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ASYMPTOTIC BEHAVIOR OF THE NEUTRAL CURVES OF THE LINEAR STABILITY PROBLEM FOR A LAMINAR BOUNDARY LAYER

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Asymptotic flow schemes corresponding to two branches of the solution for the neutral stability curve of a laminar boundary layer in an incompressible fluid are constructed. Two-term asymptotic solutions are obtained in the limit when the Reynolds number tends to infinity. The linear formulation of the problem is used and the flow is assumed to be two dimensional.

1. We consider the two-dimensional laminar flow of a viscous incompressible fluid near a flat surface. We shall assume that small two-dimensional unsteady perturbations are superimposed on the main steady flow. We also assume that the characteristic transverse scale δ of the main flow is much less than the longitudinal scale L, and that the small unsteady perturbations have characteristic wavelength $\lambda \ll L$.

Then over a distance of order λ the main flow can be regarded as plane-parallel with a relative error of order λ/L. Under this assumption, the Navier—Stokes equations, linearized with respect to the small perturbations, can be written in the dimensionless form

$$\frac{\partial u}{\partial t}+V\frac{\partial u}{\partial x}+v\frac{\partial V}{\partial y}+\frac{\partial p}{\partial x}=\frac{1}{\alpha\,\mathrm{Re}}\left(\frac{\partial^2 u}{\partial y^2}+\alpha^2\frac{\partial^2 u}{\partial x^2}\right)$$
$$\frac{\partial v}{\partial t}+V\frac{\partial v}{\partial x}+\frac{1}{\alpha^2}\frac{\partial p}{\partial y}=\frac{1}{\alpha\,\mathrm{Re}}\left(\frac{\partial^2 v}{\partial y^2}+\alpha^2\frac{\partial^2 v}{\partial x^2}\right),\quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0 \tag{1.1}$$

Here, V(y) is the longitudinal velocity of the main flow divided by $V_\delta=\lim_{y\to\infty} V$; u, v are the small corrections to the longitudinal and transverse velocity divided by V_δ and $V_\delta\alpha$; p is the perturbation of the pressure divided by ρV_δ^2 (ρ is the density); the longitudinal and transverse coordinates x and y are divided by λ and δ, respectively; $\alpha = \delta/\lambda$; $\mathrm{Re} = \delta V_\delta/\nu$; ν is the kinematic viscosity.

We shall assume that the main flow corresponds to flow in a boundary layer, and therefore $\mathrm{Re}\,\delta/L = O(1)$. From this we conclude

$$(\alpha\,\mathrm{Re})^{-1}=O(\lambda/L)\ll 1$$

Thus, to the same relative accuracy with which the main flow can be regarded as plane-parallel, the right-hand sides of (1.1) in the case $\lambda \leqslant O(1)$ can be ignored in the main part of the flow, and the viscosity need be taken into account only in certain sufficiently thin layers $\Delta y < O(1)$.

Equations (1.1) can be readily reduced to the well-known Orr—Sommerfeld equation if the perturbations are assumed to be harmonic. We note however that in the layers Δy in which the viscosity of the gas is important the right-hand sides of Eqs. (1.1) admit certain simplifications; in these layers, with relative error $\alpha^2\Delta y^2$, we omit the

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term $\alpha^2\partial^2 u/\partial x^2$ in the first equation and on the right-hand of the second. In addition, we introduce the phase velocity c of propagation of the perturbations and make the substitutions

$$\xi = x - ct,\ W = V - c,\ u = q(y)\exp(i\xi),\quad v = W(y)\tau(y)\exp(i\xi),\ p = iR(y)\exp(i\xi)$$

Then from (1.1) we obtain

$$W^2\tau' + R = -i\varepsilon^3(W\tau)''',\quad \alpha^2 W^2\tau + R' = 0 \tag{1.2}$$

Here, $\varepsilon^3 = (\alpha\ \mathrm{Re})^{-1}$, and the primes denote the derivatives with respect to y.

Assuming that p, u, $v \to 0$ as $y \to \infty$ and u = v = 0 for y = 0, for the system (1.2) we obtain the boundary conditions

$$\tau,\ R \to 0,\ y \to \infty;\ \tau = \tau' = 0,\ y = 0$$

2. Since the influence of viscosity can be ignored in the greater part of the boundary layer, it follows from Rayleigh's theorem that for V(y) profiles without points of inflection the critical layer, on which W = 0, must be situated at an asymptotically small distance y_* from the surface of the body. At the same time, estimates of the orders of magnitude by means of (1.2) give $\alpha = O(y_*) = o(1)$, and the existence of two asymptotic flow schemes can be proposed: a) a scheme with three characteristic flow regions and b) a scheme with five regions. In what follows, we shall refer to the corresponding regions in which the influence of viscosity is important or unimportant as "viscous" or "inviscid." We consider the structure of the asymptotic schemes a) and b), denoting the corresponding flow regions by numbers.

Scheme a)

1. An inviscid region with dimension $\Delta y_1 = O(1/\alpha)$ and $W \approx 1 - c = \mathrm{const}$.

2. The main inviscid part of the boundary layer, $\Delta y_2 = O(1)$, $W = V - c$.

3. The viscous region next to the surface of the body, $\Delta y_3 = O(\varepsilon) = O(\alpha)$, $W = O(\varepsilon)$.

Scheme b)

1 and 2. Regions analogous to regions 1 and 2 in Scheme a).

3. The viscous region near the critical layer, which is detached from the surface of the body, $\Delta y_3 = O(\varepsilon)$, $W = O(\varepsilon)$, $\varepsilon = o(\varepsilon)$.

4. The inviscid region between the body and region 3, $\Delta y_4 = O(\alpha)$, $W = O(\alpha)$.

5. The viscous wall layer $\Delta y_5 = o(\alpha)$, $W = O(\alpha)$.

The solution of (1.2) in region 1 after corresponding normalization has the form

$$\tau = \exp(-\alpha y);\ R = \alpha(1-c)^2\tau \tag{2.1}$$

We seek the solution in region 2 in the form of asymptotic series in the small parameter α:

$$\tau = \tau_0 + \alpha\tau_1 + \alpha^2\tau_2 + \ldots;\ R = \alpha R_1 + \alpha^2 R_2 + \ldots \tag{2.2}$$

The boundary conditions as $y \to \infty$ for this solution follow from the expansion of the relations (2.1) in powers of α.

We assume that in region 2 as $y \to 0$ we can expand V in an asymptotic series of the form

$$V = a_1 y + a_2 y^2 + a_4 y^4 = O(\alpha^5),\ y = O(\alpha) \tag{2.3}$$

Here, $a_3 = 0$, if V corresponds to flow in a boundary layer [1].

Then for y_* and c we can write

$$y_* = \alpha y_1 + \ldots,\ c = a_1 y_1 \alpha + \ldots \tag{2.4}$$

Using (2.2), we obtain from (1.2)

$$\tau_0 = 1;\quad \tau_1 = \int_y^\infty (F^{-2} - 1)\,dy - y,\quad R_1 = (1-c)^2;\quad R_2 = (1-c)^2\left[\int_y^\infty (F^2 - 1)\,dy - y\right],\quad F = W/(1-c) \tag{2.5}$$

The integrals in (2.5) converge as $y \to \infty$ if the displacement thickness and the momentum loss thickness are finite. However, the integral in the relation for τ_1 diverges as $y \to y_*$.

Therefore, separating this singularity explicitly, we obtain

$$\tau^*=1+\frac{1}{a_1^2\eta}+\frac{2a_2}{a_1^3}\alpha\ln\alpha+\alpha\left(B+\frac{2a_2}{a_1^3}\ln\eta+\frac{R_2^*-4a_2y_1a_1^{-1}}{a_1^2\eta}\right)+\ldots \tag{2.6}$$

$$\tau^*=1+\frac{1}{a_1^2\eta}+\alpha\left(B+\frac{R_2^*}{a_1^2\eta}\right)+\alpha^2\ln\alpha\frac{12y_1^2a_4}{a_1^3}+\ldots \quad \text{for } a_2=0 \tag{2.7}$$

$$R^*=\alpha+\alpha^2R_2^*+\ldots;\ R_2^*=D-2a_1y_1,\quad B=\int_0^\infty\left(V^{-2}-\frac{1}{a_1^2y^2}+\frac{2a_2}{a_1^3y(1+y)}-1\right)dy;\quad D=\int_0^\infty(V^2-1)\,dy \tag{2.8}$$

Here, $\eta=(y-y_*)/\alpha$, and y_1 is determined from (2.4).

It follows from (2.6) and (2.7) that for $y=O(\alpha)$ the solution can be represented in the form

$$\tau^*=\tau_1^*+\alpha\ln\alpha\tau_2^*+\alpha\tau_3^*+\alpha^2\ln\alpha\tau_4^*+\alpha^2\tau_5^*+\alpha^3\ln\alpha\tau_6^*+\alpha^3\tau_7^*+\alpha^4\ln\alpha\tau_8^*+\ldots$$

$$R^*=\alpha+\alpha^2R_2^*+\alpha^3R_3^*+\alpha^4R_4^*+\ldots,\quad y_*=\alpha y_1+\alpha^2\ln\alpha y_4+\alpha^2y_3+\ldots \tag{2.9}$$

Using the expansions (2.3) and (2.9) and bearing in mind that R_2^*, R_3^*, R_4^* can be regarded as constants in region 2 for $y=O(\alpha)$, we obtain from (1.2)

$$\tau_4^*=S_4(\eta);\ \tau_5^*=S_5(\eta)+2a_2a_1^{-3}R_2^*-6y_1a_2a_1^{-1}\ln\eta \tag{2.10}$$

In the case $a_2=0$, the following three terms in the expansion of τ^* have the form

$$\tau_6^*=\text{const};\qquad \tau_7^*=S_7(\eta)+12y_1^2a_4a_1^{-3}\ln\eta;\qquad \tau_8^*=\text{const} \tag{2.11}$$

Here, $S_i(\eta)$ are polynomials in integral powers of η beginning with the degree -1 and higher.

3. We find the solution in region 3 for case a).

We introduce the expansions

$$\varepsilon^3=\alpha^3(\varepsilon_1+\varepsilon_2\alpha\ln\alpha+\varepsilon_3\alpha+\ldots),\quad d\tau/d\eta=\varphi_1+a_1^{-2}\eta^{-2}+\varphi_2\alpha\ln\alpha+[\varphi_3-a_1^{-2}\eta^{-2}(R_2^*-4a_2a_1^{-1}y_1)]\alpha+\ldots \tag{3.1}$$

For $a_2=0$ we shall assume $\varepsilon_2=\varphi_2=0$ as follows from the conditions (2.6) in the outer region 2.

In region 3, the two-term expansion of W has the form

$$W=\alpha\eta a_1+\alpha^2\eta a_2(\eta+2y_1)+\ldots$$

Substituting this relation together with the expansion of R^* in (1.2), we find

$$\eta^2a_1\varphi_1=-i\varepsilon_1(\eta\varphi_1''+3\varphi_1'),\quad \eta^2a_1\varphi_k=-i\varepsilon_k(\eta\varphi_1''+3\varphi_1')-i\varepsilon_1(\eta\varphi_k''-3\varphi_k') \tag{3.2}$$

Here, $k=2$ for $a_2\neq0$ and $k=3$ for $a_2=0$. After a change of variables, (3.2) is reduced to

$$r^2\sigma_1=-i(r\sigma_1''+3\sigma_1'),\qquad r^2(\sigma_k-\sigma_1\varepsilon_k^\circ)=-i(r\sigma_k''+3\sigma_k')$$

$$r=a_1^{1/3}\varepsilon_1^{-1/4}\eta;\quad \varphi_i=a_1^{-4/3}\varepsilon_1^{-2/3}\sigma_i\ (i=1,2,3);\quad \varepsilon_k^\circ=\varepsilon_k\varepsilon_1^{-1} \tag{3.3}$$

The boundary conditions corresponding to matching to the solution in region 2 follow from (3.1) and (2.6):

$$\sigma_1\to0,\quad \sigma_k\to0,\quad r\to\infty \tag{3.4}$$

The conditions on the surface of the body follow from (1.3):

$$d\tau/d\eta=0,\quad \eta=\eta_w=-y_*/\alpha \tag{3.5}$$

Using the expression (2.8) for y_*, we obtain

$$r_w=r_1+r_2\alpha\ln\alpha+r_3\alpha+\ldots$$

(In the case $a_2=0$, $r_2=0$.)

Then from the boundary condition (3.5), going over to the variable r and σ, and expanding the solution $\sigma(r_w)$ in a series at the point $r = r_1$, we obtain

$$\sigma_1=r_1^{-2};\quad \sigma_2=-r_2(\sigma_1'+2r_1^{-3}) \quad \text{for} \quad a_2\neq 0,\quad \sigma_1=r_1^{-2};\quad \sigma_3=-r_3(\sigma_1'+2r_1^{-3}+R_2^*r_1^{-2}) \quad \text{for} \quad a_2=0 \tag{3.6}$$

Note that the value of τ in region 2 is real, from which there follows the condition

$$\operatorname{Im}\int\limits_{r_w}^{\infty}(\sigma_1+\sigma_2\alpha\ln\alpha+\sigma_3\alpha+\ldots)\,dr=0$$

The lower limit of integration can (with the accuracy needed for a three-term expansion of σ) be replaced by r_1, since $\operatorname{Im}\sigma_1(r_1) = 0$, which leads to the relation

$$\operatorname{Im}\int\limits_{r_1}^{\infty}\sigma_1\,dr=0,\quad \operatorname{Im}\int\limits_{r_1}^{\infty}\sigma_k\,dr=0 \tag{3.7}$$

For the solution of the first of Eqs. (3.3), the subsidiary condition (3.7) is satisfied by an appropriate choice of r_1, which determines the value of this parameter.

The solution of the second equation of the system (3.3) is related to the solution of the first by

$$\sigma_k=C\sigma_1-\varepsilon_k^\circ\sigma_1'r/3 \tag{3.8}$$

Here, C is an arbitrary real constant, which follows from the conditions (3.7).

At the same time, the relations (3.4) are satisfied.

To determine C and ε_k°, we use the boundary conditions (3.6). The solution for σ_1 shows that $\operatorname{Im}\sigma_1' \neq 0$ at $r = r_1$. Therefore, equating the real and imaginary parts of σ_k from (3.6) and (3.8), we obtain

$$r_k=\varepsilon_k^\circ r_1/3;\quad C=-2\varepsilon_2^\circ/3 \quad \text{for } a_2\neq 0,\quad C=-2\varepsilon_3^\circ/3+R_2^* \quad \text{for } a_2=0 \tag{3.9}$$

We find the values of ε_1 and ε_k° from the conditions of matching to the solution (2.6) in region 2. For the first term of the expansion (2.6), using (3.1), we have

$$\int\limits_{\eta_1}^{\infty}\varphi_1\,d\eta-a_1^{-2}\eta_1^{-1}=1$$

Hence, going over to σ and r,

$$\varepsilon_1^{1/3}=a_1^{-5/3}N,\quad N=\int\limits_{r_1}^{\infty}\sigma_1\,dr-r_1^{-1} \tag{3.10}$$

For the second term of the expansions (3.1) and (2.6),

$$\int\limits_{\eta_1}^{\infty}\varphi_2\,d\eta=2a_2a_1^{-3}$$

Going over to σ and r and bearing in mind that in accordance with (3.8) and (3.9) we have $\sigma_2=-\varepsilon_2^\circ[\sigma_1+(r\sigma_1)']/3$, we obtain

$$\int\limits_{r_1}^{\infty}\sigma_2\,dr=-\varepsilon_2^\circ N/3,\quad \varepsilon_2^\circ=-6a_2a_1^{-3} \tag{3.11}$$

Treating similarly the case $a_2 = 0$, when $\varphi_2 = r_2 = 0$, we obtain for ε_3°

$$\varepsilon_3^\circ=3(R_2^*-B) \tag{3.12}$$

In accordance with numerical calculations, $N = 0.9993$ and $r_1 = -2.2972$. Finally, recalling that $\varepsilon^3 = (\alpha\ \mathrm{Re})^{-1}$, and substituting ε_1 and ε_k° in (3.1), we obtain

$$\begin{aligned}\mathrm{Re}^{-1}&=0.9979a_1^{-5}\alpha^4[1-6a_2a_1^{-3}\alpha\ln\alpha+O(\alpha)],\quad a_2\neq 0\\ \mathrm{Re}^{-1}&=0.9979a_1^{-5}\alpha^4[1+3(R_2^*-B)\alpha+O(\alpha^2\ln\alpha)],\quad a_2=0\end{aligned} \tag{3.13}$$

The values of R_2^* and B can be calculated from the relations (2.8), and for R_2^* we have

$$R_2^*=D+2r_1Na_1^{-1} \tag{3.14}$$

4. We find a connection between ε and α for the flow scheme b) with critical layer detached from the surface of the body.

In this case, to satisfy the condition (1.3) on the surface of the body we need to know the solution in region 4, in which the viscosity of the gas can be ignored with an error of order $(\varepsilon/\alpha)^3$.

The relations (2.6), (2.7), (2.10), and (2.11) satisfy the equations of motion without allowance for viscosity, but the constants which occur in these expressions may change on the transition through the critical layer to negative values. If we assume that the case of neutral oscillations is a limiting case for damped ($\mathrm{Im}\, c = \mathrm{Im}\, W < 0$) or growing ($\mathrm{Im}\, c > 0$) oscillations, then on the transition through the critical layer $\ln|\eta|$ must change by πi or $-\pi i$, respectively. In accordance with [2], in the passage to the limit it is necessary to use the growing oscillations, i.e., in region 4 the relations (3.4), (3.5), (3.11), and (3.12) do not contain $\ln \eta$ but the expression $\ln|\eta| - \pi i$, if $a_1 > 0$.

Bearing in mind that $\eta_w = y_*/\alpha$ on the surface of the body ($y = 0$), we obtain from the expansion (3.10)

$$\eta_w = \eta_1 + \eta_2 \alpha \ln \alpha + \eta_3 \alpha + \ldots \tag{4.1}$$

In (4.1), the η_i are real numbers. Therefore, we can satisfy the condition $\tau_w = 0$ in each approximation by choosing appropriately the values of η_i until imaginary quantities appear in the expansions for τ.

For example,

$$\eta_1 = -a_1^{-2}, \quad \eta_2 = 2a_2 a_1^{-5} \tag{4.2}$$

The fulfillment of this condition for complex values of the expansion coefficients and the satisfaction of the condition $\tau' = 0$ for $\eta = \eta_w$ require solutions of the system (1.2) in the wall boundary layer. By means of the relations (2.6), (2.10), and (2.11), the condition $\tau_w = 0$ can be satisfied for $a_2 \neq 0$ to the term of order α and for $a_2 = 0$ to the term of order α^3.

Bearing this fact in mind, we introduce in the wall boundary layer new variables equal in order of magnitude to unity:

$$\gamma = \tau/\alpha; \quad n = (\eta - \eta_w)/\alpha; \quad \varkappa = \varepsilon^3/\alpha^5 \ \text{ for } a_2 \neq 0, \quad \gamma_0 = \tau/\alpha^3; \quad n_0 = (\eta - \eta_w)/\alpha^3; \quad \varkappa_0 = \varepsilon^3/\alpha^9 \ \text{ for } a_2 = 0 \tag{4.3}$$

Then in the boundary layer

$$W = a_1(\eta_1 + \eta_2 \alpha \ln \alpha)\alpha + O(\alpha^2) \ \text{ for } a_2 \neq 0, \quad W = a_1(\eta_1 + \eta_3 \alpha)\alpha + O(\alpha^3 \ln \alpha) \ \text{ for } a_2 = 0$$

Thus, if we restrict ourselves to two terms of the expansion, the value of W in the wall boundary layer can be regarded as constant.

Since the value of R is also constant to the necessary accuracy for two expansion terms, for the flow in the boundary layer we obtain

$$W_b^2 \gamma_b' + R_b = -i \varkappa_b W_b \gamma_b''' \tag{4.4}$$

Here, the primes denote the derivatives with respect to n and n_0:

$$W_b = a_1(\eta_1 + \eta_2 \alpha \ln \alpha); \quad \gamma_b = \gamma_1 + \gamma_2 \alpha \ln \alpha, \quad R_b = 1; \quad \varkappa_b = \varkappa_1 + \varkappa_2 \alpha \ln \alpha \ \text{ for } a_2 \neq 0$$

Similarly,

$$W_b = a_1(\eta_1 + \eta_3 \alpha); \quad \gamma_b = \gamma_{10} + \gamma_3 \alpha, \quad R_b = 1 + R_2^* \alpha; \quad \varkappa_b = \varkappa_{10} + \varkappa_3 \alpha \ \text{ for } a_2 = 0$$

The solution of (4.4) satisfying the conditions $\gamma_b = \gamma_b' = 0$ for $n = 0$ ($n_0 = 0$) and $\gamma_b''' \to 0$ as $n \to \infty$ has the form

$$W_b^2 \gamma_b = -n + (1+i)\{\exp[Qn(i-1)] - 1\}(2Q)^{-1}, \quad Q = W_b^{1/2}(2\varkappa_b)^{-1/2} \tag{4.5}$$

It follows from (4.5) that in the limit $n \to \infty$

$$\mathrm{Im}\, \gamma_b \to \sqrt{-\varkappa_b/(2W_b^5)}$$

Restricting ourselves to two-term expansions, we obtain

$$\mathrm{Im}\, \gamma_b \to \sqrt{-\frac{\varkappa_1}{2a_1^5 \eta_1^5}} \left[1 + \left(\frac{\varkappa_2}{\varkappa_1} - 5\frac{\eta_2}{\eta_1}\right)\frac{\alpha \ln \alpha}{2}\right] \ \text{ for } a_2 \neq 0$$

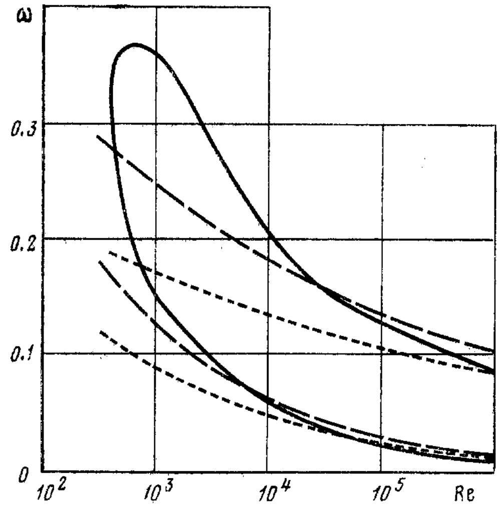

Fig. 1

$$\operatorname{Im}\gamma_b \to \sqrt{-\frac{\varkappa_{10}}{2a_1^5\eta_1^5}}\left[1+\left(\frac{\varkappa_3}{\varkappa_{10}}-5\,\frac{\eta_3}{\eta_1}\right)\frac{\alpha}{2}\right] \quad \text{for } a_2=0$$

To determine $\varkappa_1$, $\varkappa_{10}$, $\varkappa_2$, $\varkappa_3$, we use the conditions of matching to the exterior solution in region 4, i.e., the relations (2.6), (2.7), (2.10), and (2.11) with ln η replaced by ln |η| − iπ. In the cases $a_2<0$ or, respectively, $a_4<0$, using (4.2) and the two-term expansions of $\varkappa$ and $\varkappa_0$, we finally obtain

$$\mathrm{Re}^{-1}=8\pi^2 a_2^2 a_1^{-11}\alpha^6[1+10a_2a_1^{-3}\alpha\ln\alpha+O(\alpha)] \quad \text{for } a_2\neq 0$$

$$\mathrm{Re}^{-1}=288\pi^2 a_4^2 a_1^{-19}\alpha^{10}[1+(11R_2^*-9B)\alpha+O(\alpha^2\ln\alpha)] \quad \text{for } a_2=0 \tag{4.6}$$

The values of R_2^* and B can be calculated by means of (2.8) if we bear in mind that $y_1=-\eta_1=a_1^{-2}$.

5. The relations (3.13) and (4.6) determine the asymptotic behavior of the curves of neutral stability in the limit Re → ∞. The solutions (3.13) correspond to the "lower" branch of the neutral stability curve, the solutions (4.6) to the "upper" branch. Note that the first terms of the asymptotic expansions obtained above (these terms were found earlier in [3] from the Orr—Sommerfeld equations) are identical for the relations (4.6) and differ by less than 1% from (3.13).

In Fig. 1, we give an example of calculation of the asymptotic behavior of the neutral curves for the Blasius profile ($\delta=\delta^*$, $a_1=0.5714$; $a_2=0$, $a_4=-0.12014$). In this case, integration of (3.7) gave B = −3.006. The value of D was determined from the known values of the displacement thickness $\delta^* = \delta$ and the momentum loss thickness δ^{**}:

$$D=-(1+\delta^{**}/\delta^*)$$

In this case, for the lower and upper branches of the neutral stability curve we have, respectively,

$$\alpha=0.497\,\mathrm{Re}^{-1/4}(1+2.39\,\mathrm{Re}^{-1/4}+\ldots),\quad \alpha=0.340\,\mathrm{Re}^{-1/10}(1+0.91\,\mathrm{Re}^{-1/10}+\ldots)$$

In Fig. 1, the curves with long dashes show the two-term solution, and the curves with short dashes the one-term solution. The continuous curve is the Tollmien solution.

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