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MOTION OF TWO PULSATING SPHERES IN AN IDEAL INCOMPRESSIBLE FLUID

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The problem of the interaction of two pulsating spheres in an ideal incompressible fluid was first investigated in detail by Bjerknes [1]. However, his and subsequent studies on this subject [2-5] were restricted to the interaction forces between the spheres, whereas the law of their motion was not considered because of the much greater complexity of the corresponding problem. The aim of the present paper is to find an approximate analytic solution to the problem of the motion of two pulsating spheres in an ideal incompressible fluid filling the entire space exterior to the spheres under the assumption that the flow of the fluid is irrotational.

The System of Equations of Motion

Suppose two spheres with masses m_k (here and in what follows, $k = 1, 2$), whose radii a_k may depend on the time t, move along the line of their centers, which we take as coordinate axis, and we denote by x_k the coordinates of the centers of the spheres and by b the distance between the centers. We assume that the spheres can execute small pulsations, the relation $a_k \ll b$ holding throughout the motion, and their external forces are absent. We shall describe the motion of the spheres approximately by the system of equations [6]

$$\frac{d}{dt}(L_1\dot{x}_1 - M\dot{x}_2) + 6\pi\rho a_1^3 a_2^3 b^{-4}\dot{x}_1\dot{x}_2 = 0, \qquad L_k = m_k + \frac{2}{3}\pi\rho a_k^3, \qquad M = 2\pi\rho a_1^3 a_2^3 b^{-3} \tag{1}$$

(the second equation is obtained by interchanging the indices and reversing the sign of the last term), where ρ is the density of the fluid, and the dot denotes differentiation with respect to t. The accuracy of the description depends on the parameter $\varepsilon = [\max\limits_{(k,t)} \{a_k b^{-1}\}]^3$, which at moderate values of the ratio of a_k to b can already be very small (for example, for two air bubbles in water of diameter 0.1 mm separated by 1 mm we have $\varepsilon = 1.25 \cdot 10^{-4} \ll 1$).

Let l and T be a certain characteristic length and characteristic time. It follows from dimensional considerations [7] that the equations of motion (1) can be reduced to dimensionless equations containing two independent dimensionless parameters, which we take to be α_k:

$$\frac{d}{d\tau}\left(A_1\dot{u}_1 - \frac{B}{y^3}\dot{u}_2\right) + 3\frac{B}{y^4}\dot{u}_1\dot{u}_2 = 0 \tag{2}$$

$$\tau = t/T, \quad x_k = u_k l, \quad y = u_1 - u_2, \quad a_k = R_k l(4/9\pi)^{1/3}, \quad V_k = 4/3\pi R_k^3, \quad A_k = \alpha_k + V_k, \quad B = V_1 V_2, \quad \alpha_k = 9m_k/(2\pi\rho l^3)$$

(the second equation is obtained by interchanging the indices and reversing the sign of the last term), where the dot denotes differentiation with respect to τ.

Adding the equations of the system (2) term by term, we obtain the first integral $(A_1 - By^{-3})\dot{u}_1 + (A_2 - By^{-3})\dot{u}_2 = C = \text{const}$ of it, from which, using the fact that

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$\dot{y} = \dot{u}_1 - \dot{u}_2$, we find expressions for $\dot{u}_k$:

$$\dot{u}_1 = \frac{C + (A_2 - By^{-3})\dot{y}}{A_1 + A_2 - 2By^{-3}} \tag{3}$$

(the expression for $\dot{u}_2$ is obtained by interchanging the indices and reversing the sign in front of the round brackets in the numerator).

If we substitute the expressions for $\dot{u}_k$ in one of the equations of motion (2), we obtain a single equation instead of two. To avoid lengthy calculations in what follows, we restrict ourselves to the case C = 0 (when C ≠ 0, no new fundamental difficulties arise). Then, substituting (3) in the first of Eqs. (2), we obtain to accuracy ε

$$\frac{d}{d\tau}\left[D\left(1 + \frac{2E}{y^3}\right)\dot{y}\right] - \frac{3DE}{y^4}\dot{y}^2 = 0, \qquad D = \frac{A_1 A_2}{A_1 + A_2}, \qquad E = \frac{B}{A_1 + A_2} \tag{4}$$

The problem can now be formulated as follows: for given functions E(τ) and D(τ), find a solution of Eq. (4) satisfying the initial conditions $y(0) = y_0$ and $\dot{y}(0) = \dot{y}_0$. The solution of Eq. (4) will correctly describe the motion of the spheres as long as the fundamental assumption is valid, i.e.,

$$\max_{(\tau)}\{Ey^{-3}\} \sim \varepsilon \ll 1 \tag{5}$$

Integration of the Equation of Motion

If D > 0 during the whole time of the motion, then, making the substitution $\dot{y} = Z(y)$, we reduce Eq. (4) to the form

$$\frac{dZ}{dy} - 9\frac{E}{y(y^3 + 2E)}Z = -\dot{F}, \qquad F = \ln D \tag{6}$$

Assuming that the functions E and F do not depend on y explicitly, we can formally integrate Eq. (6) as a linear equation with respect to Z(y):

$$Z(y) = \left(1 + \frac{2E}{y^3}\right)^{-1.5}\left[C_1 - \dot{F}\int\left(1 + \frac{2E}{y^3}\right)^{1.5} dy\right]$$

whence, determining C_1 from the initial conditions, we obtain to order ε an equation for finding $\dot{y}$:

$$Z(y) = \dot{y} = \dot{y}_0\left(1 + \frac{3E_0}{y_0^3} - \frac{3E}{y^3}\right) + \dot{F}_0 y_0\left(1 - 1.5\frac{E_0}{y_0^3} - 3\frac{E}{y^3}\right) - \dot{F}y\left(1 - 4.5\frac{E}{y^3}\right) \tag{7}$$

Equation (7) can be formally integrated over τ and written in the integral form

$$y = Wy, \qquad Wy = y_0 + \int_0^\tau \dot{y}\, d\tau \tag{8}$$

where $\dot{y}$ is determined by the right-hand side of (7).

We shall solve the integral equation (8) iteratively, setting $y^{(0)} = y_0 \neq 0$, $y^{(n)} = Wy^{(n-1)}$; $n = 1, 2, \ldots$

Then the first approximation can be written in the form

$$y^{(1)} = Wy^{(0)} = y_0 + G(\tau) + y_0^{-3}H(\tau), \quad G(\tau) = (\dot{y}_0 + y_0\dot{F}_0)\tau - y_0(F - F_0) \qquad H(\tau) = 1.5\int_0^\tau [2\dot{y}_0(E_0 - E) - \dot{F}_0 y_0(E_0 + 2E) + 3y_0\dot{F}E]\, d\tau$$

To terms of order ε, the second approximation has the form

$$y^{(2)} = y_0 + G - \int_0^\tau \dot{F}G\, d\tau - y_0^{-3}\int_0^\tau \dot{F}H\, d\tau + 1.5\frac{E_0}{y_0^3}(2\dot{y}_0 - \dot{F}_0 y_0)\tau - 3(\dot{y}_0 + \dot{F}_0 y_0)\int_0^\tau \frac{E}{(y_0 + G)^3}\, d\tau + 4.5\int_0^\tau \frac{\dot{F}E}{(y_0 + G)^2}\, d\tau \tag{9}$$

As will be seen from what follows, the second approximation is already equal to the asymptotic behavior of the exact solution of Eq. (4) for nonpulsating spheres, so that in the case of small pulsations of the spheres we can restrict ourselves to the second approximation.

Motion of Nonpulsating Spheres

In this case D, E $\equiv$ const, and Eq. (4) can be exactly solved in quadratures:

$$\int_{y_0}^{y}\left(1+\frac{2E}{y^3}\right)^{1.5}dy=\dot{y}_0\left(1+\frac{2E}{y_0^3}\right)^{1.5}\tau$$

where the integral on the left-hand side cannot be expressed in terms of elementary functions; however, making calculations to order ε, we can obtain the approximate solution

$$y=y_0+\dot{y}_0\left(1+\frac{3E}{y_0^3}\right)\tau+1.5E\left(\frac{1}{y^2}-\frac{1}{y_0^2}\right) \tag{10}$$

which corresponds to the expression (9) if in it we set E, F = const.

The expression (10) determines the solution if three constants are known: y_0, $\dot{y}_0$, E. The number of determining constants can be reduced to two if we introduce the new variable $Y = y/y_0$ and by $q(Y)$ denote the combination $1.5E/(Y^3y_0^3)$, setting $Y_0 = y(0)/y_0 = 1$, $\dot{Y}_0 = \dot{y}_0/y_0$, $q_0 = q(Y_0)$. Then the expression (10), which determines the motion of nonpulsating spheres, can be written to order ε in the simpler form

$$Y(1-q+q_0)=1+\dot{Y}_0(1+3q_0)\tau$$

Motion of Pulsating Spheres

Suppose the spheres execute small pulsations, so that their dimensionless volumes V_k vary periodically about certain mean values $\langle V_k\rangle$. We set

$$V_k=\langle V_k\rangle(1+v_k),\quad \langle v_k\rangle=0,\quad \delta=\max_{(k,\tau)}\{|v_k(\tau)|\}\ll 1$$

Then from (2), (4), (6) we obtain to order δ

$$E=\langle E\rangle(1+e),\quad F=\langle F\rangle+f;\quad \langle e\rangle,\ \langle f\rangle=0;\quad |e|,\ |f|\sim\delta\ll 1,\quad e=\gamma_1v_1+\gamma_2v_2,\quad f=\sigma_1v_1+\sigma_2v_2$$

$$\gamma_1=1-\frac{\langle V_1\rangle}{\langle A_1\rangle+\langle A_2\rangle},\quad \sigma_1=\frac{\langle V_1\rangle\langle A_2\rangle}{\langle A_1\rangle(\langle A_1\rangle+\langle A_2\rangle)}$$

Here, γ_2 and σ_2 are obtained by interchanging the indices.

Substituting the expressions found for E and F in the right-hand side of (9), we obtain to order δ

$$y^{(2)}=g+y_0(f_0-f)+\left[\dot{y}_0+y_0\dot{f}_0\left(1-4{,}5\frac{\langle E\rangle}{y_0^3}\right)\right]\left(\tau f-\int_0^\tau f\,d\tau\right)+1.5\langle E\rangle\left\{\frac{1+e_0}{y_0^3}(2\dot{y}_0-y_0\dot{f}_0)\tau-\frac{1}{y_0^2}+\frac{1}{g^2}+2(\dot{y}_0+y_0\dot{f}_0)\left[\int_0^\tau\frac{e}{g^3}d\tau+3y_0\int_0^\tau\frac{f-f_0}{g^4}d\tau\right]+3\int_0^\tau\frac{f}{g^2}d\tau\right\},\quad g(\tau)=y_0+(\dot{y}_0+y_0\dot{f}_0)\tau \tag{11}$$

If the spheres do not pulsate, then e, f $\equiv$ 0, and (11) takes a form corresponding to that found earlier for the case of the solution (10).

If at the initial time the spheres are at rest ($\dot{y}_0 = 0$), then, retaining in (11) only the most important terms, we have $y^{(2)} = y_0(1 + \dot{f}_0\tau)$, from which we see that when $\dot{f}_0 > 0$ the pulsating spheres begin to move away from each other, while for $\dot{f}_0 < 0$ they approach each other.

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DESTABILIZATION OF FLOWS WITH FREE SURFACE BY HIGH-FREQUENCY WAVES

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When wave packets of small but finite amplitude propagate in liquids and gases average fields (average flows, average displacements of the interfaces between different liquids, etc.) arise because of the nonlinearity of the media [1, 2], their amplitude being proportional to the square of the wave amplitude. The present paper is an investigation of such fields that arise when a packet of surface waves propagates on a horizontally inhomogeneous flow. It is shown that the average flows induced by the waves can strongly destabilize or stabilize the main flow.

We investigate the following mechanism by which the packet of surface waves influences the main flow. Suppose a slowly varying flow arises as a result of flow of the considered fluid past an obstacle (see Fig. 1). The nature of the flow depends on the Froude number $Fr = U/\sqrt{gH}$, where U is the flow velocity, H is the depth of the flow, and g is the acceleration due to gravity (it is assumed that if there is only a horizontal component it is uniform over the depth). If $Fr < 1$, the flow is subcritical, a quiescent flow; if $Fr > 1$, then it is supercritical [3]. If a wave packet propagates in a horizontally nonuniform flow, and the maximal Froude number Fr is near 1, then the average quantities due to the wave (the corrections to H and U) can change Fr and, therefore, stabilize or destabilize the flow.

We calculate the mean values η_c and u_c due to propagation of a steady packet of surface waves $\eta = a(x)e^{i(\omega t-kx)}$. We shall assume that the surface waves are irrotational: $u = \partial\varphi/\partial x$, $v = \partial\varphi/\partial y$ (φ is the potential, and x and y are the horizontal and vertical components of the velocity), u_c does not depend on the vertical coordinate y, and $U = U_0 + \Omega y$, where Ω is the vorticity of the main flow. From the Euler equations averaged with respect to the phase $\theta = \omega t - kx$, we obtain

$$\frac{\partial u_c}{\partial t}+\frac{\partial(U(H_1)u_c)}{\partial x}-\Omega\frac{\partial(u_cH)}{\partial x}+g\frac{\partial\eta}{\partial x}=-\frac{\partial}{\partial x}\left\langle\left(\frac{\partial\varphi}{\partial x}\right)^2-\left(\frac{\partial\varphi}{\partial y}\right)^2\right\rangle \quad (1)$$

Here, $\langle\ \rangle$ denotes averaging over θ, $H_1 = H + h$. Note that the value of the right-hand side of (1) does not depend on y:

$$\left\langle\left(\frac{\partial\varphi}{\partial x}\right)^2-\left(\frac{\partial\varphi}{\partial y}\right)^2\right\rangle=(\omega-kU(H_1))\frac{|a|^2}{\mathrm{sh}^2\,kH}$$

If $\Omega = 0$, then (1) can be obtained by averaging the Bernoulli equation. From the kinematic condition averaged with respect to θ on the surface of the fluid η we have

$$\frac{\partial\eta_c}{\partial t}+\frac{\partial(U(H_1))\eta_c}{\partial x}+\frac{\partial}{\partial x}(HU_c)=-\frac{\partial}{\partial x}\left\langle\Omega\eta^2+\eta\frac{\partial\varphi}{\partial x}\right\rangle \quad (2)$$

Equation (2) is the law of conservation of mass during the propagation of the surface wave.

In the stationary approximation, $\partial/\partial t = 0$, Eqs. (1)-(2) can be integrated:

$$u_c=\frac{1}{\beta}\left\langle g\left(\Omega\eta^2+\frac{\partial\varphi}{\partial x}\eta\right)-U(H_1)\left(\left(\frac{\partial\varphi}{\partial x}\right)^2-\left(\frac{\partial\varphi}{\partial y}\right)^2\right)\right\rangle \quad (3)$$

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