mu+i0}\,\tilde{\sigma}_{+}\,\tilde{\tilde{\sigma}}_{-}\right)\tilde{\tilde{\mathbb{T}}}(\mu)\tilde{\mathbb{T}}(\lambda)\left(1+\frac{i\varkappa}{\lambda-\mu-i0}\,\tilde{\sigma}_{+}\,\tilde{\tilde{\sigma}}_{-}\right)R(\lambda-\mu).\end{aligned} \tag{2.3.6}$$

Proof. First we prove (2.3.4). The proof will be based on the study of the asymptotic behavior of the products $\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)$ and $\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)$ as $x_1\to-\infty$. Here we shall devote basic attention to the formal-algebraic side, not going into the analytic justifications of our calculations and making it our goal to give as simply and rapidly as possible a method of calculating the desired commutation relations.

It was proved in Sec. 2.2 that the products $\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)$ and $\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)$ satisfy, respectively, the differential equations (2.2.3) and (2.2.4). The operators $\mathcal{L}$ and $\mathcal{L}'$ (2.2.6-7) figuring in (2.2.3) and (2.2.4) do not coincide with the sum of the operators $\tilde{\mathbb{L}}(x,\lambda)$ and $\tilde{\tilde{\mathbb{L}}}(x,\mu)$, as would be so in the classical case, but differ from it by the summands $\varkappa\tilde{\sigma}_{-}\tilde{\tilde{\sigma}}_{+}$ and $\varkappa\tilde{\sigma}_{+}\tilde{\tilde{\sigma}}_{-}$, respectively, arising from the noncommutativity of the quantum operators. In connection with this, in the quantum case in describing the asymptotic behavior of the products $\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)$ and $\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)$ as $x_1\to-\infty$ or $x_2\to+\infty$ the role of the classical matrix $E(x;\lambda,\mu)$ (see Sec. 1.3) will be played, respectively, by the matrices $\mathcal{E}(x;\lambda,\mu)$ and $\mathcal{E}'(x;\lambda,\mu)$ :

$$\mathcal{E}(x;\lambda,\mu) = \exp \mathcal{L}_0(\lambda,\mu)\,x = \begin{pmatrix} e^{-i\frac{\lambda+\mu}{2}x}, & 0, & 0, & 0 \\ 0, & e^{i\frac{\mu-\lambda}{2}x}, & 0, & 0 \\ 0, & 2\varkappa\frac{\sin\frac{\lambda-\mu}{2}x}{\lambda-\mu}, & e^{i\frac{\lambda-\mu}{2}x}, & 0 \\ 0, & 0, & 0, & e^{i\frac{\lambda+\mu}{2}x} \end{pmatrix}, \qquad (2.3.7)$$

$$\mathcal{E}'(x;\lambda,\mu) = \exp \mathcal{L}'_0(\lambda,\mu)\,x = \begin{pmatrix} e^{-i\frac{\lambda+\mu}{2}x}, & 0, & 0, & 0 \\ 0, & e^{i\frac{\mu-\lambda}{2}x}, & 2\varkappa\frac{\sin\frac{\lambda-\mu}{2}x}{\lambda-\mu}, & 0 \\ 0, & 0, & e^{i\frac{\lambda-\mu}{2}x}, & 0 \\ 0, & 0, & 0, & e^{i\frac{\lambda+\mu}{2}x} \end{pmatrix}, \qquad (2.3.8)$$

where $\mathcal{L}_0(\lambda,\mu)$ and $\mathcal{L}'_0(\lambda,\mu)$ are the "asymptotic" (as $|x|\to\infty$ ) values of the operators $\mathcal{L}(x;$ $\lambda,$ and $\mathcal{L}'(x;\lambda,\mu)$, respectively,

$$\mathcal{L}_0(\lambda,\mu) = -i\frac{\lambda}{2}\tilde{\sigma}_3 - i\frac{\mu}{2}\tilde{\tilde{\sigma}}_3 + \varkappa\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+ , \qquad (2.3.9)$$

$$\mathcal{L}'_0(\lambda,\mu) = -i\frac{\lambda}{2}\tilde{\sigma}_3 - i\frac{\mu}{2}\tilde{\tilde{\sigma}}_3 + \varkappa\tilde{\sigma}_+\tilde{\tilde{\sigma}}_- . \qquad (2.3.10)$$

We note that by virtue of (2.2.15) one has

$$R(\lambda-\mu)\,\mathcal{L}_0(\lambda,\mu) = \mathcal{L}'_0(\lambda,\mu)\,R(\lambda-\mu). \qquad (2.3.11)$$

From (2.3.11) in combination with (2.3.7-8) follows the analogous equation for the matrices $\mathcal{E}(x;\lambda,\mu)$ and $\mathcal{E}'(x;\lambda,\mu)$:

$$R(\lambda-\mu)\,\mathcal{E}(x;\lambda,\mu) = \mathcal{E}'(x;\lambda,\mu)\,R(\lambda-\mu). \qquad (2.3.12)$$

Now we concern ourselves with the investigation of the asymptotic behavior of the product $\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)$ as $x_1\to-\infty$ . We note, first of all, that the differential equation (2.2.3), which, one can rewrite using the notation (2.3.9-10) and (2.1.31) in the form

$$\frac{\partial}{\partial x_2}\left(\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\right) = :\left(\mathcal{L}_0(\lambda,\mu) + \tilde{V}(x_2) + \tilde{\tilde{V}}(x_2)\right)\left(\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\right): , \qquad (2.3.13)$$

is equivalent under the initial condition (2.2.5) with the Volterra integral equation

$$\left(\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\right) = \mathcal{E}(x_2-x_1;\lambda,\mu) + \int_{x_1}^{x_2} dx :\left(\tilde{\mathbb{T}}_{x}^{x_2}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x}^{x_2}(\mu)\right)\left(\tilde{V}(x) + \tilde{\tilde{V}}(x)\right): \mathcal{E}(x-x_1;\lambda,\mu). \qquad (2.3.14)$$

We introduce into consideration the limit

$$\mathcal{T}(x;\lambda,\mu) = \lim_{x_1\to-\infty}\left(\tilde{\mathbb{T}}_{x_1}^{x}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x}(\mu)\right)\mathcal{E}(x_1;\lambda,\mu). \qquad (2.3.15)$$

Substituting (2.3.14) in (2.3.15), we get for $\mathcal{T}(x;\lambda,\mu)$ the following integral representation

$$\mathcal{T}(x;\lambda,\mu) = \mathcal{E}(x;\lambda,\mu) + \int_{-\infty}^{x} d\eta :\left(\tilde{\mathbb{T}}_{\eta}^{x}(\lambda)\tilde{\tilde{\mathbb{T}}}_{\eta}^{x}(\mu)\right)\left(\tilde{V}(\eta) + \tilde{\tilde{V}}(\eta)\right): \mathcal{E}(\eta;\lambda,\mu). \qquad (2.3.16)$$

We note that $\mathcal{T}(x;\lambda,\mu)$ as before satisfies the differential equation (2.2.3):

$$\frac{d}{dx}\,\mathcal{T}(x;\lambda,\mu)=\,\vdots\,\mathcal{L}(x;\lambda,\mu)\,\mathcal{T}(x;\lambda,\mu)\,\vdots\;. \tag{2.3.17}$$

On the other hand, arguing exactly as in the proof of Lemma 2.2.1 in Sec. 2.2, one can see that the product $\tilde{\mathbb{T}}_-(x,\lambda)\tilde{\tilde{\mathbb{T}}}_-(x,\mu)$, which we denote by $\mathcal{T}_-(x;\lambda,\mu)$, satisfies the same differential equation

$$\frac{d}{dx}\,\mathcal{T}_-(x;\lambda,\mu)=\,\vdots\,\mathcal{L}(x;\lambda,\mu)\,\mathcal{I}(x;\lambda,\mu)\,\vdots\;. \tag{2.3.18}$$

Consequently, the quantities $\mathcal{T}(x;\lambda,\mu)$ and $\mathcal{I}(x;\lambda,\mu)$ can differ only by some matrix factor $\mathbb{C}(\lambda,\mu)$:

$$\mathcal{I}(x;\lambda,\mu)=\mathcal{T}(x;\lambda,\mu)\,\mathbb{C}(\lambda,\mu)\;. \tag{2.3.19}$$

We note the obvious similarity of our arguments with the arguments made in proving Theorem 2 (Sec. 1.3). As also in Sec. 1.3, we find the matrix $\mathbb{C}(\lambda,\mu)$ , comparing the asymptotics of $\mathcal{T}(x;\lambda,\mu)$ and $\mathcal{I}(x;\lambda,\mu)$ as $x\to-\infty$. The asymptotics of $\mathcal{T}(x;\lambda,\mu)$ as $x\to-\infty$ are easily determined from the integral representation (2.3.16):

$$\mathcal{T}(x;\lambda,\mu)\underset{x\to-\infty}{\sim}\mathcal{E}(x;\lambda,\mu)\;. \tag{2.3.20}$$

It remains to investigate the asymptotics of $\mathcal{I}(x;\lambda,\mu)$. For this we use the quantum analog of the integral representation (1.1.24):

$$\tilde{\mathbb{T}}_-(x,\lambda)=\tilde{e}(x,\lambda)+\int_{-\infty}^{x}d\eta\;\vdots\;\tilde{\mathbb{T}}^{x}_{\eta}(\lambda)\,\tilde{\mathbb{V}}(\eta)\;\vdots\;\tilde{e}(\eta,\lambda) \tag{2.3.21}$$

or

$$\tilde{\tilde{\mathbb{T}}}_-(x,\mu)=\tilde{\tilde{e}}(x,\mu)+\int_{-\infty}^{x}d\eta\;\vdots\;\tilde{\tilde{\mathbb{T}}}^{x}_{\eta}(\mu)\,\tilde{\tilde{\mathbb{V}}}(\eta)\;\vdots\;\tilde{\tilde{e}}(\eta,\mu)\;. \tag{2.3.22}$$

Substituting (2.3.21) and (2.3.22) in the product $\tilde{\mathbb{T}}_-(x,\lambda)\tilde{\tilde{\mathbb{T}}}_-(x,\mu)$, we get

$$\begin{aligned}\mathcal{I}(x;\lambda,\mu)&=\mathrm{E}(x;\lambda,\mu)+\int_{-\infty}^{x}d\eta\;\vdots\;\tilde{\mathbb{T}}^{x}_{\eta}(\lambda)\,\tilde{\mathbb{V}}(\eta)\;\vdots\;\tilde{e}(\eta,\lambda)\,\tilde{\tilde{e}}(x,\mu)+\\
&+\int_{-\infty}^{x}d\eta\;\vdots\;\tilde{\tilde{\mathbb{T}}}^{x}_{\eta}(\mu)\tilde{\tilde{\mathbb{V}}}(\eta)\;\vdots\;\tilde{e}(x,\lambda)\,\tilde{\tilde{e}}(\eta,\mu)+\int_{-\infty}^{x}d\eta_1\int_{-\infty}^{x}d\eta_2\Big[i\varkappa\Psi^{+}(\eta_1)\tilde{\mathbb{T}}^{x}_{\eta_1}(\lambda)\tilde{\sigma}_{+}-i\tilde{\mathbb{T}}^{x}_{\eta_1}(\lambda)\tilde{\sigma}_{-}\Psi(\eta_1)\Big]\times\\
&\times\Big[i\varkappa\Psi^{+}(\eta_2)\tilde{\tilde{\mathbb{T}}}^{x}{}_{\eta_2}(\mu)\tilde{\tilde{\sigma}}_{+}-i\tilde{\tilde{\mathbb{T}}}^{x}{}_{\eta_2}(\mu)\tilde{\tilde{\sigma}}_{-}\Psi(\eta_2)\Big]\tilde{e}(\eta_1,\lambda)\tilde{\tilde{e}}(\eta_2,\mu)\,.\end{aligned} \tag{2.3.23}$$

Here we again use the notation introduced in Sec. 1.3 $\mathrm{E}(x;\lambda,\mu)=\tilde{e}(x,\lambda)\tilde{\tilde{e}}(x,\mu)$.

The fourth summand in (2.3.23) can be transformed completely analogously to the way the corresponding summand in (2.2.11) was transformed. Omitting the corresponding calculations, which coincide almost identically with the chain of calculations (2.2.11-14), we give only the final result:

$$\mathcal{I}(x;\lambda,\mu)=\mathrm{E}(x;\lambda,\mu)+\int_{-\infty}^{x}d\eta\;\vdots\big(\tilde{\mathbb{T}}^{x}_{\eta}(\lambda)\tilde{\tilde{\mathbb{T}}}^{x}_{\eta}(\mu)\big)\big(\tilde{\mathbb{V}}(\eta)+\tilde{\tilde{\mathbb{V}}}(\eta)+\varkappa\,\tilde{\sigma}_{-}\tilde{\tilde{\sigma}}_{+}\big)\vdots\;\mathrm{E}(\eta;\lambda,\mu)\,. \tag{2.3.24}$$

In order to find the asymptotics of $\mathcal{I}(x;\lambda,\mu)$ as $x\to-\infty$, we note that the product $\tilde{\mathbb{T}}^{x}_{\eta}(\lambda)\tilde{\tilde{\mathbb{T}}}^{x}_{\eta}(\mu)$ has the following asymptotics:

$$\widetilde{\mathbb{T}}^{x}_{\eta}(\lambda)\widetilde{\widetilde{\mathbb{T}}}^{x}_{\eta}(\mu) \underset{x,\eta\to-\infty}{\sim} \mathcal{E}(x-\eta;\lambda,\mu) \tag{2.3.25}$$

as $x,\eta\to-\infty$. Formula (2.3.25) follows from (2.3.14). Substituting (2.3.25) in (2.3.24) and discarding terms which decrease as $x\to-\infty$, we get

$$\mathcal{T}_-(x;\lambda,\mu) \underset{x\to-\infty}{\sim} \mathrm{E}(x;\lambda,\mu) + \int_{-\infty}^{x} d\eta\, \mathcal{E}(x-\eta;\lambda,\mu)\,\varkappa\,\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\,\mathrm{E}(\eta;\lambda,\mu)\,. \tag{2.3.26}$$

Calculating the integral in (2.3.26), we arrive at the following result:

$$\mathcal{I}(x;\lambda,\mu) \underset{x\to-\infty}{\sim} \mathrm{E}(x;\lambda,\mu) + \frac{i\varkappa}{\lambda-\mu+i0}\,e^{i\frac{\mu-\lambda}{2}x}\,\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+ = \mathcal{E}(x;\lambda,\mu)\,\mathbb{C}(\lambda,\mu)\,, \tag{2.3.27}$$

where

$$\mathbb{C}(\lambda,\mu) = \mathrm{I}_4 + \frac{i\varkappa}{\lambda-\mu+i0}\,\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\,. \tag{2.3.28}$$

Rewriting (2.3.19) in the form

$$\mathcal{T}(x;\lambda,\mu) = \mathcal{I}(x;\lambda,\mu)\,\mathbb{C}^{-1}(\lambda,\mu) \tag{2.3.29}$$

and using the fact that

$$\mathbb{C}^{-1}(\lambda,\mu) = \mathrm{I}_4 - \frac{i\varkappa}{\lambda-\mu-i0}\,\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+ \tag{2.3.30}$$

(since $\sigma_-^2 = \sigma_+^2 = 0$), and recalling the definition of $\mathcal{T}(x;\lambda,\mu)$ and $\mathcal{I}(x;\lambda,\mu)$, we get finally

$$\lim_{x_1\to-\infty}\left(\widetilde{\mathbb{T}}^{x}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}^{x}_{x_1}(\mu)\right)\mathcal{E}(x;\lambda,\mu) = \widetilde{\mathbb{T}}_-(x,\lambda)\widetilde{\widetilde{\mathbb{T}}}_-(x,\mu)\left(\mathrm{I}_4 - \frac{i\varkappa}{\lambda-\mu+i0}\,\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\right). \tag{2.3.31}$$

The analogous formula for $\widetilde{\widetilde{\mathbb{T}}}^{x}_{x_1}(\mu)\widetilde{\mathbb{T}}^{x}_{x_1}(\lambda)$ is obtained by interchanging in (2.3.31) $\lambda\leftrightarrow\mu$ and $\sim\,\leftrightarrow\,\approx$:

$$\lim_{x_1\to-\infty}\left(\widetilde{\widetilde{\mathbb{T}}}^{x}_{x_1}(\mu)\widetilde{\mathbb{T}}^{x}_{x_1}(\lambda)\right)\mathcal{E}'(x;\lambda,\mu) = \widetilde{\widetilde{\mathbb{T}}}_-(x,\mu)\widetilde{\mathbb{T}}_-(x,\lambda)\left(\mathrm{I}_4 + \frac{i\varkappa}{\lambda-\mu-i0}\,\tilde{\sigma}_+\tilde{\tilde{\sigma}}_-\right). \tag{2.3.32}$$

Now everything is ready for getting the commutation relation (2.3.4). For this, we multiply (2.2.1) on the right by $\mathcal{E}(x_1;\lambda,\mu)$ and use (2.3.12), and we get

$$\mathrm{R}(\lambda-\mu)\widetilde{\mathbb{T}}^{x}_{x_1}(\lambda)\widetilde{\widetilde{\mathbb{T}}}^{x}_{x_1}(\mu)\mathcal{E}(x_1;\lambda,\mu) = \widetilde{\widetilde{\mathbb{T}}}^{x}_{x_1}(\mu)\widetilde{\mathbb{T}}^{x}_{x_1}(\lambda)\mathcal{E}'(x_1;\lambda,\mu)\,\mathrm{R}(\lambda-\mu)\,. \tag{2.3.33}$$

Passing in (2.3.33) to the limit as $x_1\to-\infty$, according to (2.3.31) and (2.3.32) we get (2.3.4).

Equation (2.3.5) is proved completely analogously. Combining (2.3.4) and (2.3.5) and using the obvious equation

$$\mathbb{T}(\lambda) = \mathbb{T}_+(x,\lambda)\,\mathbb{T}_-(x,\lambda)\,, \tag{2.3.34}$$

we get (2.3.6), thus completing the proof of Theorem 4.

We proceed to discuss the results obtained. We note first of all that a calculation, completely analogous to that given at the end of Sec. 2.2, allows us to get in the classical limit of (2.3.4) the formula (1.3.1) and analogous formulas for $\mathrm{T}_+$ and $\mathrm{T}$. Thus, the results of Theorem 4 generalize the results of Theorem 2 to the quantum case.

A summary of the commutation relations between the matrix elements of the matrices $\mathbb{T}_\pm(x,\lambda)$ and $\mathbb{T}(\lambda)$ is given in the Supplement (formulas (S31-48)). In order to show how one gets these formulas, we calculate, for example, the commutation relation between the operators $A(x,\mu)$ and $B_-^+(x,\lambda)$ (formula (S34)). For this we write in (2.3.4) the matrix element found at the intersection of the first row and third column:

$$\left(1-\frac{i\varkappa}{\lambda-\mu}\right)B_-^+(x,\lambda)A_-(x,\mu)=-\frac{i\varkappa}{\lambda-\mu}B_-^+(x,\mu)A_-(x,\lambda)+\frac{i\varkappa}{\lambda-\mu-i0}B_-^+(x,\mu)A_-(x,\lambda)+A_-(x,\mu)B_-^+(x,\lambda). \quad (2.3.35)$$

Regrouping terms, we get

$$A_-(x,\mu)B_-^+(x,\lambda)=B_-^+(x,\lambda)A_-(x,\mu)-\frac{i\varkappa}{\lambda-\mu-i0}B_-^+(x,\mu)A_-(x,\lambda)+\frac{i\varkappa}{\lambda-\mu}\left[B_-^+(x,\mu)A_-(x,\lambda)-B_-^+(x,\lambda)A_-(x,\mu)\right]. \quad (2.3.36)$$

We note that in the denominator of the third term on the right side of (2.3.36) it is unnecessary to regularize for $\lambda=\mu$, since the numerator here vanishes. This means that we can choose the regularization of the denominator arbitrarily, in particular, replace $(\lambda-\mu)^{-1}$ by $(\lambda-\mu-i0)^{-1}$ (see the analogous argument in Sec. 1.3 in connection with (1.3.11)). Then terms containing the product $B_-^+(x,\mu)A_-(x,\lambda)$, are preserved, and we get (S34).

Analogously one also gets the remaining formulas (S31-48). The calculations here, however, turn out to be rather complicated. It turns out that if one is interested in commutation relations only for $\lambda\neq\mu$ then (2.3.4-6) can be essentially simplified.

In fact, for $\lambda\neq\mu$ the regularization of $\pm i0$ in the denominator $(\lambda-\mu)$ is inessential, and we can divide, for example, (2.3.4) on the right by $\left(I-\frac{i\varkappa}{\lambda-\mu+i0}\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\right)$, obtaining here the following equation:

$$R(\lambda-\mu)\tilde{\mathbb{T}}_-(x,\lambda)\tilde{\tilde{\mathbb{T}}}_-(x,\mu)=\tilde{\tilde{\mathbb{T}}}_-(x,\mu)\tilde{\mathbb{T}}_-(x,\lambda)R_0(\lambda-\mu), \quad (2.3.37)$$

where

$$R_0(\lambda)=\left(1+\frac{i\varkappa}{\lambda}\tilde{\sigma}_+\tilde{\tilde{\sigma}}_-\right)R(\lambda)\left(1+\frac{i\varkappa}{\lambda}\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\right)=\begin{pmatrix}1-\frac{i\varkappa}{\lambda}, & 0, & 0, & 0\\ 0, & 1+\frac{\varkappa^2}{\lambda^2}, & 0, & 0\\ 0, & 0, & 1, & 0\\ 0, & 0, & 0, & 1-\frac{i\varkappa}{\lambda}\end{pmatrix}. \quad (2.3.38)$$

We note that without making any preliminary statement about $\lambda\neq\mu$, we would get in (2.3.38) a meaningless product of generalized functions of the form $(\lambda-\mu-i0)^{-1}(\lambda-\mu+i0)^{-1}$. Analogously from (2.3.5) and (2.3.6) one gets

$$R_0(\lambda-\mu)\tilde{\mathbb{T}}_+(x,\lambda)\tilde{\tilde{\mathbb{T}}}_+(x,\mu)=\tilde{\tilde{\mathbb{T}}}_+(x,\mu)\tilde{\mathbb{T}}_+(x,\lambda)R(\lambda-\mu), \quad (2.3.39)$$

$$R_0(\lambda-\mu)\tilde{\mathbb{T}}(\lambda)\tilde{\tilde{\mathbb{T}}}(\mu)=\tilde{\tilde{\mathbb{T}}}(\mu)\tilde{\mathbb{T}}(\lambda)R_0(\lambda-\mu). \quad (2.3.40)$$

Equation (2.3.40) reproduces a result obtained by Faddeev in [16].

We note one interesting thing. Although the commutation relations for the matrix elements of $\mathbb{T}_{\pm}(x,\lambda)$, obtained from (2.3.37) and (2.3.39), are only defined for $\lambda \neq \mu$, we can, using the analytic properties of the matrix elements of the matrices $\mathbb{T}_{\pm}(x,\lambda)$, extend the corresponding commutation relations to the real axis and find thus their proper regularizations for $\lambda = \mu$. We clarify what has been said with an example.

We write the matrix element lying at the intersection of the first row and second column in (2.3.37):

$$\left(1-\frac{i\varkappa}{\lambda-\mu}\right) B_-^+(x,\lambda) A_-(x,\mu) = A_-(x,\mu) B_-^+(x,\lambda). \tag{2.3.41}$$

By virtue of (2.3.1) the operator functions $A_-(x,\mu)$ and $B_-^+(x,\lambda)$ have the same analytic properties as the corresponding classical quantities $a_-(x,\mu)$ and $\bar{b}_-(x,\lambda)$. Thus, (2.3.41) is initially defined for $\operatorname{Im}\mu > 0$ and $\operatorname{Im}\lambda < 0$. When $\lambda$ and $\mu$ leave the real axis we must regularize the denominator $(\lambda-\mu)$ in (2.3.41) in the following way:

$$A_-(x,\mu) B_-^+(x,\lambda) = \left(1-\frac{i\varkappa}{\lambda-\mu-i0}\right) B_-^+(x,\lambda) A_-(x,\mu), \tag{2.3.42}$$

thus getting the proper commutation relation

In the same way from (2.3.37) and (2.3.39) one can reproduce all the commutation relations (S31-42). For the quantum transition matrix $\mathbb{T}(\lambda)$ on the infinite interval, analogously from (2.3.40) one can reproduce the commutation relations (S43-46), i.e., those commutation relations, in which at least one factor admits analytic continuation to the real axis. Exceptions are the commutation relations (S47-48), since the functions $B(\lambda)$ and $B^+(\lambda)$ are defined only for real $\lambda$. These commutation relations can be obtained only from (2.3.6).

The commutation relation (S48) between $B(\lambda)$ and $B^+(\mu)$ deserves special commentary. We write it separately:

$$B(\lambda)B^+(\mu) = \left(1-\frac{i\varkappa}{\lambda-\mu+i0}\right)\left(1-\frac{i\varkappa}{\lambda-\mu-i0}\right)B^+(\mu)B(\lambda) + 2\pi\delta(\lambda-\mu)A^+(\lambda)A(\lambda). \tag{2.3.43}$$

On the right side of (2.3.43) we see, generally speaking, the undefined product of generalized functions $(\lambda-\mu+i0)^{-1}(\lambda-\mu-i0)^{-1}$. This indicates the highly singular operator character of $B(\lambda)$ and $B^+(\mu)$. It turns out, however, that one can be saved from the singularities in the ratio (2.3.46) by regularizing the operators $B(\lambda)$ and $B^+(\mu)$ in a definite way.

Namely, we define operators $\Phi(\lambda)$ and $\Phi^+(\lambda)$ by

$$\begin{aligned} \Phi(\lambda) &= (2\pi A^+(\lambda)A(\lambda))^{-1/2} B(\lambda), \\ \Phi^+(\lambda) &= B^+(\lambda)(2\pi A^+(\lambda)A(\lambda))^{-1/2}, \end{aligned} \tag{2.3.44}$$

and we formulate the following proposition.

Proposition 2.3.1. The operators $\Phi(\lambda)$ and $\Phi^+(\lambda)$ introduced by (2.3.44) satisfy the canonical commutation relations:

$$[\Phi(\lambda), \Phi(\mu)] = [\Phi^+(\lambda), \Phi^+(\mu)] = 0, \quad [\Phi(\lambda), \Phi^+(\mu)] = \delta(\lambda-\mu). \tag{2.3.45}$$

Proof. We note first of all that from (S46)

$$A(\lambda)B^{+}(\mu) = \left(1+\frac{i\varkappa}{\lambda-\mu+i0}\right)B^{+}(\mu)A(\lambda) \tag{2.3.46}$$

and from (S43) follows the analogous relation

$$f(A(\lambda))B^{+}(\mu) = B^{+}(\mu)f\left[\left(1+\frac{i\varkappa}{\lambda-\mu+i0}\right)A(\lambda)\right] \tag{2.3.47}$$

for any analytic function $f(\lambda)$. In fact, from (2.3.46) the validity of (2.3.47) follows directly for polynomial functions $f(\lambda)$, and consequently also for analytic functions $f(\lambda)$, considered as infinite power series in $\lambda$ (we recall that all our arguments carry formal algebraic character). We can also extend (2.3.47) to functions $f(\lambda)$ of the form $f(\lambda)=\lambda^{-1/2}$, by decomposing them into power series near any point $\lambda=\lambda_0\neq 0$.

Analogous arguments allow us to get from (S45) the following relation

$$B(\lambda)f(A(\mu)) = f\left[\left(1-\frac{i\varkappa}{\lambda-\mu-i0}\right)A(\mu)\right]B(\lambda). \tag{2.3.48}$$

Now one can enter upon the proof of Proposition 2.3.1. We derive, e.g., the commutation relation between $\Phi(\lambda)$ and $\Phi^{+}(\mu)$. For this we substitute in the product $\Phi(\lambda)\Phi^{+}(\mu)$ the expression (2.3.44). We get

$$\Phi(\lambda)\Phi^{+}(\mu) = (2\pi)^{-1}(A^{+}(\lambda)A(\lambda))^{-1/2}B(\lambda)B^{+}(\mu)(A^{+}(\mu)A(\mu))^{-1/2}. \tag{2.3.49}$$

Now using (2.3.43),

$$\begin{aligned}\Phi(\lambda)\Phi^{+}(\mu) &= (2\pi)^{-1}(A^{+}(\lambda)A(\lambda))^{-\frac{1}{2}}B(\mu)B(\lambda)(A^{+}(\mu)A(\mu))^{-\frac{1}{2}}\left(1+\frac{i\varkappa}{\lambda-\mu+i0}\right)\left(1-\frac{i\varkappa}{\lambda-\mu-i0}\right) + \\ &+ (2\pi)^{-1}(A^{+}(\lambda)A(\lambda))^{-\frac{1}{2}}(2\pi)A^{+}(\lambda)A(\lambda)\delta(\lambda-\mu)(A^{+}(\mu)A(\mu))^{-1/2}.\end{aligned} \tag{2.3.50}$$

In order to get the answer needed, it remains to transform (2.3.50), using (2.3.47-48) and the commutativity of $A(\lambda)$ and $A^{+}(\mu)$ (Eqs. (S43-44)):

$$\Phi(\lambda)\Phi^{+}(\mu) = (2\pi)^{-1}B^{+}(\mu)(A^{+}(\mu)A(\mu)A^{+}(\lambda)A(\lambda))^{-1/2}B(\lambda) + \delta(\lambda-\mu) = \Phi^{+}(\mu)\Phi(\lambda) + \delta(\lambda-\mu).$$

The remaining relations from (2.3.45) are obtained analogously.

We note to conclude this section that the quantum operators $\Phi(\lambda)$ and $\Phi^{+}(\lambda)$ correspond to the classical variables of action-angle type $\varphi(\lambda)$ and $\bar{\varphi}(\lambda)$ (Eqs. (1.4.8)).

## 2.4. Spectral Decomposition

In the present section we shall show how, using the commutation relations between the matrix elements of the transition matrix $\mathbb{T}(\lambda)$, one can study the spectra of the integrals of motion of the quantum n.S.e.

First, however, we consider the connection of the quantum method of the inverse problem and the Bethe substitution method. This connection is given by the following proposition.

Proposition 2.4.1. The wave function of an $\mathcal{N}$-particle state

$$|k_1,\ldots,k_{\mathcal{N}}\rangle_B = B^{+}(k_1)\ldots B^{+}(k_{\mathcal{N}})|0\rangle$$

coincides with the wave function defined by (2.1.11) for the following choice of coefficients $C_{\ell_1 \ldots \ell_N}$:

$$C^{(B)}_{\ell_1 \ldots \ell_N} = (2\pi)^{N/2} \prod_{r<s} \frac{k_{\ell_r} - k_{\ell_s} + i\varkappa}{k_{\ell_r} - k_{\ell_s}} . \tag{2.4.1}$$

The assertion just formulated was first announced in the author's paper [14]. The proof of this fact available to the author is quite complicated and reduces, essentially, to the direct calculation of the result of the action of a segment of the series (2.1.34) on a wave function of the form (2.1.11). An analogous proof was recently published in [27]. Almost simultaneously with [27] there appeared [23], containing an elegant and short proof of an assertion, equivalent with Proposition 2.4.1. Hence we shall not give here the proof of Proposition 2.4.1, but proceed directly to the discussion of the consequences following from it.

Comparing (2.4.1) with (2.1.13), we see that one has

$$|k_1, \ldots, k_N\rangle_B = (2\pi)^{N/2} \prod_{r<s} \left| \frac{k_r - k_s + i\varkappa}{k_r - k_s} \right| \, |k_1, \ldots, k_N\rangle_{norm} , \tag{2.4.2}$$

where $|k_1, \ldots, k_N\rangle_{norm}$ is the $N$-particle state which is associated by (2.1.4) with the wave function $f_N^{(norm)}(x_1, \ldots, x_N | k_1, \ldots, k_N)$ (Eq. (2.1.11)). Equation (2.4.2) shows that the wave functions generated by operators $B^+(k_j)$ are not normalized on the $\delta$-function. Moreover, the denominators $(k_r - k_s)^{-1}$ make the normalizations of these wave functions so singular that $B^+(\lambda)$ cannot be defined even as a generalized operator-valued function.* This fact allows us to clarify the singular commutation relations (2.4.43).

Now we consider the $N$-particle state generated by the normalized operators $\Phi^+(k_j)$ (Eqs. (2.3.44)):

$$|k_1, \ldots, k_N\rangle = \Phi^+(k_1) \ldots \Phi^+(k_j) |0\rangle . \tag{2.4.3}$$

Substituting (2.3.44) in (2.4.3), we get

$$|k_1, \ldots, k_N\rangle = B^+(k_1)(2\pi A^+(k_1)A(k_1))^{-\frac{1}{2}} \ldots B^+(k_N)(2\pi A^+(k_N)A(k_N))^{-\frac{1}{2}} |0\rangle . \tag{2.4.4}$$

With the help of (2.3.47-48) moving the factor $(2\pi A^+(k_j)A(k_j))^{-1/2}$ to the right in (2.4.4) and using the equation

$$A(k_j)|0\rangle = A^+(k_j)|0\rangle = |0\rangle , \tag{2.4.5}$$

which follows directly from (2.1.32-33), we arrive at

$$|k_1, \ldots, k_N\rangle = \frac{1}{(2\pi)^{N/2}} \prod_{r<s} \left| \frac{k_r - k_s}{k_r - k_s + i\varkappa} \right| \, |k_1, \ldots, k_N\rangle_B , \tag{2.4.6}$$

or by virtue of (2.4.2),

$$|k_1, \ldots, k_N\rangle = |k_1, \ldots, k_N\rangle_{norm} .$$

*A. K. Pogrebkov pointed this out to the author.

Thus, the operators $\Phi^+(k_j)$ give birth to the normalized eigenfunctions of the Hamiltonian (2.1.5).

We proceed now to the consideration of a circle of questions connected with the quantum integrals of motion for (2.1.6).

We shall show that, analogously to the classical case, the role of the generating function of the quantum integrals of motion is played by $\ln A(\lambda)$. In fact, from (S43) follows the commutation relation

$$[\ln A(\lambda), \ln A(\mu)] = 0 . \tag{2.4.7}$$

Moreover, substituting in (2.3.47) $f(\lambda)=\ln\lambda$, we get

$$(\ln A(\lambda))B^+(\mu) = B^+(\mu)\left[\ln A(\lambda) + \ln\left(1+\frac{i\varkappa}{\lambda-\mu+i0}\right)\right] \tag{2.4.8}$$

or

$$[\ln A(\lambda), \Phi^+(\mu)] = \Phi^+(\mu)\ln\left(1+\frac{i\varkappa}{\lambda-\mu+i0}\right) . \tag{2.4.9}$$

We let the operator $\ln A(\lambda)$ act on the $N$-particle state $|k_1,\ldots,k_N>$

$$\ln A(\lambda)\,\Phi^+(k_1)\ldots\Phi^+(k_N)\,|\,0> . \tag{2.4.10}$$

Using (2.4.9), we can move $\ln A(\lambda)$ in (2.4.10) to the right. Noting, in addition, that by virtue of (2.4.5) one has

$$\ln A(\lambda)|0> = 0 , \tag{2.4.11}$$

we arrive at the following result. The state $|k_1,\ldots,k_N>$ is an eigenfunction of the operator $\ln A(\lambda)$:

$$\ln A(\lambda)|k_1,\ldots,k_N> = \sum_{j=1}^{N}\ln\left(1+\frac{i\varkappa}{\lambda-k_j+i0}\right)|k_1,\ldots,k_N> , \tag{2.4.12}$$

where the corresponding eigenvalue is additive with respect to the momenta $k_j$;

Decomposing both sides of (2.4.12) in powers of $\lambda^{-1}$, we get that the state $|k_1,\ldots,k_N>$ is proper also for the operators $A_m$, defined as coefficients of the expansion

$$\ln A(\lambda) = i\varkappa\sum_{m=1}^{\infty} A_m\lambda^{-m} . \tag{2.4.13}$$

The corresponding eigenvalues $c_m(k)$

$$A_m|k_1,\ldots,k_N> = \sum_{j=1}^{N} c_m(k_j)|k_1,\ldots,k_N> \tag{2.4.14}$$

are defined from the expansion

$$\ln\left(1+\frac{i\varkappa}{\lambda-k}\right) = i\varkappa\sum_{m=1}^{\infty} c_m(k)\lambda^{-m} \tag{2.4.15}$$

and have the form

$$c_m(k) = \frac{k^m-(k-i\varkappa)^m}{im\varkappa} = \sum_{s=1}^{m}\frac{(m-1)!}{s!(m-s)!}(-i\varkappa)^{s-1}k^{m-s+1} . \tag{2.4.16}$$

Unfortunately, for the quantum case a method of calculating the operators $A_m$, analogous to the method of the Riccati equation (1.4.2-4) in the classical case, is still unknown. Hence, in order to connect the operators $A_m$ with the local integrals of motion for (2.1.6), we have to use a result of Tsvetkov [38]. As shown in [38], the classical integrals of motion $\mathcal{I}_m$ (1.4.2) for (1.1.1) after Wick quantization become quantum self-adjoint operators $\mathbb{I}_m$ in the space $\mathbb{F}$:

$$\mathbb{I}_m = \; : \mathcal{I}_m : \tag{2.4.17}$$

commuting with the Hamiltonian $\mathbb{H}$. In particular,

$$\mathbb{I}_1 = \; : N : = \mathbb{N} \; , \tag{2.4.18}$$

$$\mathbb{I}_2 = \; : P : = \mathbb{P} \; , \tag{2.4.19}$$

$$\mathbb{I}_3 = \; : H : = \mathbb{H} \; . \tag{2.4.20}$$

(For the definition of $\mathbb{N}, \mathbb{P}$, and $\mathbb{H}$, see Eqs. (2.1.5, 7, 8).)

The eigenvalues of the operators $\mathbb{I}_m$ on the states $|k_1,\dots,k_N>$ have the form

$$\mathbb{I}_m | k_1,\dots,k_N> = \sum_{j=1}^{N} k_j^m \, | k_1,\dots,k_N> . \tag{2.4.21}$$

Comparing (2.4.21) and (2.4.14), we get the relation

$$A_m = \sum_{s=1}^{m} \frac{(m-1)!}{s!(m-s)!} (-i\varkappa)^{s-1} \mathbb{I}_{m-s+1} \; . \tag{2.4.22}$$

In particular,

$$A_1 = \mathbb{I}_1 \; , \tag{2.4.23}$$

$$A_2 = \mathbb{I}_2 - \frac{i\varkappa}{2} \mathbb{I}_1 \; , \tag{2.4.24}$$

$$A_3 = \mathbb{I}_3 - i\varkappa \mathbb{I}_2 - \frac{\varkappa^2}{3} \mathbb{I}_1 \; . \tag{2.4.25}$$

Formulas (2.4.23-25) allow one to express $\mathbb{N}, \mathbb{P}$, and $\mathbb{H}$ in terms of $A_1, A_2$, and $A_3$:

$$\mathbb{N} = A_1 \; , \tag{2.4.26}$$

$$\mathbb{P} = A_2 + \frac{i\varkappa}{2} A_1 \; , \tag{2.4.27}$$

$$\mathbb{H} = A_3 + i\varkappa A_2 - \frac{\varkappa^2}{6} A_1 \; . \tag{2.4.28}$$

For positive values of the connection constant $\varkappa$ (the case of repulsion), the states $| k_1,\dots,k_N>$ ($N = 0,1,2\dots$), as indicated in Sec. 2.1, form a complete system of eigenfunctions of $\mathbb{H}$ in the space $\mathbb{F}$. This fact, and also the additivity of the eigenvalues of $\ln A(\lambda)$ in (2.4.12), allow us to write for the generating function of the quantum integrals of motion $\mathbb{I}_m$ the following spectral decomposition:

$$\ln A(\lambda) = \int_{-\infty}^{\infty} d\mu \ln\left(1 + \frac{i\varkappa}{\lambda-\mu}\right) \Phi^+(\mu)\Phi(\mu), \quad \operatorname{Im}\lambda > 0. \tag{2.4.29}$$

The analogous decompositions for $\mathbb{N}, \mathbb{P}$, and $\mathbb{H}$ have the form

$$\mathbb{N}=\int_{-\infty}^{\infty} d\mu\ \Phi^{+}(\mu)\Phi(\mu), \tag{2.4.30}$$

$$\mathbb{P}=\int_{-\infty}^{\infty} d\mu\cdot\mu\ \Phi^{+}(\mu)\Phi(\mu), \tag{2.4.31}$$

$$\mathbb{H}=\int_{-\infty}^{\infty} d\mu\cdot\mu^{2}\Phi^{+}(\mu)\Phi(\mu). \tag{2.4.32}$$

Equations (2.4.30-32) show that $\Phi^{+}(\mu)$ and $\Phi(\mu)$ are operators of birth and annihilation of elementary particles with momentum $\mu$ and energy $\mu^2$. The operator $\Phi^{+}(\mu)\Phi(\mu)$ can here be interpreted as the operator of the density of the number of particles with momentum $\mu$. We note the obvious similarity of (2.4.29-32) with (1.4.10, 12-14) for the classical n.S.e.

To conclude this section, we discuss the case of attraction ($\varkappa<0$). Here, as noted in Sec. 2.1, in the spectrum there appear connected states, which can be obtained from the scattering states $|k_1,\ldots,k_N\rangle$ by analytic continuation with respect to the momenta (2.1.15).

We calculate the eigenvalues of the integrals of motion for the connected states. This, of course, can be done by simply substituting (2.1.15) in (2.4.21), but we choose another method of calculation, which allows us at the same time to get interesting deductions of general character.

First, we find the eigenvalue of the operator $A(\lambda)$ on the state $|k_1,\ldots,k_N\rangle$. This is easy to do, letting the operator $A(\lambda)$ act on the expression $\Phi^{+}(k_1)\ldots\Phi^{+}(k_N)|0\rangle$ and moving $A(\lambda)$ to the right with the help of (S46). As a result we have

$$A(\lambda)|k_1,\ldots,k_N\rangle=\prod_{j=1}^{N}\frac{\lambda-k_j+i\varkappa}{\lambda-k_j}|k_1,\ldots,k_N\rangle,\ \mathrm{Im}\,\lambda>0\ . \tag{2.4.33}$$

We note two things in connection with (2.4.33). Firstly, the eigenvalues of $A(\lambda)$ are multiplicative (hence additive as eigenvalues of $\ln A(\lambda)$) in the momenta $k_j$. Secondly, the eigenvalue $\prod_{j=1}^{N}\frac{\lambda-k_j+i\varkappa}{\lambda-k_j}$ has in the upper half-plane with respect to $\lambda$ exactly $N$ zeros $\lambda=k_j-i\varkappa=k_j+i|\varkappa|$ (we recall that we are considering the case $\varkappa<0$).

An eigenvalue of the operator $A(\lambda)$ on a connected state of $N$ particles is obtained from (2.4.33) by analytic continuation of (2.1.15) with respect to the momenta $k_j$. Here in the product

$$\prod_{j=1}^{N}\frac{\lambda-k_j-i|\varkappa|}{\lambda-k_j}=\frac{\lambda-\frac{P}{N}+i|\varkappa|\frac{N-3}{2}}{\lambda-\frac{P}{N}+i|\varkappa|\frac{N-1}{2}}\cdot\frac{\lambda-\frac{P}{N}+i|\varkappa|\frac{N-5}{2}}{\lambda-\frac{P}{N}+i|\varkappa|\frac{N-3}{2}}\cdot\ldots\cdot\frac{\lambda-\frac{P}{N}-i|\varkappa|\frac{N+1}{2}}{\lambda-\frac{P}{N}-i|\varkappa|\frac{N-1}{2}} \tag{2.4.34}$$

there occurs consecutive cancellation of numerators and denominators, and as a result there remains the factor

$$\frac{\lambda-\frac{P}{N}-i|\varkappa|\frac{N+1}{2}}{\lambda-\frac{P}{N}+i|\varkappa|\frac{N-1}{2}} \tag{2.4.35}$$

having a unique zero in the upper half-plane with respect to $\lambda$ at the point $\lambda=\frac{P}{N}+i|\varkappa|\frac{N+1}{2}$. It is interesting to note that on the other hand (2.1.15) can be obtained, by requiring that the eigenvalue $A(\lambda)$ have a unique zero in the half-plane $\operatorname{Im}\lambda > 0$ and that the momenta $k_j$ be distributed symmetrically with respect to the real axis. In fact, the condition of cancellation of numerators and denominators in (2.4.34) leads to the requirement of equidistance of momenta $k: k_{j+1}-k_j = i|\varkappa|$, which in combination with the requirement of symmetry $k_{\mathcal{N}} = \bar{k}_1$ gives (2.1.15).

This result has interesting analogs in the theory of the classical nonlinear Schrödinger equation. It is known [11, 41], that connected states of quantum particles correspond in the classical limit to solutions for (1.1.1). The classical coefficient of passage $a(\lambda)$ to a one-soliton solution, characterized by the momentum $P$ and the number of particles $\mathcal{N}$, has the form

$$a(\lambda) = \frac{\lambda - \frac{P}{\mathcal{N}} - i|\varkappa|\frac{\mathcal{N}}{2}}{\lambda - \frac{P}{\mathcal{N}} + i|\varkappa|\frac{\mathcal{N}}{2}} \quad . \tag{2.4.36}$$

Comparing (2.4.36) and (2.4.35), we see that they coincide up to the translation $\lambda \to \lambda - i\frac{|\varkappa|}{2}$, which in the quasiclassical limit is inessential.

The eigenvalues $c_m^{(\mathcal{N})}(p)$ of the integrals of motion $A_m$ on an $\mathcal{N}$-particle connected state are obtained, as earlier, by the expansion of the generating function

$$\ln \frac{\lambda - \frac{P}{\mathcal{N}} - i|\varkappa|\frac{\mathcal{N}+1}{2}}{\lambda - \frac{P}{\mathcal{N}} + i|\varkappa|\frac{\mathcal{N}-1}{2}} = -i|\varkappa| \sum_{m=1}^{\infty} c_m^{(\mathcal{N})}(p)\lambda^{-m}, \tag{2.4.37}$$

$$c_m^{(\mathcal{N})}(p) = \frac{i}{m|\varkappa|}\left[\left(\frac{P}{\mathcal{N}} - i|\varkappa|\frac{\mathcal{N}-1}{2}\right)^m - \left(\frac{P}{\mathcal{N}} + i|\varkappa|\frac{\mathcal{N}+1}{2}\right)^m\right], \tag{2.4.38}$$

$$c_m^{(1)}(p) = c_m(p) \; .$$

Using (2.4.26-28) it is easy to get the eigenvalues of the integrals of motion $\mathbb{N}, \mathbb{P}$, and $\mathbb{H}$ on an $\mathcal{N}$-particle connected state $|p,\mathcal{N}>$:

$$|p,\mathcal{N}> = |k_1,\dots,k_{\mathcal{N}}>, \; k_j = \frac{P}{\mathcal{N}} + i|\varkappa|\left(j - \frac{\mathcal{N}+1}{2}\right), \quad j = 1,\dots,\mathcal{N} \; . \tag{2.4.39}$$

These eigenvalues have the form

$$\mathbb{N}|p,\mathcal{N}> = \mathcal{N}|p,\mathcal{N}>, \tag{2.4.40}$$

$$\mathbb{P}|p,\mathcal{N}> = p|p,\mathcal{N}>, \tag{2.4.41}$$

$$\mathbb{H}|p,\mathcal{N}> = \left(\frac{p^2}{\mathcal{N}} - \frac{\varkappa^2}{12}(\mathcal{N}^3 - \mathcal{N})\right)|p,\mathcal{N}> . \tag{2.4.42}$$

Unfortunately, we still do not have available a method based on the quantum method of the inverse problem for constructing the canonical operators $\Phi_{\mathcal{N}}^+(p)$ and $\Phi_{\mathcal{N}}(p)$ of birth and annihilation of normalized connected states of $\mathcal{N}$ particles with total momentum $p$. If, however, one admits that such operators are constructed, then the proper generalization of the spectral decompositions (2.4.29-32) to the case $\varkappa<0$ must assume the form:

$$\ln A(\lambda) = \sum_{\mathcal{N}=1}^{\infty} \int_{-\infty}^{\infty} dp \ln \frac{\lambda - \frac{P}{\mathcal{N}} - i|\varkappa|\frac{\mathcal{N}+1}{2}}{\lambda - \frac{P}{\mathcal{N}} + i|\varkappa|\frac{\mathcal{N}-1}{2}} \Phi_{\mathcal{N}}^+(p)\Phi_{\mathcal{N}}(p), \tag{2.4.29'}$$

$$\mathbb{N}=\sum_{\mathcal{N}=1}^{\infty}\int_{-\infty}^{\infty} dp\,\mathcal{N}\,\Phi^{+}_{\mathcal{N}}(p)\Phi_{\mathcal{N}}(p)\;, \tag{2.4.30'}$$

$$\mathbb{P}=\sum_{\mathcal{N}=1}^{\infty}\int_{-\infty}^{\infty} dp\cdot p\,\Phi^{+}_{\mathcal{N}}(p)\Phi_{\mathcal{N}}(p)\,, \tag{2.4.31'}$$

$$\mathbb{H}=\sum_{\mathcal{N}=1}^{\infty}\int_{-\infty}^{\infty} dp\Big(\frac{p^2}{\mathcal{N}}-\frac{\varkappa^2}{12}(\mathcal{N}^3-\mathcal{N})\Big)\,\Phi^{+}_{\mathcal{N}}(p)\Phi_{\mathcal{N}}(p). \tag{2.4.32'}$$

## 2.5. Quantum $\mathbb{M}$-Operator

All arguments of the preceding sections were carried out for a fixed moment of time $t$. The introduction of temporal evolution does not present any difficulty in quantum mechanics. In fact, the solution of the Heisenberg equation of motion

$$X_t = i\,[X,\mathbb{H}] \tag{2.5.1}$$

for any observable quantity $X$ is given by the formula

$$X(t)=e^{i\mathbb{H}t}\,X(0)\,e^{-i\mathbb{H}t}\;. \tag{2.5.2}$$

In particular, for matrix elements of the quantum transition matrix $\mathbb{T}(\lambda)$, we get, using the commutation relations

$$[A(\lambda),\,\mathbb{H}] = 0\,, \tag{2.5.3}$$

$$[B^{+}(\lambda),\mathbb{H}] = \lambda^2 B^{+}(\lambda) \tag{2.5.4}$$

the following result

$$A(t,\lambda)=A(0,\lambda)\;, \tag{2.5.5}$$

$$B^{+}(t,\lambda)=e^{i\lambda^2 t}\,B^{+}(0,\lambda)\;. \tag{2.5.6}$$

Nevertheless, there is definite methodological interest in the following question: Does there exist in the quantum case an operator $\mathbb{M}(x,\lambda)$, allowing one to describe the temporal evolution of the transition matrix $\mathbb{T}^{x_2}_{x_1}(\lambda)$, analogous to the operator $M$ in the classical case (Sec. 1.5)?

A positive answer to this question is given by the following proposition.

Proposition 2.5.1. The Heisenberg equation of motion for the quantum transition matrix on a finite interval

$$\frac{d}{dt}\,\mathbb{T}^{x_2}_{x_1}(\lambda) = i\,[\,\mathbb{H}\,,\mathbb{T}^{x_2}_{x_1}(\lambda)]$$

can be represented in the form analogous to (1.5.24):

$$\frac{d}{dt}\mathbb{T}^{x_2}_{x_1}(\lambda)=\,:\mathbb{M}(x_2,\lambda)\mathbb{T}^{x_2}_{x_1}(\lambda)-\mathbb{T}^{x_2}_{x_1}(\lambda)\mathbb{M}(x_1,\lambda):\;, \tag{2.5.7}$$

where the operator $\mathbb{M}(x,\lambda)$ has the form

$$\mathbb{M}(x,\lambda)=\,:M(x,\lambda):\;\;\Big(i\frac{\lambda^2}{2}+i\varkappa\Psi^{+}(x)\Psi(x)\Big)\sigma_3+\varkappa\big(\Psi^{+}_x(x)-i\lambda\Psi^{+}(x)\big)\sigma_+ +\big(\Psi_x(x)+i\lambda\Psi(x)\big)\sigma_-\;.$$

Proof. We denote the commutator $i[\mathbb{H},\mathbb{T}^{x_2}_{x_1}(\lambda)]$ by the symbol $\mathcal{M}^{x_2}_{x_1}(\lambda)$ and we find the differential equation with respect to the variable $x_2$, to which this quantity is subordinate.

For this we differentiate $\mathcal{M}_{x_1}^{x_2}(\lambda)$ with respect to $x_2$, using (2.1.24). We get

$$\frac{\partial}{\partial x_2}\mathcal{M}_{x_1}^{x_2}(\lambda) = i\left[\mathbb{H}, \vdots\, \mathbb{L}(x_2,\lambda)\mathbb{T}_{x_1}^{x_2}(\lambda)\,\vdots\,\right] =$$

$$= \vdots\, \mathbb{L}(x_2,\lambda)\mathcal{M}_{x_1}^{x_2}(\lambda)\,\vdots - i\sigma_-\mathbb{T}_{x_1}^{x_2}(\lambda)\left[i\mathbb{H}, \Psi(x_2)\right] + i\varkappa\sigma_+\left[i\mathbb{H}, \Psi^+(x_2)\right]\mathbb{T}_{x_1}^{x_2}(\lambda). \tag{2.5.8}$$

Using the equation of motion (2.1.6) for $\Psi(x)$ and its conjugate equation, we reduce (2.5.8) to the following form

$$\frac{\partial}{\partial x_2}\mathcal{M}_{x_1}^{x_2}(\lambda) = \vdots\, \mathbb{L}(x_2,\lambda)\mathcal{M}_{x_1}^{x_2}(\lambda)\,\vdots\; - i\sigma_-\mathbb{T}_{x_1}^{x_2}(\lambda)\left(i\Psi_{xx}(x_2) - 2i\varkappa\Psi^+(x_2)\Psi(x_2)\Psi(x_2)\right) +$$
$$+ i\varkappa\sigma_+\left(-i\Psi^+_{xx}(x_2) + 2i\varkappa\Psi^+(x_2)\Psi^+(x_2)\Psi(x_2)\right)\mathbb{T}_{x_1}^{x_2}(\lambda). \tag{2.5.9}$$

Before moving further, we formulate the following lemma.

LEMMA 2.5.1. One has the equation:

$$\sigma_-\left[\mathbb{T}_{x_1}^{x_2}(\lambda), \Psi^+(x_2)\right] = \sigma_+\left[\Psi(x_2), \mathbb{T}_{x_1}^{x_2}(\lambda)\right] = 0. \tag{2.5.10}$$

We shall not give the proof of Lemma 2.5.1, since it is carried out with the help of the same method of "extension" which was used in Proof 1 of Lemma 2.2.1 in Sec. 2.2. Here by virtue of the equations $\sigma_-^2 = \sigma_+^2 = 0$ the result, as also in Sec. 2.2, is independent of the sign.

Using Lemma 2.5.1, we transform (2.5.9), moving $\Psi^+(x_2)$ to the right, and $\Psi(x_2)$ to the left. We get:

$$\frac{\partial}{\partial x_2}\mathcal{M}_{x_1}^{x_2}(\lambda) = \vdots\, \mathbb{L}(x_2,\lambda)\mathcal{M}_{x_1}^{x_2}(\lambda)\,\vdots + \sigma_-\mathbb{T}_{x_1}^{x_2}(\lambda)\Psi_{xx}(x_2) - 2\varkappa\sigma_-\Psi^+(x_2)\mathbb{T}_{x_1}^{x_2}(\lambda)\Psi(x_2)\Psi(x_2) +$$
$$+ \varkappa\sigma_+\Psi^+_{xx}(x_2)\mathbb{T}_{x_1}^{x_2}(\lambda) - 2\varkappa^2\sigma_+\Psi^+(x_2)\Psi^+(x_2)\mathbb{T}_{x_1}^{x_2}(\lambda)\Psi(x_2) =$$
$$= \vdots\, \mathbb{L}(x_2,\lambda)\mathcal{M}_{x_1}^{x_2}(\lambda)\,\vdots + :\left(M_x(x_2,\lambda) + \left[M(x_2,\lambda), L(x_2,\lambda)\right]\right)\mathrm{T}_{x_1}^{x_2}(\lambda): \tag{2.5.11}$$

On the other hand, the right side of (2.5.7), which we denote by $\mathcal{M}'^{x_2}_{x_1}(\lambda)$, satisfies exactly the same differential equation. In fact, differentiating the right side of (2.5.7) with respect to $x_2$ and using (1.5.28), we get

$$\frac{\partial}{\partial x_2}\mathcal{M}'^{x_2}_{x_1}(\lambda) = \frac{\partial}{\partial x_2}:\mathbf{M}(x_2,\lambda)\mathbb{T}_{x_1}^{x_2}(\lambda) - \mathbb{T}_{x_1}^{x_2}(\lambda)\mathbf{M}(x_1,\lambda)\,\vdots =$$
$$= \frac{\partial}{\partial x_2}:\mathrm{M}(x_2,\lambda)\mathrm{T}_{x_1}^{x_2}(\lambda) - \mathrm{T}_{x_1}^{x_2}(\lambda)\mathrm{M}(x_1,\lambda): \; = \vdots\, \mathbb{L}(x_2,\lambda)\mathcal{M}'^{x_2}_{x_1}(\lambda)\,\vdots + :\left(M_x(x_2,\lambda) + \left[M(x_2,\lambda), L(x_2,\lambda)\right]\right)\mathrm{T}_{x_1}^{x_2}(\lambda): \tag{2.5.12}$$

Since the quantities $\mathcal{M}_{x_1}^{x_2}(\lambda)$ and $\mathcal{M}'^{x_2}_{x_1}(\lambda)$ satisfy the same differential equation and the same initial condition

$$\mathcal{M}_{x_1}^{x_1}(\lambda) = \mathcal{M}'^{x_1}_{x_1}(\lambda) = 0, \tag{2.5.13}$$

we conclude that they in fact coincide, which is what had to be proved.

Analogously, one can prove the quantum analogs of (1.5.25-27).

## CONCLUSIONS

We summarize. In the present paper, for the example of the nonlinear Schrödinger equation we developed a new method of exact quantization of completely integrable field-theoretic models. This method allowed not only the reproduction of known results for the quantum nonlinear Schrödinger equation, obtained earlier with the help of the Bethe substitution, but also getting a series of new results, namely, constructing a generating function for the quantum integrals of motion and operators of birth—annihilation of elementary excitations. In comparison with the method of Bethe substitution our method has the advantage that it allows one to construct and study eigenvectors of the Hamiltonian by a purely algebraic method, without writing down the explicitly corresponding wave functions in a coordinate representation.

A central role in the method we propose is played, as we saw, by the $\mathrm{R}$-matrix, which gives its name to the method. The use of the $\mathrm{R}$-matrix allows us compactly and effectively to calculate the commutation relations between the matrix elements of the quantum transition matrix, without resorting to infinite series, as was done, e.g., in [22, 26], which appeared after the author's paper [14].

We list some problems concerning the quantum nonlinear Schrödinger equation, which still remain unsolved:

1) It would be desirable to find an effective method of construction of quantum integrals of motion analogous to the Riccati equation in the classical case. This would allow one to definitively free oneself in studying quantum integrals of motion from references to results obtained with the help of Bethe substitution.

2) To construct in the realms of the method of the $\mathrm{R}$-matrix operators of birth and annihilation of connected states of $\mathcal{N}$ particles $\Phi^{+}_{\mathcal{N}}(k)$ and $\Phi_{\mathcal{N}}(k)$.

3) To construct the generating function of the quantum $\mathbf{M}$-operators analogous to the way this was done for the classical case in Sec. 1.5.

After the publication of [13, 14], problems connected with the quantum generalization of the method of the inverse problem attracted the attention of a large number of investigators, both in the USSR and abroad. In the Soviet Union work on the quantum method of the inverse problem was conducted at the Leningrad Branch of the Mathematics Institute (LOMI) under the direction of Faddeev [13-21]. Of the foreign authors one should single out Thacker (USA, Batavia) [22-25] and Honerkamp (GFR, Freiburg) [26, 27].

We list the basic directions in which the quantum method of the inverse problem is developing at the present time:

1) Quantum relativistically invariant completely integrable models [17], in which the method of the $\mathrm{R}$-matrix was successfully applied to the quantization of the sin-Gordon equation.

2) The study of completely integrable lattice spin models, such as the Heisenberg ferromagnet [18] and the $XYZ$-model [19].

3) The investigation of models with several kinds of particles, having isotopic symmetry [20, 21].

4) And, finally, the very long-range direction, intensively developed recently — the attempt to solve the inverse scattering problem for the auxiliary linear equation, i.e., to express the field operators, for example, $\Psi(x)$ and $\Psi^{+}(x)$, for the n.S.e. in terms of the

scattering data $A(\lambda)$ and $B^{+}(\lambda)$. The solution of this problem is of great interest for quantum field theory, since it would allow the effective study of Green's functions of completely integrable quantum field systems. Some results in this direction are obtained in [23-25, 28] for the nonlinear Schrödinger equation.

In conclusion, it is necessary to mention the classical version of the $R$-matrix method developed in Chap. I of the present paper. This method allowed one not only to simplify the calculations connected with the computation of the Poisson brackets, but also to get a new result such as the expression for the generating function of the $M$-operators. Thus, in the theory of classical completely integrable equations there arises a new object, the $r$-matrix. The place which the $r$-matrix occupies in the method of the inverse problem is still not entirely clear. One does not know, e.g., the precise class of $L$-operators, which have a $r$-matrix. In connection with this there is great interest in the problem of generalizing the method of the $r$-matrix to nonultralocal $L$-operators in the terminology of Faddeev [16], i.e., $L$-operators, the Poisson brackets between whose matrix elements contain derivatives of the $\delta$-function.

## SUPPLEMENT

In the Supplement we gather together the Poisson brackets (in the classical case) and commutation relations (in the quantum case) between the matrix elements of the transition matrices for finite, semi-infinite, and infinite intervals. All the formulas are written for real values of $\lambda$ and $\mu$.

1. Summary of Poisson brackets between matrix elements of the classical transition matrix $T_{x_1}^{x_2}(\lambda)$ for the finite interval $[x_1, x_2]$.

We recall that the matrix $T_{x_1}^{x_2}(\lambda)$ has the form (1.1.11):

$$T_{x_1}^{x_2}(\lambda)=\begin{pmatrix} a_{x_1}^{x_2}(\lambda), & \varkappa \bar{b}_{x_1}^{x_2}(\lambda) \\ b_{x_1}^{x_2}(\lambda), & \bar{a}_{x_1}^{x_2}(\lambda) \end{pmatrix}, \quad \lambda=\bar{\lambda}.$$

The desired Poisson brackets are given by (1.2.11):

$$\{\tilde{T}_{x_1}^{x_2}(\lambda), \tilde{\tilde{T}}_{x_1}^{x_2}(\mu)\} = [r(\lambda-\mu), \tilde{T}_{x_1}^{x_2}(\lambda)\tilde{\tilde{T}}_{x_1}^{x_2}(\mu)].$$

Below are written six independent matrix elements of (1.2.11) of the 16 possible ones.

$$\{a_{x_1}^{x_2}(\lambda), a_{x_1}^{x_2}(\mu)\}=0, \tag{S1}$$
$$\{a_{x_1}^{x_2}(\lambda), \bar{a}_{x_1}^{x_2}(\mu)\}=\frac{\varkappa^2}{\lambda-\mu}\left(\bar{b}_{x_1}^{x_2}(\lambda)b_{x_1}^{x_2}(\mu)-b_{x_1}^{x_2}(\lambda)\bar{b}_{x_1}^{x_2}(\mu)\right), \tag{S2}$$
$$\{a_{x_1}^{x_2}(\lambda), b_{x_1}^{x_2}(\mu)\}=\frac{\varkappa}{\lambda-\mu}\left(a_{x_1}^{x_2}(\lambda)b_{x_1}^{x_2}(\mu)-b_{x_1}^{x_2}(\lambda)a_{x_1}^{x_2}(\mu)\right), \tag{S3}$$
$$\{a_{x_1}^{x_2}(\lambda), \bar{b}_{x_1}^{x_2}(\mu)\}=\frac{\varkappa}{\lambda-\mu}\left(\bar{b}_{x_1}^{x_2}(\lambda)a_{x_1}^{x_2}(\mu)-a_{x_1}^{x_2}(\lambda)\bar{b}_{x_1}^{x_2}(\mu)\right), \tag{S4}$$
$$\{b_{x_1}^{x_2}(\lambda), b_{x_1}^{x_2}(\mu)\}=0, \tag{S5}$$
$$\{b_{x_1}^{x_2}(\lambda), \bar{b}_{x_1}^{x_2}(\mu)\}=\frac{1}{\lambda-\mu}\left(\bar{a}_{x_1}^{x_2}(\lambda)a_{x_1}^{x_2}(\mu)-a_{x_1}^{x_2}(\lambda)\bar{a}_{x_1}^{x_2}(\mu)\right). \tag{S6}$$

The remaining 10 relations are obtained from the ones listed by complex conjugation, interchange of $\lambda$ and $\mu$, and the use of the antisymmetry of the Poisson brackets.

2. Summary of Poisson brackets between matrix elements of the classical transition matrix $T_-(x,\lambda)$ for the semi-infinite interval $(-\infty, x]$.

The matrix $T_-(x,\lambda)$ has the form

$$T_-(x,\lambda) = \begin{pmatrix} a_-(x,\lambda), & \varkappa\bar{b}_-(x,\lambda) \\ b_-(x,\lambda), & \bar{a}_-(x,\lambda) \end{pmatrix}, \quad \lambda = \bar{\lambda}.$$

We recall that the matrix elements $a_-(x,\lambda)$ and $b_-(x,\lambda)$ admit analytic continuation with respect to $\lambda$ to the upper half-plane, and $\bar{b}_-(x,\lambda)$ and $\bar{a}_-(x,\lambda)$ to the lower (see Sec. 1.1).

Original formula for Poisson brackets (1.3.11):

$$\{\tilde{T}_-(x,\lambda), \tilde{\tilde{T}}_-(x,\mu)\} = r(\lambda-\mu)\tilde{T}_-(x,\lambda)\tilde{\tilde{T}}_-(x,\mu) - \tilde{T}_-(x,\lambda)\tilde{\tilde{T}}_-(x,\mu)r_-(\lambda-\mu).$$

Independent matrix elements:

$$\{a_-(x,\lambda), a_-(x,\mu)\} = 0, \tag{S7}$$

$$\{a_-(x,\lambda), \bar{a}_-(x,\mu)\} = -\frac{\varkappa^2}{\lambda-\mu+i0} b_-(x,\lambda)\bar{b}_-(x,\mu), \tag{S8}$$

$$\{a_-(x,\lambda), b_-(x,\mu)\} = \frac{\varkappa}{\lambda-\mu}(a_-(x,\lambda)b_-(x,\mu) - b_-(x,\lambda)a_-(x,\mu)), \tag{S9}$$

$$\{a_-(x,\lambda), \bar{b}_-(x,\mu)\} = -\frac{\varkappa}{\lambda-\mu+i0} a_-(x,\lambda)\bar{b}_-(x,\mu), \tag{S10}$$

$$\{b_-(x,\lambda), b_-(x,\mu)\} = 0, \tag{S11}$$

$$\{b_-(x,\lambda), \bar{b}_-(x,\mu)\} = -\frac{1}{\lambda-\mu+i0} a_-(x,\lambda)\bar{a}_-(x,\mu). \tag{S12}$$

In formula (S9) regularization of the denominator is not necessary, since the numerator vanishes for $\lambda = \mu$. Here the Poisson bracket admits analytic continuation to the same half-plane with respect to $\lambda$ and $\mu$.

3. Summary of Poisson brackets between matrix elements of the classical transition matrix $T_+(x,\lambda)$ for the semi-infinite interval $[x, +\infty)$.

The matrix $T_+(x,\lambda)$ has the form:

$$T_+(x,\lambda) = \begin{pmatrix} a_+(x,\lambda), & \varkappa\bar{b}_+(x,\lambda) \\ b_+(x,\lambda), & \bar{a}_+(x,\lambda) \end{pmatrix}, \quad \lambda = \bar{\lambda}.$$

The matrix elements $a_+(x,\lambda)$ and $\bar{b}_+(x,\lambda)$ admit analytic continuation with respect to $\lambda$ to the upper half-plane, and $b_+(x,\lambda)$ and $\bar{a}_+(x,\lambda)$ to the lower (see Sec. 1.1).

Original formula for Poisson brackets (1.3.12):

$$\{\tilde{T}_+(x,\lambda), \tilde{\tilde{T}}_+(x,\mu)\} = r_+(\lambda-\mu)\tilde{T}_+(x,\lambda)\tilde{\tilde{T}}_+(x,\mu) - \tilde{T}_+(x,\lambda)\tilde{\tilde{T}}_+(x,\mu)r(\lambda-\mu).$$

Independent matrix elements:

$$\{a_+(x,\lambda), a_+(x,\mu)\} = 0, \tag{S13}$$

$$\{a_+(x,\lambda), \bar{a}_+(x,\mu)\} = \frac{\varkappa^2}{\lambda-\mu+i0} \bar{b}_+(x,\lambda)b_+(x,\mu), \tag{S14}$$

$$\{a_+(x,\lambda), b_+(x,\mu)\} = \frac{\varkappa}{\lambda-\mu+i0} a_+(x,\lambda)b_+(x,\mu), \tag{S15}$$

$$\{a_+(x,\lambda), \bar{b}_+(x,\mu)\} = \frac{\varkappa}{\lambda-\mu}(\bar{b}_+(x,\lambda)a_+(x,\mu) - a_+(x,\lambda)\bar{b}_+(x,\mu)), \tag{S16}$$

$$\{ b_+(x,\lambda), b_+(x,\mu)\} = 0, \tag{S17}$$

$$\{ b_+(x,\lambda), \bar{b}_+(x,\mu)\} = \frac{1}{\lambda-\mu+i0}\, \bar{a}_+(x,\lambda)\, a_+(x,\mu). \tag{S18}$$

In connection with (S16) one can make a remark analogous to that made in Paragraph 2 about (S9).

4. Summary of Poisson brackets between matrix elements of the classical transition matrix $T(\lambda)$ on the infinite interval $(-\infty,\infty)$.

The matrix $T(\lambda)$ has the form

$$T(\lambda) = \begin{pmatrix} a(\lambda), & \varkappa \bar{b}(\lambda) \\ b(\lambda), & \bar{a}(\lambda) \end{pmatrix}, \quad \lambda = \bar{\lambda}.$$

The matrix element $a(\lambda)$ admits analytic continuation with respect to $\lambda$ to the upper half-plane, $\bar{a}(\lambda)$ to the lower. The matrix elements $b(\lambda)$ and $\bar{b}(\lambda)$, generally speaking, do not admit analytic continuation (see Sec. 1.1).

Original formula for Poisson brackets (1.3.14):

$$\{ \tilde{T}(\lambda), \tilde{\tilde{T}}(\mu)\} = r_+(\lambda-\mu)\tilde{T}(\lambda)\tilde{\tilde{T}}(\mu) - \tilde{T}(\lambda)\tilde{\tilde{T}}(\mu)\, r_-(\lambda-\mu).$$

Independent matrix elements:

$$\{ a(\lambda), a(\mu)\} = 0, \tag{S19}$$

$$\{ a(\lambda), \bar{a}(\mu)\} = 0, \tag{S20}$$

$$\{ a(\lambda), b(\mu)\} = \frac{\varkappa}{\lambda-\mu+i0}\, a(\lambda)\, b(\mu); \tag{S21}$$

$$\{ a(\lambda), \bar{b}(\mu)\} = -\frac{\varkappa}{\lambda-\mu+i0}\, a(\lambda)\, \bar{b}(\mu), \tag{S22}$$

$$\{ b(\lambda), b(\mu)\} = 0, \tag{S23}$$

$$\{ b(\lambda), \bar{b}(\mu)\} = 2\pi i\, |a(\lambda)|^2 \delta(\lambda-\mu). \tag{S24}$$

5. Summary of commutation relations between matrix elements of the quantum transition matrix $\mathbb{T}_{x_1}^{x_2}(\lambda)$ for the finite interval $[x_1, x_2]$.

The matrix $\mathbb{T}_{x_1}^{x_2}(\lambda)$ has the form (2.1.18)

$$\mathbb{T}_{x_1}^{x_2}(\lambda) = \begin{pmatrix} A_{x_1}^{x_2}(\lambda), & \varkappa B_{x_1}^{+x_2}(\lambda) \\ B_{x_1}^{x_2}(\lambda), & A_{x_1}^{+x_2}(\lambda) \end{pmatrix}, \quad \lambda = \bar{\lambda}.$$

Original formula (2.2.1):

$$R(\lambda-\mu)\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)\tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu) = \tilde{\tilde{\mathbb{T}}}_{x_1}^{x_2}(\mu)\tilde{\mathbb{T}}_{x_1}^{x_2}(\lambda)R(\lambda-\mu).$$

Independent commutation relations:

$$A_{x_1}^{x_2}(\lambda)A_{x_1}^{x_2}(\mu) = A_{x_1}^{x_2}(\mu)A_{x_1}^{x_2}(\lambda), \tag{S25}$$

$$A_{x_1}^{x_2}(\lambda)A_{x_1}^{+x_2}(\mu) = A_{x_1}^{+x_2}(\mu)A_{x_1}^{x_2}(\lambda) + \frac{i\varkappa^2}{\lambda-\mu}\left(B_{x_1}^{+x_2}(\mu)B_{x_1}^{x_2}(\lambda) - B_{x_1}^{+x_2}(\lambda)B_{x_1}^{x_2}(\mu)\right), \tag{S26}$$

$$B_{x_1}^{x_2}(\mu)A_{x_1}^{x_2}(\lambda)=\left(1+\frac{i\varkappa}{\lambda-\mu}\right)A_{x_1}^{x_2}(\lambda)B_{x_1}^{x_2}(\mu)-\frac{i\varkappa}{\lambda-\mu}A_{x_1}^{x_2}(\mu)B_{x_1}^{x_2}(\lambda), \tag{S27}$$

$$A_{x_1}^{x_2}(\lambda)B_{x_1}^{+x_2}(\mu)=\left(1+\frac{i\varkappa}{\lambda-\mu}\right)B_{x_1}^{+x_2}(\mu)A_{x_1}^{x_2}(\lambda)-\frac{i\varkappa}{\lambda-\mu}B_{x_1}^{+x_2}(\lambda)A_{x_1}^{x_2}(\mu), \tag{S28}$$

$$B_{x_1}^{x_2}(\lambda)B_{x_1}^{x_2}(\mu)=B_{x_1}^{x_2}(\mu)B_{x_1}^{x_2}(\lambda), \tag{S29}$$

$$B_{x_1}^{x_2}(\lambda)B_{x_1}^{+x_2}(\mu)=B_{x_1}^{+x_2}(\mu)B_{x_1}^{x_2}(\lambda)+\frac{i}{\lambda-\mu}\left(A_{x_1}^{+x_2}(\mu)A_{x_1}^{x_2}(\lambda)-A_{x_1}^{+x_2}(\lambda)A_{x_1}^{x_2}(\mu)\right), \tag{S30}$$

All other commutation relations are obtained by Hermitian conjugation and interchange of $\lambda$ and $\mu$.

Here and later all commutation relations are given in one form: on the left side is the product of two matrix elements, on the right side is a linear combination of matrix elements with coefficients having $(\lambda-\mu)$ in the denominator.

6. Summary of commutation relations between matrix elements of the quantum transition matrix $\mathbb{T}_-(x,\lambda)$ for the semi-infinite segment $(-\infty,x]$.

The matrix $\mathbb{T}_-(x,\lambda)$ has the form:

$$\mathbb{T}_-(x,\lambda)=\begin{pmatrix} A_-(x;\lambda), & \varkappa B_-^+(x,\lambda) \\ B_-(x,\lambda), & A_-^+(x,\lambda) \end{pmatrix}, \quad \lambda=\bar{\lambda} .$$

The analytic properties of the matrix elements of $\mathbb{T}_-(x,\lambda)$ are the same as in Paragraph 2.

Original formula (2.3.4):

$$R(\lambda-\mu)\tilde{\mathbb{T}}_-(x,\lambda)\tilde{\tilde{\mathbb{T}}}_-(x,\mu)\left(1-\frac{i\varkappa}{\lambda-\mu+i0}\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\right)=\tilde{\tilde{\mathbb{T}}}_-(x,\mu)\tilde{\mathbb{T}}_-(x,\lambda)\left(1+\frac{i\varkappa}{\lambda-\mu-i0}\tilde{\sigma}_+\tilde{\tilde{\sigma}}_-\right)R(\lambda-\mu) .$$

Independent commutation relations:

$$A_-(x,\lambda)A_-(x,\mu)=A_-(x,\mu)A_-(x,\lambda), \tag{S31}$$

$$A_-(x,\lambda)A_-^+(x,\mu)=A_-^+(x,\mu)A_-(x,\lambda)+\frac{i\varkappa^2}{\lambda-\mu+i0}B_-^+(x,\mu)B_-(x,\lambda), \tag{S32}$$

$$B_-(x,\mu)A_-(x,\lambda)=\left(1+\frac{i\varkappa}{\lambda-\mu}\right)A_-(x,\lambda)B_-(x,\mu)-\frac{i\varkappa}{\lambda-\mu}A_-(x,\mu)B_-(x,\lambda), \tag{S33}$$

$$A_-(x,\lambda)B_-^+(x,\mu)=\left(1+\frac{i\varkappa}{\lambda-\mu+i0}\right)B_-^+(x,\mu)A_-(x,\lambda) . \tag{S34}$$

$$B_-(x,\lambda)B_-(x,\mu)=B_-(x,\mu)B_-(x,\lambda) . \tag{S35}$$

$$B_-(x,\lambda)B_-^+(x,\mu)=B_-^+(x,\mu)B_-(x,\lambda)+\frac{i}{\lambda-\mu+i0}A_-^+(x,\mu)A_-(x,\lambda). \tag{S36}$$

7. Summary of commutation relations between matrix elements of the quantum transition matrix $\mathbb{T}_+(x,\lambda)$ for the semi-infinite interval $[x,+\infty)$.

The matrix $\mathbb{T}_+(x,\lambda)$ has the form:

$$\mathbb{T}_+(x,\lambda)=\begin{pmatrix} A_+(x,\lambda), & \varkappa B_+^+(x,\lambda) \\ B_+(x,\lambda), & A_+^+(x,\lambda) \end{pmatrix}, \quad \lambda=\bar{\lambda} .$$

The analytic properties of the matrix elements are the same as in Paragraph 3.

Original formula (2.3.5):

$$R(\lambda-\mu)\left(1+\frac{i\varkappa}{\lambda-\mu-i0}\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\right)\tilde{\mathbb{T}}_+(x,\lambda)\tilde{\tilde{\mathbb{T}}}_+(x,\mu)=\left(1-\frac{i\varkappa}{\lambda-\mu+i0}\tilde{\sigma}_+\tilde{\tilde{\sigma}}_-\right)\tilde{\tilde{\mathbb{T}}}_+(x,\mu)\tilde{\mathbb{T}}_+(x,\lambda)R(\lambda-\mu).$$

Independent commutation relations:

$$A_+(x,\lambda)A_+(x,\mu)=A_+(x,\mu)A_+(x,\lambda), \tag{S37}$$

$$A_+(x,\lambda)A_+^+(x,\mu)=A_+^+(x,\mu)A_+(x,\lambda)-\frac{i\varkappa^2}{\lambda-\mu+i0}B_+^+(x,\lambda)B_+(x,\mu), \tag{S38}$$

$$B_+(x,\mu)A_+(x,\lambda)=\left(1+\frac{i\varkappa}{\lambda-\mu+i0}\right)A_+(x,\lambda)B_+(x,\mu), \tag{S39}$$

$$A_+(x,\lambda)B_+^+(x,\mu)=\left(1+\frac{i\varkappa}{\lambda-\mu}\right)B_+^+(x,\mu)A_+(x,\lambda)-\frac{i\varkappa}{\lambda-\mu}B_+^+(x,\lambda)A_+(x,\mu), \tag{S40}$$

$$B_+(x,\lambda)B_+(x,\mu)=B_+(x,\mu)B_+(x,\lambda), \tag{S41}$$

$$B_+(x,\lambda)B_+^+(x,\mu)=B_+^+(x,\mu)B_+(x,\lambda)-\frac{i}{\lambda-\mu-i0}A_+^+(x,\lambda)A_+(x,\mu). \tag{S42}$$

8. Summary of commutation relations between matrix elements of the quantum transition matrix $\mathbb{T}(\lambda)$ for the infinite interval $(-\infty,\infty)$.

The matrix $\mathbb{T}(\lambda)$ has the form:

$$\mathbb{T}(\lambda)=\begin{pmatrix} A(\lambda), & \varkappa B^+(\lambda) \\ B(\lambda), & A^+(\lambda) \end{pmatrix}, \quad \lambda=\bar{\lambda}.$$

The analytic properties of the matrix elements are the same as in Paragraph 4.

Original formula (2.3.6)

$$R(\lambda-\mu)\left(1+\frac{i\varkappa}{\lambda-\mu-i0}\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\right)\tilde{\mathbb{T}}(\lambda)\tilde{\tilde{\mathbb{T}}}(\mu)\left(1-\frac{i\varkappa}{\lambda-\mu+i0}\tilde{\sigma}_-\tilde{\tilde{\sigma}}_+\right)=\left(1-\frac{i\varkappa}{\lambda-\mu+i0}\tilde{\sigma}_+\tilde{\tilde{\sigma}}_-\right)\tilde{\tilde{\mathbb{T}}}(\mu)\tilde{\mathbb{T}}(\lambda)\left(1+\frac{i\varkappa}{\lambda-\mu-i0}\tilde{\sigma}_+\tilde{\tilde{\sigma}}_-\right)R(\lambda-\mu).$$

Independent commutation relations:

$$A(\lambda)A(\mu)=A(\mu)A(\lambda), \tag{S43}$$

$$A(\lambda)A^+(\mu)=A^+(\mu)A(\lambda), \tag{S44}$$

$$B(\mu)A(\lambda)=\left(1+\frac{i\varkappa}{\lambda-\mu+i0}\right)A(\lambda)B(\mu), \tag{S45}$$

$$A(\lambda)B^+(\mu)=\left(1+\frac{i\varkappa}{\lambda-\mu+i0}\right)B^+(\mu)A(\lambda), \tag{S46}$$

$$B(\lambda)B(\mu)=B(\mu)B(\lambda), \tag{S47}$$

$$B(\lambda)B^+(\mu)=\left(1+\frac{i\varkappa}{\lambda-\mu+i0}\right)\left(1-\frac{i\varkappa}{\lambda-\mu-i0}\right)B^+(\mu)B(\lambda)+2\pi A^+(\lambda)A(\lambda)\delta(\lambda-\mu). \tag{S48}$$

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# SOLUTIONS OF THE YANG—BAXTER EQUATION

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UDC 517.43+530.145

We give the basic definitions connected with the Yang—Baxter equation (factorization condition for a multiparticle S-matrix) and formulate the problem of classifying its solutions. We list the known methods of solution of the Y—B equation, and also various applications of this equation to the theory of completely integrable quantum and classical systems. A generalization of the Y—B equation to the case of $Z_2$-graduation is obtained, a possible connection with the theory of representations is noted. The supplement contains about 20 explicit solutions.

0. By the Yang—Baxter equation [1, 2] is meant the following functional equation:

$$ {}_{\alpha\alpha'}R_{\gamma\gamma'}(u-v)\,{}_{\gamma\alpha''}R_{\beta\gamma''}(u)\,{}_{\gamma'\gamma''}R_{\beta'\beta''}(v) = {}_{\alpha'\alpha''}R_{\gamma'\gamma''}(v)\,{}_{\alpha\gamma''}R_{\gamma\beta''}(u)\,{}_{\gamma\gamma'}R_{\beta\beta'}(u-v) \tag{1} $$

for a collection of functions ${}_{\alpha\beta}R_{\gamma\delta}(u)$ of a complex parameter $u$, depending on four indices $\alpha, \beta, \gamma, \delta$, running through values from 1 to some natural number $N$. In (1) and later we understand summation over repeated indices.

Equation (1), which first appeared in [1, 2], has many applications to the theory of completely integrable quantum and classical systems and exactly solvable models of statistical physics. In recent years it has undergone intensive study. Here the profound connection of (1) with such areas of mathematics as group theory and algebraic geometry has become more and more apparent.

The present paper is an (apparently the first) attempt to give a systematic survey of the facts accumulated at the time it is written relating to the solutions of (1). The account is structured in the following way. In Sec. 1 we give the basic definitions and we

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR, Vol. 95, pp. 129-160, 1980.

 0090-4104/82/1905-1596$07.50 