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On a Model of Frictional Sliding

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Abstract—A model of frictional sliding with an N-shaped curve for the sliding velocity dependence of the coefficient of friction is considered. This type of friction law is shown to be related to dynamic i.e., velocity dependent 'ageing' of asperity junctions. Mechanisms of 'ageing' for ductile (Bowden-Tabor) and brittle (Byerlee) materials, though different in nature, lead to qualitatively similar N-shaped velocity dependencies of the coefficient of friction. Estimates for the velocities limiting the range of negative velocity sensitivity of the coefficient of friction are obtained for the ductile case and—albeit with a lesser degree of reliability—for the brittle one. It is shown by linear stability analysis that discontinuous sliding (stick-slip) is associated with the *descending* portion of the N-shaped curve. An instability criterion is obtained. An expression for the period of the attendant relaxation oscillations of the sliding velocity is given in terms of the calculated velocity dependence of the coefficient of friction. It is suggested that the micromechanically motivated friction law proposed should be used in models of earthquakes due to discontinuous frictional sliding on a crustal fault.

Key words: Frictional sliding, crustal fault, stick-slip, stability analysis, models of earthquakes.

Introduction

A commonly accepted model for earthquakes on a pre-existing seismogenic crustal fault (DIETERICH, 1979) is based on rate- and state-dependent frictional sliding. The coefficient of friction is considered to depend on the sliding velocity as well as on an internal variable characterizing the state of the contacting surfaces. The main feature of the model, giving rise to unstable frictional sliding and attendant earthquakes, is a negative sensitivity of the shear stress τ to the sliding velocity V in the steady-state regime of sliding (DIETERICH, 1979; RICE 1983; GU *et al.*, 1984).

The underlying mechanism is attributed to some form of 'ageing' or strengthening of surface contacts (DIETERICH, 1979; SCHOLZ, 1990). In this respect, the only distinction between static and dynamic friction is that in the latter case, the ageing time is related to the sliding velocity. The ageing time, which is identified with

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the average time of contact of asperities on the sliding surfaces, is inversely proportional to V (DIETERICH, 1979). This implies that the average sliding velocity may be represented as

$$V = D_c / t_w \tag{1}$$

where D_c is a velocity-independent sliding distance (of the order of the asperity size, cf. DIETERICH, 1979 and DIETERICH and KILGORE, 1994) required for temporarily rearresting the sliding surfaces by a new population of contacting asperities upon the breakaway from the previous one and t_w is the average arrest time. It is during this time that the asperity contacts are exposed to the ageing process, whatever its physical nature. The strengthening effect associated with ageing increases with the arrest time and thus decreases with the sliding velocity.

The above dynamic process of asperity ageing is concurrent with the breakaway process (e.g., by shearing of asperities for a ductile material or by asperity breaking for a brittle one). This is commonly characterized by a positive rate sensitivity as revealed in an 'isostructural' velocity increment test which invariably exhibits a positive shear stress increment (DIETERICH, 1979). This jump in stress is followed by a transient leading to a steady-state stress level corresponding to the new value of V. It was suggested by DIETERICH (1979) to describe the transient by an equation for a 'relaxation' with time t of the ageing time t_a towards its steady-state value t_w given by (1):

$$\frac{dt_a}{dt} = -\frac{t_a - t_w}{t_w}, \tag{2}$$

with a relaxation time equal to t_w. This leads to an expression for the shear stress of the form

$$\tau = \mu(V, t_a)\sigma \tag{3}$$

where σ is the normal stress and $\mu(V, t_a)$ is the rate (V) and state (t_a) dependent coefficient of friction which in steady-state, when $t_a = t_w$, depends only on sliding velocity. (In the above equation, both stresses are nominal, i.e., they refer to the total area of the sliding object.)

The velocity dependence of τ at steady-state (ss) is of paramount importance as instability of a sliding regime depends primarily on the sign of the derivative $d\tau_{SS}/dV$. In a generally accepted picture (DIETERICH, 1979; RICE, 1983; DIETERICH, 1993; CHESTER and HIGGS, 1992), τ_{SS} monotonically decreasing with V is considered in the context of unstable sliding ('stick-slip') on a crustal fault. However, some experimental data (SHIMAMOTO, 1986; TEUFEL, 1981; DIETERICH, 1993), indicate that $d\tau_{SS}/dV$ ceases to be negative and a slight increase in τ_{SS} is recognizable at high sliding velocities. Since the existence of ascending portions of the τ_{SS} vs. V curve will have significant consequences for the characteristics of unstable frictional sliding, notably for the period of 'stick-slip', we shall discuss possible mechanisms giving rise to such a feature. It will be shown that an N-shaped τ_{SS} vs. V curve should be expected quite generally, for both ductile and brittle materials in frictional sliding. The period of stick-slip type relaxation oscillations in the unstable sliding regime will be related to the above-mentioned N-shaped curve.

The Case of a Ductile (Bowden-Tabor) Material

Direct quantitative observations of frictional contacts during slip (DIETERICH and KILGORE, 1994) suggest that ageing of contacting asperities can be related to an increase of contact area with the logarithm of time. Here it will be shown how creep of asperity contacts under the normal load can give rise to an N-shape friction characteristic for a ductile material. The frictional sliding process is assumed to be of the Bowden-Tabor type (BOWDEN and TABOR, 1950). Forces normal to the surfaces deform the contacting asperities causing them to 'weld' together. Forces parallel to the surfaces are determined by the resistance to shear of the junctions thus formed. It is further assumed that the normal load P is supported by N asperity junctions which constitute only a small fraction of the total asperity population (Fig. 1).

Initial 'instantaneous' loading leads to flattening of asperities so that the height of a junction is decreased from h_i down to h_o. The relation between h_o/h_i and the lateral size, a, of a columnar asperity junction can be evaluated from the relation

$$\frac{P}{Na^2} \cdot \frac{h_o}{h_i} \cong \sigma_y^c \tag{4}$$

where σ_y^c is the yield stress under compression. This relation is based on the assumption that the area of contact for a single asperity junction is inversely proportional to its height implying constancy of volume. Since these columnar junctions undergo creep under a compressive normal stress, their cross-sectional area increases as they shorten with the ageing time t_a (Fig. 2) as does the resistance to shear.

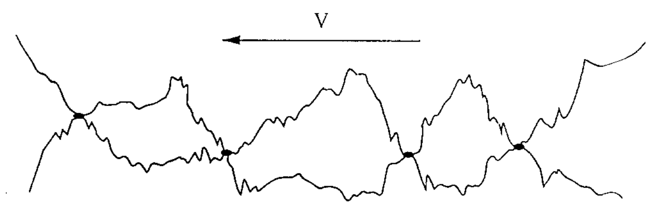

Figure 1
Geometry of asperity junctions (schematic).

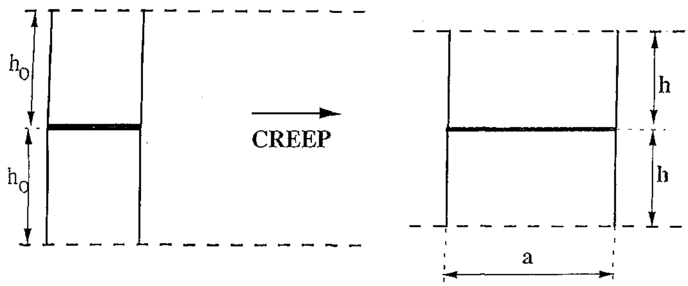

Figure 2
Creep of contacting asperities (schematic).

The creep rate is given by

$$\dot{\varepsilon}_{\text{creep}} = \dot{\varepsilon}_o \exp(\sigma_{\text{loc}}/S) \tag{5}$$

where S is the strain rate sensitivity of the flow stress, σ_{loc} is the local normal stress acting in an individual asperity junction, and $\dot{\varepsilon}_o$ is a pre-exponential factor considered to be stress independent. Both S and $\dot{\varepsilon}_o$ are temperature dependent. In terms of the variation of the height h of a junction with the ageing time t_a equation (5) can be rewritten as

$$\frac{dh/dt_a}{h_o} = -\dot{\varepsilon}_o \exp(\sigma_{\text{loc}}/S). \tag{6}$$

Using equation (4) the varying stress σ_{loc} is expressed as

$$\sigma_{\text{loc}} = \frac{P/N}{a^2(h_i/h)} = \sigma_y^c \frac{h}{h_o} \tag{7}$$

leading to

$$\frac{dh/dt_a}{h_o} = -\dot{\varepsilon}_o \exp\left(\frac{\sigma_y^c}{S} \cdot \frac{h}{h_o}\right). \tag{8}$$

In this derivation it has been assumed that the variation of the asperity height is sufficiently small, and h in the denominator on the left-hand side of (8) has been replaced by h_o. A further assumption is that no additional asperity junctions arise during the ageing time when the distance between the two surfaces decreases as the length of existing junctions decreases by creep. Under this assumption, a continual reduction of normal stress in an asperity results from the increase of cross-sectional area which is proportional to the creep rate. If we were to include the occurrence of new asperity contacts resulting from the sliding surfaces coming closer together due to the asperity flattening, the stress would also decrease as a function of an increase in N. For small decrements in h, the increment in N and consequently that in the total contact area, would be proportional to the creep rate. The concomitant increase in shear force stemming from the growing number of asperity junctions would be proportional to the creep rate as well. A simple way to take into account both effects is to include a multiplicative correcting factor $(1 + \xi)$ in the term describing the changes in the cross-section of an asperity junction, thus introducing a model parameter ξ.

Integration of (8) yields

$$h = h_o\left\{1 - \frac{S}{\sigma_y^c}\ln\left[1 + \left(\frac{\sigma_y^c}{S}\right)(\dot{\varepsilon}_o t_a)\exp\left(\frac{\sigma_y^c}{S}\right)\right]\right\}. \tag{9}$$

In the 'static ageing' case t_a is synonymous with the actual ageing time, while in the dynamic case, t_a obeys (2) and tends towards the steady-state value t_w given by (1). Notice that equation (9) states that the total contact area increases with time logarithmically. Accordingly the static coefficient of friction will obey a logarithmic time dependence in compliance with the classic work by RABINOWICZ (1965).

Equation (9) was obtained under the assumption that no strain hardening occurs during creep. As a consequence, it loses its validity at large times when strain hardening accumulates to inhibit creep of asperity junctions. A characteristic reduction of the asperity junction for creep to be suppressed, characterized by the asperity height h^*, can be estimated by equating the stress associated with strain hardening, $\theta \ln(h/h_o)$, to the normal stress $\sigma_y^c(h/h_o)$ acting in an asperity junction:

$$h^* \cong \frac{h_o}{1 + \sigma_y^c/\theta}. \tag{10}$$

Here θ denotes the strain hardening coefficient. As σ_y^c is smaller than θ, the quantity h/h_o is always sufficiently close to unity, which justifies the first-order expansions in $h - h_o$ made throughout this paper.

In a somewhat simplified description we will consider creep of asperity junctions to reduce their height down to the critical value h^* following equation (9) and then to terminate at that value of h. The above condition of termination of creep can be expressed in terms of a critical ageing time t_a^* related to h^* via equation (9) which, in turn, is related to a critical imposed sliding velocity V^* according to equation (1). That is to say, for velocities V below V^* the total contact area will be velocity independent. An estimate for V^* can be obtained by equating the asperity height h given by equation (9) to h^* expressed by equation (10) and inserting $t_a = t_w \cong a/V$:

$$V^* \cong (a\dot{\varepsilon}_o)\left(\frac{\sigma_y^c}{S}\right)\exp\left(\frac{\sigma_y^c}{S}\right)\left[\exp\left(\frac{\sigma_y^c}{\theta}\cdot\frac{\sigma_y^c}{S}\right) - 1\right]^{-1}. \tag{9'}$$

An asperity junction is plastically sheared with a local plastic strain rate of

$$\dot{\varepsilon}_{\text{loc}} \cong (a/h_o)\cdot(1/t_w) \tag{11}$$

where, as mentioned above, t_w can be estimated as a/V. As the process of shearing of asperity junctions and the process of asperity creep are governed by a common mechanism of plastic deformation, the shear stress necessary to produce the local strain rate given by (11) can be expressed in a form related to that of eq. (5):

$$\tau_{\text{loc}} = \tfrac{1}{2} S \ln\left(\frac{\dot{\varepsilon}_{\text{loc}}}{\dot{\varepsilon}_o}\right) = \tfrac{1}{2} S \ln\left(\frac{V}{h_o \dot{\varepsilon}_o}\right). \tag{12a}$$

The factor 1/2 stems from the Tresca approximation relating the shear strength to the tensile strength. Equation (12a) is valid for sufficiently high shear rates. At low shear rates, the dependence of the shear rate on shear stress is better represented by a linear than an exponential relation. A convenient interpolation of the two limits is given by a hyperbolic sine function. Accordingly, in this regime (12a) is replaced by

$$\tau_{\text{loc}} = \tfrac{1}{2} S sh^{-1}\left(\frac{\dot{e}_{\text{loc}}}{\dot{\varepsilon}_o}\right) = \tfrac{1}{2} S sh^{-1}\left(\frac{V}{h_o \dot{\varepsilon}_o}\right) \tag{12b}$$

where sh^{-1} denotes the inverse of the sh function. Obviously, for $V/(h_o \dot{\varepsilon}_o) \gg 1$, eq. (12b) reduces to eq. (12a). With the above equations, we can now calculate the velocity dependence of the coefficient of friction. Assuming that the variation in the height of an asperity junction during the 'ageing' time t_a is small, i.e., that $(h_o - h)/h_o \ll 1$, the nominal shear stress acting parallel to the sliding surfaces reads:

$$\tau = \frac{N \tau_{\text{loc}} a^2 (h_o/h)}{L^2} = \frac{N}{2}\left(\frac{a}{L}\right)^2 S \left\{1 + (1+\xi)\frac{S}{\sigma_y^c} \times \ln\left[1 + \frac{\sigma_y^c}{S}(\dot{\varepsilon}_o t_a) \exp\left(\frac{\sigma_y^c}{S}\right)\right] sh^{-1}\left(\frac{V}{h_o \dot{\varepsilon}_o}\right)\right\} \tag{13}$$

where L^2 is the total surface area. In equation (13), the parameter ξ accounts for the increase in the number of asperity contacts due to creep of the existing ones, as discussed above. The dynamic coefficient of friction,

$$\mu = L^2 \tau / P, \tag{14}$$

can now be calculated by combining eqs. (13) and (4):

$$\mu = \mu_o \left\{1 + (1+\xi)\frac{S}{\sigma_y^c} \ln\left[1 + \frac{\sigma_y^c}{S}(\dot{\varepsilon}_o t_a) \exp\left(\frac{\sigma_y^c}{S}\right)\right]\right\} \cdot \frac{S}{\sigma_y^c} sh^{-1}\left(\frac{V}{h_o \dot{\varepsilon}_o}\right) \tag{15a}$$

where

$$\mu_o = \frac{1}{2}\frac{h_o}{h_i} \cong \frac{1}{2}.$$

Equation (15a) only holds as long as creep continues, i.e., for V > V*. For $V < V^*$, the corresponding expression,

$$\mu = \mu_o \cdot \frac{h_o}{h^*}\frac{S}{\sigma_y^c} sh^{-1}\left(\frac{V}{h_o \dot{\varepsilon}_o}\right), \tag{15b}$$

describes a monotonic increase of the coefficient of friction with V. For the case of $V > V^*$ the steady-state (ss) coefficient of friction is given by

$$\mu_{SS} = \mu_o\left(1 + (1+\xi)\frac{S}{\sigma_y^c}\ln\left[1 + \left(\frac{\sigma_y^c}{S}\right)\left(\frac{h_o\dot{\varepsilon}_o}{V}\right)\exp\left(\frac{\sigma_y^c}{S}\right)\right]\right\}\cdot\frac{S}{\sigma_y^c}sh^{-1}\left(\frac{V}{h_o\dot{\varepsilon}_o}\right) \quad (16)$$

which yields a nonmonotonic, U-shaped μ versus V curve. A minimum of the coefficient of friction corresponds to a sliding velocity V_2.

Combining equations (15b) and (16), which together give the coefficient of friction in the entire velocity range, one recognizes that the resulting μ versus V curve is N-shaped (Fig. 3). Qualitatively, the three branches of the N-shaped curve can be understood as follows: In the low and the high velocity limits, the total contact area does not depend on the sliding velocity, and the velocity dependence of the coefficient of friction stems solely from the strain rate dependence of the local shear stress: In the intermediate regime, the behavior is dominated by an increase in the contact area with an increase of the ageing time associated with a decrease in the velocity. It should be noted that in the entire velocity range the *isostructural* ($t_a = \text{const}$) response of τ to a jump in V is positive, $(\partial\tau/\partial \ln V)_{\text{structure}} = S > 0$.

To get a feel for the orders of magnitude of the velocities V_1 and V_2 corresponding to the maximum and the minimum of the coefficient of friction, respectively, we may take values of parameters representative of pure metals, e.g.,

$$\dot{\varepsilon}_o \cong 10^{-7}\,\text{s}^{-1}, \quad \sigma_y^c/\theta \cong 0.25, \quad \text{and} \quad \sigma_y^c/S \cong 10.$$

The results also depend on the condition of the contacting surfaces. For example, for laboratory tests on metals one may take $h_o \cong 1\ \mu$m. Estimating V_1 by simultaneously solving eqs. (15a) and (15b) yields $V_1 \cong \dot{\varepsilon}_o h_o \exp(\sigma_c^y/\theta) \cong 2.2\cdot 10^{-3}\ \mu$m/s—a value in proximity of V^* as evaluated from eq. (9′). By minimizing the expression for μ given by eq. (16) one obtains $V_2 \approx 2.5\cdot 10^5 h_o\dot{\varepsilon}_o \approx 2.5\cdot 10^{-2}\ \mu$m/s. Thus, in this estimate, the range of negative velocity sensitivity of the coefficient of friction spreads over one decade in velocity. The magnitude of the coefficient of friction at

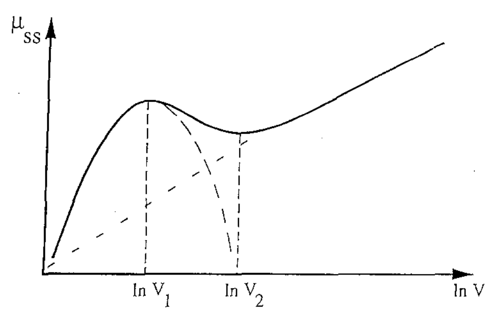

Figure 3
Sliding velocity dependence of the coefficient of friction in the steady-state regime. (Ductile case; schematic.)

the maximum, evaluated using equations (15b) with the above estimate for V_1, just slightly exceeds μ_o. i.e., it is about 1/2, which is consistent with the reported values of the static friction coefficient. It is worth emphasizing that what is commonly regarded as the static coefficient of friction is in fact the value of the dynamic one corresponding to a low velocity at the limit of the measurement sensitivity.

The Case of a Brittle (Byerlee) Material

In his paper on the friction based on brittle fracture BYERLEE (1967), suggested that it is more appropriate to assume that frictional sliding of many geologic materials is governed by brittle fracture of asperities. In other words, an asperity is assumed to fail in a brittle way, rather than by plastic shear. In the brittle case, the asperities do not yield plastically, and the number of contacting asperities is primarily governed by the surface topography. In this situation one has to distinguish between two principally different velocity regimes. In the high velocity case, the initial contact, which is 'tip-on-tip', has no time to evolve, and the stress required for sliding is that for crushing the asperities. This is the regime discussed by BYERLEE in his original paper (1967). As suggested in the same paper, interlocking of asperities may occur, and we shall consider this as an important effect in the low velocity range. Given enough time, contacting asperities will slide on each other creating fresh contact area. These new adhesive contacts have to be sheared for macroscopic sliding to proceed. In what follows we consider these high velocity (crushing) and low velocity (interlocking) regimes.

(i) *'Crushing' Regime (High Velocity).* The tip of a normally loaded conical asperity (Fig. 4) is crushed (in compression) under the load p and the tangential force f needed to break it (in tension) when the following condition (TIMOSHENKO and GOODIER, 1951) is satisfied:

$$f = \left(1 + \frac{T}{C}\right) kp \tag{17}$$

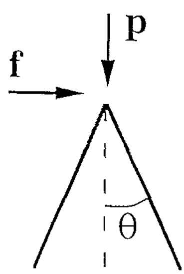

Figure 4
Normal (p) and tangential (f) forces on a brittle asperity.

where T and C denote the tensile and the compressive strength of the material respectively, and k is a proportionality constant. It is well known for brittle materials such as ceramics that the tensile strength is time dependent. This effect is referred to as 'static fatigue' [cf., ASHBY and JONES, 1986] and the time dependence can be described by a power-law

$$T = T_o(t_o/t)^{1/n} \tag{18}$$

where T_o and t_o are scaling quantities with the dimensions of stress and time respectively, and the exponent n is typically on the order of ten. For steady-state conditions, the time t in the above expression is obviously the waiting time at the asperity, $t = t_w \propto 1/V$. Hence, the shear stress τ is expressed in terms of the normal stress σ as

$$\tau = k^*\left[1 + \frac{T_o}{C}\left(\frac{t_o}{t_w}\right)^{1/n}\right] . \sigma \tag{19}$$

where $k^* = \text{const}$. This yields the steady-state coefficient of frictions

$$\mu_{ss} = k^*\left[1 + \frac{T_o}{C}\left(\frac{t_o}{t_w}\right)^{1/n}\right]. \tag{20}$$

(ii) *'Interlocking Regime' (Low Velocity).* We assume that the rate of sliding of two contacting asperities on each other and thus the rate of creation of new adhesive area, is proportional to some power of local shear stress acting along the common surface. One possible mechanism of this process is breaking of what we may call 'secondary asperities' on the surface of the primary ones. Considering this particular mechanism which is illustrated by Figure 5 will allow the analysis to be more focussed, while capturing the main features which are common to any mechanism by which new adhesive surface is created.

We assume that the upper asperity can slip down the slope of the lower asperity by breaking 'secondary asperities' on its surface. The rate of growth of the length of contact l with the dwell time at a 'primary asperity', i.e., the steady-state ageing time, t_w, is given by

$$\frac{dl}{dt_w} = b/t_b \tag{21}$$

where b is the average distance between secondary asperities (considered to be of the same order of magnitude as their size) and t_b is the average time required to break a secondary asperity. The latter quantity is related to the force p via equation (18) where

$$T \approx \frac{p \cos \theta}{lb} . \tag{22}$$

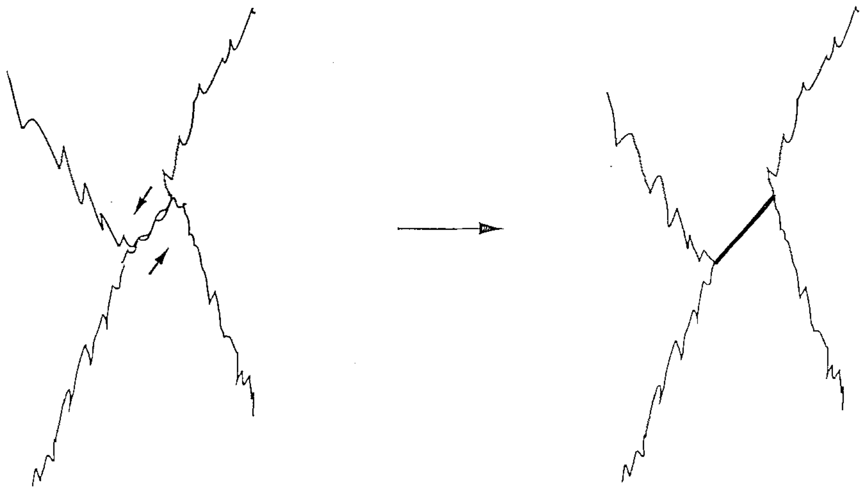

Figure 5
Formation of an adhesive junction between contacting primary asperities by breakage of secondary asperities (schematic).

Here lb represents the contact area and θ is the asperity angle measured from its axis. Equation (21) is then rewritten as

$$\frac{dl}{dt_w} = \frac{b}{t_o}\left(\frac{p\cos\theta}{T_o bl}\right)^n \tag{23}$$

which yields, upon integration,

$$l = (n+1)^{1/(n+1)}\left(\frac{t_w}{t_o}\right)^{1/(n+1)}\left(\frac{p\cos\theta}{T_o b^2}\right)^{n/(n+1)} \cdot b. \tag{24}$$

The force necessary to shear an adhesive junction along the contact of length l is given by

$$f\sin\theta = lb\tau_j \tag{25}$$

where f is the force to be applied in the direction normal to the asperity axis, i.e., parallel to the fault. The local shear stress in the junction, τ_j, is related to the local strain rate in the junction, $\dot{\gamma}_{\text{loc}}$, through an Arrhenius-type equation yielding

$$\tau_j = Ssh^{-1}\left(\frac{\dot{\gamma}_{\text{loc}}}{\dot{\gamma}_o}\right) \tag{26}$$

where S denotes the strain rate sensitivity of the shear stress and $\dot{\gamma}_o$ is the pre-exponential function in the Arrhenius equation.

The local shear stress in a junction can be evaluated as the ratio of the strain ($\sim h/(h \cdot 2\tan\theta) = 1(2\tan\theta)$) associated with shearing the junction and the time t_w,

$$\dot{\gamma}_{\text{loc}} = (2t_w \tan\theta)^{-1}. \tag{27}$$

Using equations (24)–(26), the following relation between the total shear force $F = Nf$ and the total normal force $P = Np$ (where N is the number of adhesive junctions) is obtained

$$\mu_{SS} = F/P = (n+1)^{1/(n+1)} \frac{S \operatorname{cotan} \theta}{T_o} \left(\frac{t_w}{t_o}\right)^{1/(n+1)} sh^{-1}\left(\frac{\operatorname{cotan} \theta}{\dot{\gamma}_o t_w}\right). \tag{28}$$

It is readily seen that the coefficient of friction in steady state, μ_{SS}, given by equation (28) depends on the waiting time and on the sliding velocity in a nonmonotonic way. At low velocities the velocity dependence is described by the power-law dependence $\mu_{SS} \sim V^{1-1/(n+1)}$ and is dominated by the rate dependence of the shear strength. At larger velocities, but still in the 'interlocking' regime, a decrease of μ_{SS} with V stemming from the other time dependent factor prevails.

The combined velocity dependence of the coefficient of friction arising from the two different velocity regimes is again N-shaped. Qualitatively, for high sliding velocities (Byerlee regime proper) no junctions have time to form and primary asperities have to be crushed. Since the tensile strength increases with strain rate, μ_{SS} is an increasing function of V. At lower velocities (interlocking regime), secondary asperities are broken, and the longer the waiting time, the larger is the length of a junction formed along a smooth 'cleaved' slope of a primary asperity. In this range of sliding velocities the coefficient of friction increases as V decreases. Finally, at very low sliding velocities, asperity junctions are fully formed, and only the velocity dependence of the tensile strength is relevant. Here, again, the coefficient of friction decreases with decreasing sliding velocity.

It follows from equation (28) describing friction in the interlocking regime that the maximum on the μ_{SS} vs. V diagram (Fig. 6) corresponds to the following value of the sliding velocity:

$$V_1 = \frac{a\dot{\gamma}_o}{\operatorname{cotan} \theta} e^{n+1}. \tag{29}$$

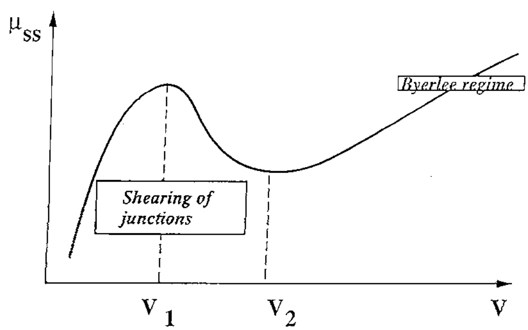

Figure 6
Sliding velocity dependence of the coefficient of friction in the steady-state regime. (Brittle case; schematic. Note that the diagram is drawn strongly not to scale in order to display both ascending branches of the curve.)

(In this derivation, the sh^{-1} function has been replaced with the logarithmic one.) With $\dot{\gamma}_o$ of the order of 10^{-7}–$10^{-6}\,\mathrm{s}^{-1}$, $a \cong h_o \cong 1\,\mu\mathrm{m}$ and n of the order of ten, the velocity V_1 is estimated to be of the order of 10^{-2}–$10^{-1}\,\mu\mathrm{m/s}$ and to be significantly larger than for ductile (Bowden-Tabor) materials.

A fortunate circumstance that made it possible to evaluate V_1 is that the time t_o determining 'static ageing' does not enter eq. (29). Unfortunately, a reliable evaluation of the velocity V_2 corresponding to the minimum of the coefficient of friction is more difficult. An estimation of this velocity signifying a transition from the interlocking to the crushing regime and from the intercept of the curves described by equations (20) and (28) involves parameters whose values are uncertain, notably t_o. Using, as an example, the parameter values of $S/T_o = 0.1$, $T_o/C = 0.1$, $a = 1\,\mu\mathrm{m}$, $\dot{\gamma}_o = 10^{-3}\,\mathrm{s}^{-1}$, $n = 10$ and $t_o = 0.1$ s, one finds that $V_2 \cong 10^2\,\mu\mathrm{m/s}$. However, for the reason mentioned, this value cannot be used with any confidence, and the above estimate was only meant to demonstrate that an N-shaped characteristic is possible for plausible values of the parameters in the brittle case as well. Although this is an unsatisfactory situation, we can still conclude that the 'N-shapedness' of the μ_{SS} vs. V characteristic found in the previous section for ductile materials also holds for brittle ones and thus is a rather general feature of frictional sliding. This is true even though the whole range of velocities in which this shape can be revealed in a particular system may not always be accessible to tests.

Stick-slip on a Fault

Both the Bowden-Tabor and the Byerlee types of materials will be treated within a common frame: the shear stress τ is considered to depend on the 'ageing time' t_a that is identical to the waiting time t_w for steady-state sliding and relaxes to t_w according to equation (2) otherwise. The waiting time t_w is given by a/V. In steady state, the dependence of τ on V is represented by an N-shaped function $\sigma\mu_{SS}(V)$ shown in Figure 7, along with the curve for $\sigma\mu_{SS}(t_w)$.

We adopt the model of a fault patch embedded in an elastic body (RICE, 1983; DIETERICH, 1993). Stress acting on the patch is determined by remote stressing, slip on the patch, and the elastic compliance. The patch is considered as a one-degree-of-freedom system. The equation of motion for the slip displacement δ reads

$$\tau(t) - K\delta = \sigma \cdot \mu(t_a) \tag{30}$$

where $\tau(t)$ is the remotely applied stress and—$K\delta$ is the decrease in stress due to fault slip, and K being the elastic stiffness. The right-hand side of the equation represents the frictional resistance to sliding that depends on the ageing time t_a. The function $\mu(t_a)$ has the property that it reduces to the function $\mu(t_w)$ depicted above when t_a relaxes to its quasi-steady state value, $t_w = D_c V \cong a/\dot{\delta}$ according to equation (2). The inertia effects have been neglected.

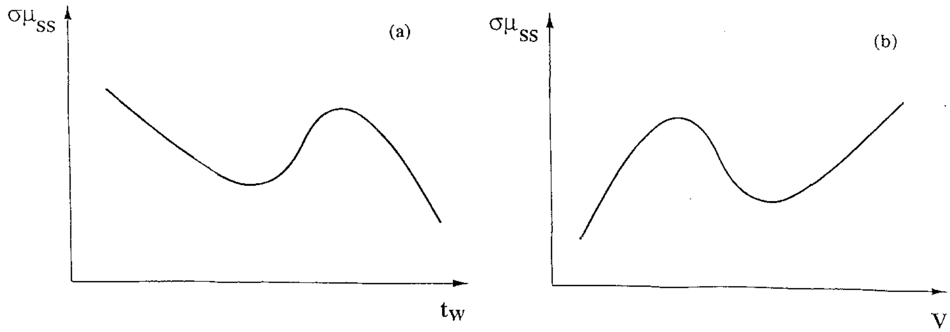

Figure 7
Dynamic coefficient of friction as a function of (a) the waiting time t_w and (b) the sliding velocity V.

It is interesting to note (RICE, 1992; ESTRIN, 1995) that mathematically, the system described is similar to another system undergoing dynamic ageing, namely, to a material exhibiting the so-called Portevin-Le Chatelier effect (KUBIN and ESTRIN, 1985; MCCORMICK, 1988). In such a material, dislocations temporarily arrested at localized obstacles are additionally pinned by diffusing solutes, the pinning effect being an increasing function of the waiting time at the obstacles i.e., a decreasing function of the rate of straining. A theory of this phenomenon (KUBIN and ESTRIN, 1985) can account for the effects observed. The model of frictional sliding under consideration can be treated using the parallel with the Portevin-Le Chatelier effect. Stability of sliding with a constant velocity can easily be tested by linear stability analysis, i.e., by considering the behavior with time of a small perturbation of such a motion. Following a common approach (see, e.g., GU *et al.*, 1984; DIETERICH, 1992), we denote a perturbation in δ by $\Delta\delta$ and a perturbation in t_a by Δt_a, and linearize equations (30) and (2) with respect to these perturbations which are considered to be small. The resulting set of two linear differential equations is solved by taking both $\Delta\delta$ and Δt_a to be proportional to $\exp(\omega t)$ where the sign of the 'growth parameter' ω indicates whether the perturbation will grow or decay with time. The set of the linearized equations mentioned reads:

$$K\Delta\delta + \sigma \frac{d\mu}{dt_a} \Delta t_a = 0$$

$$\frac{t_a \omega}{\Lambda} \Delta\delta + \left(\frac{V}{\Lambda} + \omega\right) \Delta t_a = 0. \tag{31}$$

The condition for this set of equations to have nontrivial solutions yields for ω:

$$\omega = \frac{KV}{\sigma d\mu/d \ln t_a - K\Lambda}. \tag{32}$$

The onset of instability is signified by the condition that perturbations be growing, i.e., that ω be positive. This means that

$$\sigma \frac{d\mu}{d \ln t_a} > K\Lambda. \tag{33}$$

In terms of the sliding velocity dependence of the coefficient of friction this inequality is tantamount to

$$\frac{d\mu_{ss}}{d \ln V} < -\frac{K\Lambda}{\sigma}. \tag{34}$$

The rate sensitivity of the coefficient of friction has thus to be negative and sufficiently large for unstable sliding to occur. For zero stiffness just the negativeness of the rate sensitivity is sufficient for instability to occur.

Let us now go beyond the linear stability analysis and try to elucidate the time dependence of frictional sliding. Following DIETERICH (1992) we consider the case of constant rate of remote stressing, $\tau(t) = \tau_o + \dot{\tau}t$ ($\tau_o = \text{const}$), and assume for simplicity that 'relaxation' of t_a to t_w is quasi-instantaneous, so that $\mu(t_a)$ in equation (30) can be replaced by $\mu_{SS}(t_w)$. Differentiation of (30) with respect to time then gives

$$\dot{\tau} - KV = \sigma \frac{d\mu_{ss}}{dV} \dot{V}. \tag{35}$$

A steady-state solution corresponding to $\dot{V} = 0$ is

$$V^* = \frac{\dot{\tau}}{K}. \tag{36}$$

Rewriting equation (35) as

$$\dot{V} = -\frac{V - V^*}{\dfrac{\sigma}{K}\dfrac{d\mu_{ss}}{dV}} \tag{37}$$

we recognize that the velocity tends to its steady-state value. This steady state cannot be reached if V^* happens to fall within the interval (V_1, V_2) of negative rate sensitivity of the coefficient of friction. Indeed, suppose $V_1 < V^* < V_2$. For velocity $V < V_1$, the denominator in equation (37) is positive, and the velocity will monotonically increase until at the point $V = V_1$ the time derivative becomes infinite, and the system has to jump onto the opposite ascending branch of the μ_{SS} vs. V characteristic. The velocity then acquires the value $V = \tilde{V}_1$. As seen from (37), the time derivative of V is now negative and the velocity decreases monotonically reaching the point $V = V_2$ where the derivative diverges and the system jumps onto the low-velocity ascending branch to acquire the value of $V = \tilde{V}_2$. This 'looping' around the region of negative rate sensitivity will be repeated over and over again

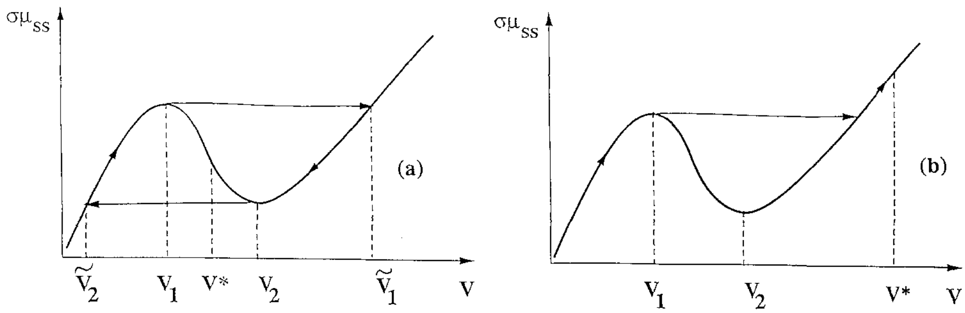

Figure 8
The trajectory of the representing point of the system on the $\sigma\mu_{SS}$ vs. V diagram: (a) relaxation oscillations (stick-slip) for the case of the imposed velocity V^* falling within the descending branch of the curve; (b) a single 'catastrophic' velocity jump for the case of V^* corresponding to the high-velocity ascending branch.

with the variation of V following the "slow-fast-slow-fast" sequence. The type of discontinuous oscillations described is commonly referred to as relaxation oscillations and it is obvious that the model proposed exhibits stick-slip behavior. It can be described as follows: If the rate of remote stressing imposes a steady-state strain rate which the system cannot support it chooses to maintain the prescribed velocity on the average by spending part of a cycle in low velocity regime (stick) and part in high velocity regime (slip). The transitions between these regimes of sliding are discontinuous (Fig. 8a). The time spent in each of the two regimes can easily be calculated once the $\mu_{SS} - V$ characteristic is known. Thus the time T_{stick} spent in the stick regime is given by

$$T_{\text{stick}} = -\frac{\sigma}{K}\int_{\tilde{V}_2}^{V_1} \frac{(d\mu_{SS}/dV)\,dV}{V - V^*} \tag{38}$$

and the time T_{slip} spent in the slip regime by

$$T_{\text{slip}} = -\frac{\sigma}{K}\int_{\tilde{V}_1}^{V_2} \frac{(d\mu_{SS}/dV)\,dV}{V - V^*}. \tag{39}$$

The period T of the relaxation oscillations is given by the cycle duration

$$T = T_{\text{stick}} + T_{\text{slip}}. \tag{40}$$

While periodic seismic acitivity is predicted by the model in the range of $KV_1 < \dot{\tau} < KV_2$, it is also interesting to mention that for high rates of stressing corresponding to $\dot{\tau} > KV_2$ a singular seismic activity event will occur. After a jump from V_1 to $\tilde{V}_1$, the velocity will increase monotonically until the steady-state velocity $V^* = \dot{\tau}/K > V_2$ will be reached with which subsequent sliding will continue (Fig. 8b).

Discussion and Conclusion

The idea that the velocity dependence of frictional sliding may be represented by an N-shaped curve is not entirely new. A friction law of this type was proposed by BARENBLATT *et al.* (1981). CARLSON and LANGER (1989) also indicated that such type of model might be appropriate, though they did not use this form of frictional law in their analysis of earthquake statistics. In the present paper, micromechanical models giving rise to N-shaped friction characteristic have been considered. It has been shown that for a wide range of materials including ductile (of Bowden-Tabor type) and brittle (of Byerlee type), the sliding velocity dependence of the steady-state coefficient of friction should be given by this type of characteristic curve (Fig. 7). For the Bowden-Tabor case, the N-shapedness of the friction characteristic directly follows from the fact that the contact area affected by creep under normal load is velocity independent in the limit cases of high and low sliding velocities. The descending portion of the curve is associated with the intermediate velocity range where the contact area increases with the asperity contact time and decreases with growing sliding velocity. The positions of the extrema on this curve can be fairly reliably estimated once the surface conditions, particularly, the average asperity size, are known. Depending on the material undergoing frictional sliding, the predicted general features of the $\mu - V$ characteristic may or may not be accessible to laboratory tests. In the Byerlee case, the N-shaped friction law has been predicted for brittle materials with a time dependent tensile strength ('static fatigue'). A combination of this effect occurring in the interlocking regime with direct asperity crushing mechanism in the high velocity regime has been shown to result in this type of characteristic.

We are not aware of any laboratory measurements that would substantiate the N-shape of the friction law. However, nonmonotonic behavior of the coefficient of friction with the sliding velocity has been reported repeatedly. We can only speculate that the 'inverted U' shape of the μ vs. V curve for the case of metal-on-metal friction (RABINOWITZ, 1965), represents a portion of an N-shaped curve near the maximum. The location of the maximum is consistent with the estimate made above for the ductile case. Singular observations for minerals or rocks tend to suggest N-shapedness, cf. SHIMAMOTO's (1986) observations on friction of halite, but there can be a multitude of reasons for that. DIETERICH and LINKER (1992) have found a minimum in the $\mu - V$ curve for granite which is perhaps again a portion of an N-shaped characteristic possibly around V_2. The variation of the minimum with the normal stress shows a tendency consistent with that observed by Dieterich and Linker. Of course all this does not warrant the validity of the friction law proposed in the present paper and more focused experiments targeting the nonmonotonic velocity dependence of friction are needed to verify it.

In the context of earthquake modeling, the N-shapedness of the friction characteristic appears to be of significance, as the periodicity of earthquakes on a fault associated with relaxation oscillations of the sliding velocity (stick-slip) is determined by the shape of the ascending branches of the characteristic rather than by its behavior in the range of negative rate sensitivity. It is therefore suggested that the shape of the coefficient of friction vs. sliding velocity characteristic proposed above be employed in earthquake modeling. All other ingredients of the Dieterich-Rice-Ruina approach, notably the kinetics of relaxation of the ageing time t_a to its steady-state value t_w (cf., equation (2)), should be retained. A nice feature of the model proposed is that for the prediction of the periodicity of seismic activity on a fault, only the shape of the ascending branches of the $\mu_{SS} - V$ characteristic together with the positions of the extrema, have to be known in the case of remotely applied stress. In this stress-driven case, the system never enters the interval of negative rate sensitivity. This is, of course, a consequence of neglecting dynamic effects in the model. Another consequence is that the velocity $\tilde{V}_1$ is overestimated in this approach. Inclusion of inertia would also affect the calculation of the integrals in eqs. (38) and (39) and make the analysis more complicated. In the present paper we chose to keep this part of the problem as simple as possible and concentrated on the effect of the form of the friction law. Thermal effects were not included for the same reason.

It should be mentioned that in the relaxation oscillations regime periodic occurrence of velocity jumps (i.e., periodic seismicity) is expected from the simple model outlined above. In recent years, stochastic models (BURRIDGE and KNOPOFF, 1967; CARLSON and LANGER, 1989; BÅK and TANG, 1989; CHRISTIANSEN and OLAMI, 1992), using frictional laws with $d\mu_{SS}/dV < 0$ in conjunction with spatial coupling were put forward to explain the earthquake statistics, viz. the Gutenberg-Richter law. A local μ vs. V characteristic of the type proposed in the present note, combined with such spatial coupling models, leads to interesting statistics of earthquakes. The mentioned parallel of both dynamic strain ageing and frictional sliding phenomena being associated with negative rate sensitivity was pursued further (LEBYODKIN *et al.*, 1995). Similarity of the statistics of stress drops during tensile deformation of a material exhibiting negative strain-rate sensitivity and the earthquake statistics was demonstrated.

We should like to conclude by stating that by all the existing uncertainties with the estimates of the extremal points of the proposed N-shaped friction characteristic this kind of velocity dependence of friction has a micromechanical justification and does not contradict the experimental evidence, however limited it might be. The model deserves further experimental scrutiny, in particular in view of the sensitivity of the instability condition and of the intrinsic periodicity of the relaxation oscillations, and hence of the associated seismicity, to the details of the friction vs. sliding velocity characteristic. Though spatial coupling will lead to a complex spatio-temporal pattern of seismic events, the core of the problem is the local friction law, and we believe that the N-shaped characteristic proposed in this article has a better micromechanical foundation than a monotonic μ vs. V law commonly used.

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