CROFTON'S FUNCTION AND INVERSION FORMULAS IN REAL INTEGRAL GEOMETRY

I. M. Gel'fand and M. I. Graev

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0. Introduction

This paper is devoted to inversion formulas for an integral transform $f \mapsto \mathscr{R}f$, carrying a function f on $\mathbf{R}^n$ to its integrals over k-dimensional planes in $\mathbf{R}^n$ (its precise definition is given below). Inversion formulas are solutions of the problem of reconstructing the function f on $\mathbf{R}^n$ from its integral transform $\mathscr{R}f$. The transform carrying a function on $\mathbf{R}^n$ to its integrals over hyperplanes (the case $k = n - 1$) is called the Radon transform; inversion formulas for it are well known (cf. [1, 2]). In [3] inversion formulas are given for the case of planes of arbitrary even dimension k, and in [4] there is a class of inversion formulas for the case of odd k. The difference between even and odd k is that for even k the inversion formulas are local. This term means that to reconstruct the function at a point it is only necessary to know its integrals over planes near this point; now for odd k the inversion formulas are nonlocal. For example, on $\mathbf{R}^3$ to reconstruct a function at a point it is necessary to know its integrals over all lines in $\mathbf{R}^3$. However in an astonishing way for odd k one can also get local inversion formulas. This is precisely the goal of the present paper. We note that numerical calculations with the nonlocal formulas from [4] require n-fold integration at the same time that the calculations with the formulas of the present paper require (3k)-fold integration.

1. Definition of the Transform $f \mapsto \mathscr{R}f$

Let f be a finite function on $\mathbf{R}^n$. We take arbitrary $\beta \in \mathbf{R}^n$ and a k-frame $\alpha = (\alpha_1, \ldots, \alpha_k)$, where $\alpha_i \in \mathbf{R}^n$ $(i = 1, \ldots, k)$. Then the collection of vectors $x = \alpha_1 t_1 + \ldots + \alpha_k t_k + \beta$, where $t = (t_1, \ldots, t_k)$ runs through $\mathbf{R}^k$, forms a k-dimensional plane h in $\mathbf{R}^n$. The transform $f \mapsto \mathscr{R}f$ is defined by the following formula:

$$(\mathscr{R}f)(\alpha, \beta) = \int_{\mathbf{R}^k} f(\alpha_1 t_1 + \ldots + \alpha_k t_k + \beta)\, dt_1 \ldots dt_k. \tag{1}$$

It is easy to see that $(\mathscr{R}f)(\alpha, \beta)$ depends only on the plane h itself and the measure of h defined by the choice of the k-frame α.

2. Inversion Formulas for the Case of Even k [3]

First we note that the problem of reconstructing the function f from the function $\mathscr{R}f$ is overdetermined for $k < n - 1$ since the manifold $H_{k,n}$ of k-dimensional planes in $\mathbf{R}^n$ has dimension greater than n for $k < n$; to reconstruct f it suffices to know the restrictions of $\mathscr{R}f$ to certain n-dimensional submanifolds of $H_{k,n}$. For this reason there exists many different inversion formulas. For even k one can get the class of inversion formulas of [3] in this way.

From the function $\varphi = \mathscr{R}f$ we construct a differential k-form $\varkappa\varphi$ on the manifold $\overset{+}{H}_{k,n}$ of oriented k-dimensional planes in $\mathbf{R}^n$. First it is convenient to define $\varkappa\varphi$ not on $\overset{+}{H}_{k,n}$ but on the manifold $E_{k,n} \times \mathbf{R}^n$, where $E_{k,n}$ is the manifold of k-frames α:

$$\varkappa\varphi = \sum_{i_1, \ldots, i_k = 1}^{n} \frac{\partial^k \varphi(\alpha, \beta)}{\partial\beta^{i_1} \ldots \partial\beta^{i_k}}\, d\alpha_1^{i_1} \wedge \ldots \wedge d\alpha_k^{i_k} \tag{2}$$

(β^i, α_j^i being the coordinates, respectively, of the vectors β and α_j). It is easy to see that under the natural projection π: $E_{k,n} \times \mathbf{R}^n \to \overset{+}{H}_{k,n}$ this form lowers from $E_{k,n} \times \mathbf{R}^n$ to $\overset{+}{H}_{k,n}$ [the plane $h = \pi(\alpha, \beta)$ is the collection of vectors $x = \alpha_1 t_1 + \ldots + \alpha_k t_k + \beta$].

It is proved in [3] that the restriction of $\varkappa\varphi$ to the submanifold $H_x \subset \overset{+}{H}_{k,n}$ of planes passing through an arbitrary fixed point $x \in \mathbf{R}^n$ is a closed form on H_x. Here for any k-di-

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mensional cycle $\gamma \subset H_x$ the following equation holds:

$$\int_\gamma \varkappa\varphi = c(\gamma) f(x) \tag{3}$$

and there exist cycles γ (for example the Euler cycle) for which $c(\gamma) \neq 0$. The inversion formula (3) obtained is local, i.e., to reconstruct the value of the function f at the point x from it it suffices to know the values of $\varphi = \mathcal{R}f$ only on the set of planes infinitely close to x.

3. Inversion Formulas for the Case of Odd k [4]

The definition of the form $\varkappa\varphi$ and (3) are also preserved for odd k. However, for odd k we have $c(\gamma) = 0$ for any k-dimensional cycle $\gamma \subset H_x$ and for the transformation $f \mapsto \mathcal{R}f$ there only exist nonlocal inversion formulas. A class of such formulas is constructed in [4]; in them instead of the differential form $\varkappa\varphi$ the density $\chi_x\varphi$ on the manifold of k-dimensional planes passing through the point x is used.*

We give the construction of $\chi_x\varphi$. We shall define k-dimensional planes in $\mathbf{R}^n$ by equations of the form $x_i = a_{i1}x_{m+1} + \ldots + a_{ik}x_n + b_i$ $(i = 1, \ldots, m)$, where $m = n - k$. In abbreviated notation: $x' = Ax'' + b$, where $x' = (x_1, \ldots, x_m)$, $x'' = (x_{m+1}, \ldots, x_n)$, $b = (b_1, \ldots, b_m)$, $A = \| a_{ij} \|_{i=1,\ldots,m;\ j=1,\ldots,k}$.

We use a_{ij} and b_i as coordinates on the manifold $H_{k,n}$ of k-dimensional planes. In these coordinates the function $\mathcal{R}f$ is defined by the following equation:

$$(\mathcal{R}f)(A, b) = \int f(Ax'' + b, x'')\, dx'', \quad dx'' = dx_{m+1} \ldots dx_n.$$

We note that the submanifold H_{x_0} of planes passing through the fixed point $x_0 = (x_0', x_0'')$ consists of planes with coordinates (A, b), where $b = x_0' - Ax_0''$. In view of this we can identify the elements of H_{x_0} with (m × k)-matrices A, and vectors from the tangent spaces $T_A H_{x_0}$ are also naturally identified with (m × k)-matrices.

Let $A^1, \ldots, A^k$ be linearly independent vectors in $T_A H_{x_0}$. We introduce a pseudodifferential operator $\hat{c}(A^1, \ldots, A^k)$ in $\mathbf{R}^m$, whose symbol $c(A^1, \ldots, A^k; \xi)$ is defined by the following equations:

$$c(A^1, \ldots, A^k; \xi) = |\det \| c_{ij}(A^1, \ldots, A^k; \xi) \| |,$$

where

$$c_{ij}(A^1, \ldots, A^k; \xi) = \sum_{s=1}^{m} \xi_s a_{si}^j, \quad i, j = 1, \ldots, k.$$

We define the k-density $\chi_{x_0}\varphi$ on H_{x_0}, where $\varphi = \mathcal{R}f$, as follows:

$$(\chi_{x_0}\varphi)(A; A^1, \ldots, A^k) = \hat{c}(A^1, \ldots, A^k)\, \varphi(A, b)\,|_{b = x_0' - Ax_0''} \tag{4}$$

(the operator $\hat{c}$ acts on φ as on a function of $b \in \mathbf{R}^m$).

One can get the value of the function f at the point x_0 by integrating the k-density $\chi_{x_0}\varphi$ over certain special k-dimensional submanifolds $\gamma \subset H_{x_0}$. Let γ be such that the number of planes $h \in \gamma$, belonging to the hyperplane $\langle \xi, x - x_0 \rangle = 0$ is identical for almost all $\xi \in (\mathbf{R}^n)' \setminus 0$ and nonzero. Then

$$\int_\gamma \chi_{x_0}\varphi = c(\gamma) f(x_0), \quad \text{where } c(\gamma) \neq 0. \tag{5}$$

We note that by (4) calculation of the density at a point reduces to (n − k)-fold integration. Thus the reconstruction of the function f at a point by means of (5) requires n-fold integration.†

Note. The inversion formula (5) is also valid for odd k.

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*By a k-density on a manifold X is meant a function $\nu(x; y_1, \ldots, y_k)$ of $x \in X$ and k linearly independent vectors $y_1, \ldots, y_k \in T_x X$, satisfying the following condition: if $y_i' = \sum_j g_{ij} y_j$ $(i = 1, \ldots, k)$, then $\nu(x; y_1', \ldots, y_k') = |\det \| g_{ij} \| |\, \nu(x, y_1, \ldots, y_k)$. Similarly to differential k-forms one can integrate any k-density over k-dimensional submanifolds.

†For special choice of γ (for example, if γ is the Euler cycle) the inversion formula (5) can be reduced to calculation of an integral of multiplicity less than n.

4. The Operator $\mathcal{I}_\gamma$

In this paper, for arbitrary k we shall obtain a new class of inversion formulas. Just as for the local inversion formulas (3) the integrals of the differential form $\varkappa\varphi$ lie at the base of their construction.

Let H_0 be the manifold of oriented k-dimensional planes in $\mathbf{R}^n$, passing through the point 0. We fix an arbitrary k-dimensional oriented submanifold $\gamma \subset H_0$. We denote by $\gamma + x$ the submanifold of planes obtained by parallel translation of planes from γ by the vector x. We define an operator $\mathcal{I}_\gamma$ from the space of functions $\varphi = \mathcal{R}f$ to the space of functions on $\mathbf{R}^n$ by the following formula:

$$(\mathcal{I}_\gamma\varphi)(x) = (2\pi i)^{-k} \int\limits_{\gamma+x} \varkappa\varphi. \tag{6}$$

Remark. For odd k $\mathcal{I}_\gamma \equiv 0$ for any cycle γ. Hence for odd k the operator $\mathcal{I}_\gamma$ depends only on the boundary $\partial\gamma$ of the manifold γ.

5. Crofton's Function

We want to describe the composition $\mathcal{I}_\gamma\mathcal{R}$ of the operators $\mathcal{R}$ and $\mathcal{I}_\gamma$. To this end we associate with each k-dimensional oriented submanifold $\gamma \subset H_0$ a function on $(\mathbf{R}^n)' \setminus 0$, which we call the Crofton function and denote by $\mathrm{Crf}_\gamma(\xi)$.

By G_ξ, where $\xi \in (\mathbf{R}^n)' \setminus 0$, we denote the manifold of all subspaces $h \in H_0$, belonging to the hyperplane $\langle\xi, x\rangle = 0$. Let us assume that $k < n - 1$; then G_ξ is an orientable cycle in H_0 endowed like H_0 itself with a canonical orientation. (For odd k this orientation is defined by the orientation of $\mathbf{R}^n$ and the direction of the vector ξ and for even k it is independent of the orientation of $\mathbf{R}^n$ and the direction of ξ.) Since $\dim\gamma = \operatorname{codim} G_\xi = k$, for submanifolds γ and G_ξ in H_0 in general position their intersection index $\gamma\cdot G_\xi$ is defined.

Definition of the Crofton Function for the Case $k < n - 1$:

$$\mathrm{Crf}_\gamma(\xi) = \gamma\cdot G_\xi. \tag{7}$$

In what follows we assume about γ that the function $\mathrm{Crf}_\gamma(\xi)$ is defined for almost all $\xi \in (\mathbf{R}^n)' \setminus 0$. This automatically holds if γ is in an algebraic k-dimensional submanifold of H_0.

Remark. The definition of the Crofton function is compatible with the definition of the Crofton symbol $\mathrm{Cr}_\gamma(\xi)$ introduced in [4] in the construction of nonlocal inversion formulas: $\mathrm{Cr}_\gamma(\xi)$ is defined there as the number of points of intersection of γ and G_ξ where, in contrast with the present situation, γ and G_ξ are manifolds of nonoriented subspaces.

We define the Crofton function for the exceptional case $k = n - 1$. In this case $\dim\gamma = \dim H_0$ and G_ξ consists of two points (the subspace $\langle\xi, x\rangle = 0$ with the two opposite orientations). For even k we define $\mathrm{Crf}_\gamma(\xi)$ as the number of points of intersection of γ and G_ξ: $\mathrm{Crf}_\gamma(\xi) = |\gamma \cap G_\xi|$. For odd k we set $\mathrm{Crf}_\gamma(\xi) = 0$ if $|\gamma \cap G_\xi| = 0, 2$; now if $\gamma \cap G_\xi$ consists of a unique oriented space h, then we set $\mathrm{Crf}_\gamma(\xi) = \pm 1$ where the sign is determined by whether or not the orientation of h is compatible with the direction of the vector ξ.

It follows from the definition of the Crofton function that

$$\mathrm{Crf}_\gamma(-\xi) = (-1)^k \,\mathrm{Crf}_\gamma(\xi). \tag{8}$$

6. THEOREM 1. The composition $\mathcal{I}_\gamma\mathcal{R}$ of the operators $\mathcal{R}$ and $\mathcal{I}_\gamma$ defined by (1) and (6) is a pseudodifferential operator on the space of functions on $\mathbf{R}^n$ with symbol

$$c(\xi) = \mathrm{Crf}_\gamma(\xi).$$

Proof. We have

$$(\mathcal{I}_\gamma\mathcal{R}f)(x) = (2\pi i)^{-k} \int\limits_\gamma \int\limits_{\mathbf{R}^k} \sum \frac{\partial^k f(\alpha t + x)}{\partial x^{i_1} \dots \partial x^{i_k}} \, dt \, d\alpha_1^{i_1} \wedge \dots \wedge d\alpha_k^{i_k} =$$

$$= (2\pi)^{-n/2-k} \int\limits_\gamma \int\limits_{\mathbf{R}^k} \int\limits_{(\mathbf{R}^n)'} \tilde f(\xi) e^{i\langle\xi, \alpha t + x\rangle} d\xi \wedge dt \wedge \omega_\xi(\alpha) = (2\pi)^{-n/2} \int\limits_\gamma \int\limits_{(\mathbf{R}^n)'} \tilde f(\xi) e^{i\langle\xi, x\rangle} \prod_{j=1}^{k} \delta(\langle\xi, \alpha_j\rangle) \, d\xi \wedge \omega_\xi(\alpha),$$

where $\tilde f$ is the Fourier transform of the function f, $\delta(\cdot)$ is the delta-function, and

$$\omega_\xi(\alpha) = \langle\xi, d\alpha_1\rangle \wedge \dots \wedge \langle\xi, d\alpha_k\rangle.$$

It follows from this that $\mathcal{I}_\gamma\mathcal{R}$ is a pseudodifferential operator with symbol

$$c(\xi)=\int_\gamma\prod_{j=1}^{k}\delta(\langle\xi,\alpha_j\rangle)\,\omega_\xi(\alpha). \tag{9}$$

We see that the integral (9) is equal to $\mathrm{Crf}_\gamma(\xi)$ for almost all $\xi\in(\mathbf{R}^n)'\setminus 0$. For this it suffices to consider the case when γ is a small disc intersecting G_ξ in at most one point and hence either $\gamma\cap G_\xi=\varnothing$ and $\mathrm{Crf}_\gamma(\xi)=0$ or $|\gamma\cap G_\xi|=1$ and $\mathrm{Crf}_\gamma(\xi)=\pm 1$. We introduce coordinates $s=(s_1,\ldots,s_k)$ on γ and define in each oriented subspace $h(s)\in\gamma$ a basis $\alpha_1(s),\ldots,\alpha_k(s)$ depending smoothly on s. Then it follows from (9) that

$$c(\xi)=\int_\gamma\prod_{j=1}^{k}\delta(u_j(s))\frac{D(u)}{D(s)}ds,\quad ds=ds_1\ldots ds_k,$$

where $u_j(s)=\langle\xi,\alpha_j(s)\rangle$ $(j=1,\ldots,k)$, $D(u)/D(s)$ is the Jacobian. When $\gamma\cap G_\xi=\varnothing$ the functions $u_j(s)$ do not vanish simultaneously for any s; consequently, $c(\xi)=0=\mathrm{Crf}_\gamma(\xi)$. Now if γ and G_ξ intersect in the point s_0 and are transverse at this point, then

$$c(\xi)=\mathrm{sgn}\,(D(u)/D(s))\,|_{s=s_0},$$

and hence $c(\xi)=\pm\mathrm{Crf}_\gamma(\xi)$. It is easy to see that the signs of $D(u)/D(s)|_{s=s_0}$ and $\mathrm{Crf}_\gamma(\xi)$ also coincide.

COROLLARY. The composition of operators $(\mathcal{I}_{\gamma_1}\mathcal{R})(\mathcal{I}_{\gamma_2}\mathcal{R})$, where γ_1, γ_2 are two oriented k-dimensional submanifolds of H_0, is a pseudodifferential operator with symbol

$$c(\xi)=\mathrm{Crf}_{\gamma_1}(\xi)\,\mathrm{Crf}_{\gamma_2}(\xi).$$

In particular, the symbol of the operator $(\mathcal{I}_\gamma\mathcal{R})^2$ is equal to $(\mathrm{Crf}_\gamma(\xi))^2$.

7. Quasicycles and the Inversion Formula

We call an oriented k-dimensional submanifold $\gamma\subset H_0$ a quasicycle if $|\mathrm{Crf}_\gamma(\xi)|$ is almost everywhere a nonzero constant; we denote this constant by $c(\gamma)$. In particular, γ is a quasicycle if $|\gamma\cap G_\xi|=1$ for almost all $\xi\in(\mathbf{R}^n)'\setminus 0$.

Remark. For odd k the Crofton function itself is not constant for any quasicycle γ.

The next theorem follows directly from Theorem 1 (Corollary).

THEOREM 2. If γ is a quasicycle in H_0, then $(\mathcal{I}_\gamma\mathcal{R})^2=c^2(\gamma)E$, where E is the identity operator. Thus, for the integral transform $f\mapsto\varphi=\mathcal{R}f$ one has the following inversion formula:

$$\mathcal{I}_\gamma\mathcal{R}\mathcal{I}_\gamma\varphi=c^2(\gamma)f. \tag{10}$$

We note that reconstruction of the function f at one point according to this formula reduces to the calculation of a (3k)-fold integral.

8. Example: the Case k = 1

In this case H_0 can be identified naturally with the unit sphere S^{n-1} in $\mathbf{R}^n$, $\varkappa\varphi$ is a 1-form on S^{n-1}. As γ one can take an arbitrary smooth path on S^{n-1} from the point a to the point b. Then

$$(\mathcal{I}_\gamma\mathcal{R}f)(x)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}t^{-1}(f(bt+x)-f(at+x))\,dt,$$

where it is necessary to understand the integral in the sense of principal value. The symbol of the operator $\mathcal{I}_\gamma\mathcal{R}$ is equal to $c(\xi)=1/2\,(\mathrm{sgn}\,\langle\xi,b\rangle-\mathrm{sgn}\,\langle\xi,a\rangle)$. The path γ is a quasicycle if and only if its ends a and b are diametrically opposite: b = −a.

9. Note

Theorem 1 easily generalizes to the case of operators of more general form:

$$(\mathcal{I}\varphi)(x)=(2\pi i)^{-k}\int_{\gamma_x+x}\varkappa\varphi, \tag{11}$$

where the submanifold $\gamma_x\subset H_0$ generally depends on the point x.

THEOREM 1'. The composition $\mathcal{I}\mathcal{R}$ of the operators $\mathcal{R}$ and $\mathcal{I}$ defined by (1) and (11) is a pseudodifferential operator with symbol

$$c(x, \xi) = \mathrm{Crf}_{\gamma_x}(\xi).$$

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SPECTRAL ASYMPTOTICS OF POLYNOMIAL PENCILS OF DIFFERENTIAL OPERATORS IN BOUNDED DOMAINS

K. Kh. Boimatov and A. G. Kostyuchenko

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1. Introduction. The Formulation of the Fundamental Result

The foundations of the theory of polynomial operator pencils have been established by Keldysh in his basic works [1, 2]. Some results of the investigations in this direction have been presented by Shkalikov and Markus [13].

This paper is devoted to the calculation of the spectral asymptotics of polynomial pencils of differential operators, defined on bounded domains.

1. Let $\Omega \subset R_n$ be a bounded domain with a boundary of class C^∞. In the space $L_2(\Omega)^\ell$ we consider the operator

$$L(\lambda) = \sum_{|\alpha|+jm\leqslant km} a_{\alpha,j}(x)\lambda^j D_x^\alpha, \tag{1.1}$$

where $a_{\alpha,j}(x) \in C^\infty(\bar{\Omega}; \mathrm{End}\, \mathbf{C}_q)$ $(|\alpha| + jm \leqslant km)$, $\lambda \in \mathbf{C}$, $D_x = (\partial/i\partial x_1, \ldots, \partial/i\partial x_n)$, $k, m \geqslant 1$ are integers. We assume that $a_{0,k}(x)$ is the identity matrix for all $x \in \Omega$.

We consider the operator

$$A' = \sum_{|\alpha|\leqslant mk} a_{\alpha,0}(x) D_x^\alpha, \quad D(A') = C_0^\infty(\Omega)^l,$$

and the closed extension A of the operator A' such that

$$D(A) \subset W_2^{mk}(\Omega)^l. \tag{1.2}$$

For the domain of definition of the operator pencil $L(\lambda)$ we take the domain of definition D(A) of the operator A.

The spectrum of the pencil $L(\lambda)$ consists of the set of points $\eta \in \mathbf{C}$ such that $L(\eta)$ does not have a continuous inverse. If $\ker L(\lambda_0) \neq 0$, then the number λ_0 is called an eigenvalue of the pencil $L(\lambda)$. By the multiplicity of the eigenvalue λ_0 we mean (see [2]) the number of vectors in the canonical system of eigen- and associated vectors, corresponding to the eigenvalue λ_0.

The symbol of the pencil (1.1) is defined by the formula

$$L(\lambda, x, s) = \sum_{j=0}^{k} \sum_{|\alpha|=m(k-j)} a_{\alpha,j}(x)\lambda^j s^\alpha.$$

Institute of Mathematics, Computer Center, Academy of Sciences of the Tadzhik SSR. M. V. Lomonosov Moscow State University.

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