THE REPRESENTATION OF HARMONIC MAPPINGS BY MONOGENIC FUNCTIONS

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§ 1. Posing of the Problem

By harmonic mappings we mean continuously differentiable mappings $(x_1, x_2, x_3) \to u = (u_1, -u_2/2, -u_3/2)$ defined by the system of equations

$$\frac{\partial u_1}{\partial x_1} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right), \quad \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = 0,$$
$$\frac{\partial u_1}{\partial x_2} + \frac{1}{2}\frac{\partial u_2}{\partial x_1} = 0, \quad \frac{\partial u_1}{\partial x_3} + \frac{1}{2}\frac{\partial u_3}{\partial x_1} = 0, \tag{1}$$

where the values of u_i and x_j $(i, j = 1, 2, 3)$ belong to the real field R [1]. It is known that $u_i = u_i(x_1, x_2, x_3)$ satisfy the three-dimensional Laplace equation. It is customary to call the vector $(u_1, -u_2/2, -u_3/2)$ harmonic.

The system of equations (1) can be written in another way:

$$\operatorname{div} u = 0, \quad \operatorname{rot} u = 0.$$

In the two-dimensional case these equations are the Cauchy – Riemann conditions for an analytic function of a complex variable. In this connection, the question arises: does there exist an algebra the differentiable functions on which have components satisfying (1)? Such a question involves a certain indefiniteness since the concept of differentiable function on an algebra can be introduced in various ways. From these various possibilities it is advisable to choose that in which the concepts of the theory of analytic functions generalizes so as to preserve the basic properties of such operations as differentiation and integration. Finally, from the arithmetic point of view, an algebra of minimal rank is more convenient. In the case considered, the least rank of the algebra desired is equal to three.

Such a problem already appeared in Hamilton's work. It is true that in considering this problem as well as many others, Hamilton restricted himself to the algebra of quaternions and the question of constructing another algebra was not posed [2].

At the present time Bergman [3] has sufficiently developed the theory of harmonic vectors based on the algebra of complex numbers. However, the methods of integral operators of Bergman for the three-dimensional Laplace equation have a narrow sphere of application.

§ 2. Analytic and Monogenic Hypercomplex Functions

Let A be a linear normed associative algebra with principal unit of rank n over the field F (in this paper the case of the real R and the complex C fields will be considered).

Let the law of multiplication of basis elements have the form

$$e_i e_j = \sum_{k=1}^{n} \gamma_{jk}^{i} e_k, \quad \gamma_{jk}^{i} \in F\,(i, j = 1, 2, \ldots, n),$$

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where γ_{jk}^{i} are the structural constants of the algebra. Along with the algebra A one considers the images α and $\tilde{\alpha}$ of its right and left regular representations. Bases $\{E_i\}$ and $\{\tilde{E}_i\}$ in α and $\tilde{\alpha}$ are defined by the isomorphisms

$$e_i \leftrightarrow [\gamma^i_{jk}] = \widetilde{E}_i \text{ and} [\gamma^i_{kj}] = E_i,$$

where $[\gamma_{jk}^{i}]$ for each fixed i is a matrix of size $n \times n$.

We now introduce into consideration functions $f(\zeta)$ on the algebra A whose values also lie in A. Each such function can be represented in the form

$$f(\zeta) = \sum_{k=1}^{n} u_k e_k, \ \zeta = \sum_{k=1}^{n} x_k e_k, \ x_k \in R, \ u_k \in F. \tag{2}$$

The hypercomplex function $f(\zeta)$ will be called analytic or a (hypercomplex) potential in the domain $D \subseteq A$, if its Frechet derivative for each fixed $\zeta \in D$ lies in α or $\tilde{\alpha}$. We distinguish between left and right potentials. If the derivative lies in α, the potential will be called right (if in $\tilde{\alpha}$ – left).

In our (finite-dimensional) case the Frechet derivative is the Jacobian matrix $[\partial u_m / \partial x_k]$. Conditions under which the Jacobian matrix lies in α or $\tilde{\alpha}$, are given, respectively, by:

$$\left[\frac{\partial u_i}{\partial x_k}\right] = \sum_{i=1}^{n} \varphi_i E_i, \tag{3}$$

$$\left[\frac{\partial u_i}{\partial x_i}\right] = \sum_{i=1}^{n} \varphi_i \widetilde{E}_i. \tag{3'}$$

Here $\varphi_i = \varphi_i(\zeta)$ is a function of the point ζ at which the Frechet derivative is computed.

Equations (3) and (3') will be called the Cauchy – Riemann conditions, and their right sides Cauchy – Riemann matrices (respectively, for right and left potentials) [4].

We shall also introduce the concept of monogenic function on the algebra A. For convenience in defining the derivative we shall require that A be commutative.

By a monogenic hypercomplex function on a domain $D \subseteq A$ will be meant a function $f(\zeta)$ defined on D with values in A, for which at each point $\zeta \in D$ there exists a function $f'(\zeta)$ with values in A, independent of h, and such that

$$hf'(\zeta) = \lim_{\varepsilon \to 0} [f(\zeta + \varepsilon h) - f(\zeta)] \varepsilon^{-1} \quad \forall h \in A, \tag{4}$$

where $\varepsilon > 0$.

The function $f'(\zeta)$ will be called the derivative of the function $f(\zeta)$ with respect to the variable ζ.

This definition of monogenic hypercomplex function differs from the conventional one in that the derivative is not defined directly by means of the limit

$$f'(\zeta) = \lim_{(h\varepsilon) \to 0} [f(\zeta + \varepsilon h) - f(\zeta)] (h\varepsilon)^{-1},$$

but by an indirect method, using the derivative of Gato (4). Here it is not necessary to exclude from consideration values of the increment of the independent variable which are noninvertible elements. Hence the definition based on (4) is more general than the usual one: when the function is monogenic in the usual sense, it is monogenic in the sense of (4) [the values of $f'(\zeta)$ coincide here], and besides (4) makes sense when h is a noninvertible element.

In the special case when h coincides with one of the basis vectors, (4) assumes the form

$$f'(\zeta)\, e_i = \frac{\partial f}{\partial x_i}, \tag{5}$$

where $\partial f / \partial x_i$ is the usual partial derivative, taken of $f(\zeta(x_1, \ldots, x_n))$ as of a function of several real variables.

From now on it will be assumed that e_1 is a principal unit of A (here there is no loss of generality).

Using (5), it is easy to compute the derivative $f'(\zeta)$. Taking $e_i = e_1$ in (5), we have

$$f'(\zeta) = \frac{\partial f}{\partial x_1}. \tag{6}$$

Now one can prove the following lemma which will be useful in what follows.

LEMMA. A monogenic function on the algebra A is a right potential.

Let $f(\zeta)$ be a monogenic function. Then, according to (6), there exists a derivative equal to $\partial f/\partial x_1$. We shall find the image of the derivative in the right regular representation. This will be the matrix

$$\left[\frac{\partial f}{\partial x_1}\right] = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_1} [\gamma_{jk}^i] \quad (j, k = 1, 2, \ldots, n),$$

in which the first column coincides with the first column of the Jacobian matrix, since under the right regular representation $\gamma_{j1}^i = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Thus, in the case considered, $[\partial f/\partial x_1]$ is the Cauchy–Riemann matrix, obtained by decomposition of the Jacobi matrix with respect to the system of functions $\varphi_i = \partial u_i/\partial x_1$. This proves the lemma.

§ 3. The Problem of Construction of Harmonic Algebras

A potential is sought whose Cauchy–Riemann conditions are the equations (1). For this it is necessary to find a commutative, i.e., such that

$$\gamma_{ik}^j = \gamma_{jk}^i \quad (i, j, k = 1, \ldots, n), \tag{7}$$

and associative, i.e., for which one has the relations

$$\sum_{i=1}^{n} \gamma_{ks}^i \gamma_{ji}^m = \sum_{i=1}^{n} \gamma_{is}^m \gamma_{kj}^i \quad (k, s, m, j = 1, \ldots, n), \tag{8}$$

algebra of the third rank (if this is possible to do in general), such that the potential (2) has as Cauchy–Riemann conditions (1).

First we shall consider a broader problem – we shall find what conditions an algebra must satisfy in order to have at least one hypercomplex function, defined in the following way:

1) at some point $\zeta \in D$ there exists a nonzero second derivative $f''(\zeta)$ in the sense of (4);

2) at the same point ζ the components of $f(\zeta)$ satisfy the three-dimensional Laplace equation.

Under the conditions indicated we have

$$\frac{\partial^2 f}{\partial x_i^2} = f''(\zeta) e_i^2 \quad (i = 1, 2, 3)$$

and hence

$$\Delta f \equiv f''(\zeta)(e_1^2 + e_2^2 + e_3^2) = 0. \tag{9}$$

From (9) it follows that the components of the twice differentiable hypercomplex function $f(\zeta)$ satisfy the Laplace equation if

$$e_1^2 + e_2^2 + e_3^2 = 0. \tag{10}$$

Here one disregards the trivial case $f''(\zeta) = 0$, which is realized on any algebra A by a linear function $f(\zeta) = a\zeta + b$, where a and b are constant elements of the algebra considered.

In the matrix algebra $\tilde{\alpha}$, which is the left regular representation of the algebra sought, (10) has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 + \begin{bmatrix} 0 & 1 & 0 \\ \gamma_{21}^2 & \gamma_{22}^2 & \gamma_{23}^2 \\ \gamma_{21}^3 & \gamma_{22}^3 & \gamma_{23}^3 \end{bmatrix}^2 + \begin{bmatrix} 0 & 0 & 1 \\ \gamma_{21}^3 & \gamma_{22}^3 & \gamma_{23}^3 \\ \gamma_{31}^3 & \gamma_{32}^3 & \gamma_{33}^3 \end{bmatrix}^2 = 0. \tag{11}$$

Here as E_1 a principal unit is taken and (7) is taken into consideration.

The matrix equation (11) represents a system of algebraic equations for the nine unknown structural constants. Using the condition of associativity the number of unknowns can be reduced to six. We have, from (10) and (8)

$$\sum_{s=1}^{3}\sum_{i=1}^{3}\gamma_{ik}^{s}\gamma_{ji}^{s}=\sum_{i=1}^{3}\gamma_{jk}^{i}\sum_{s=1}^{3}\gamma_{si}^{s}=0\ \ (k, j=1, 2, 3).$$

Whence, for j = 1 taking account of (11), we get: $\sum_{s=1}^{3}\gamma_{si}^{s}=0$ (i = 1, 2, 3) and then (11) assumes the form

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}^2+\begin{bmatrix}0&1&0\\\gamma_{21}^{2}&\gamma_{22}^{2}&\gamma_{23}^{2}\\\gamma_{21}^{3}&\gamma_{22}^{3}&\gamma_{23}^{3}\end{bmatrix}^2+\begin{bmatrix}0&0&1\\\gamma_{21}^{3}&\gamma_{22}^{3}&\gamma_{23}^{3}\\-\gamma_{21}^{2}-1&-\gamma_{22}^{2}&-\gamma_{23}^{2}\end{bmatrix}^2=0. \tag{12}$$

Thus, for the structural constants γ_{jk}^{i} one obtains a subdefinite system of equations [in the system (12)] of five equations and six unknowns. It should be noted that in posing the problem one is pursuing the goal of constructing algebras, the differentiable functions on which have definite harmonic components, namely those which satisfy (1). This gives the possibility of obtaining supplementary conditions to (12) for the structural constants.

§ 4. Harmonic Algebras

Suppose given a normed commutative and associative algebra A with principal units over the field F with basis $\{e_i\}_1^n$. If the set $\sigma=\{e_{i_s}\}_{s=1}^{m}$ consisting of m ≤ n elements satisfies the equation

$$e_{i_1}^2+e_{i_2}^2+\ldots+e_{i_m}^2=0,$$

then let us agree to denote the algebra A by the symbol $H_n^m(F, \sigma)$ and call it harmonic. When the set σ is fixed and there is no possibility of ambiguity, one will use the reduced notation $H_n^m(F)$. The algebra of complex numbers then should be denoted by: $H_2^2(R)$.

We shall study the question of the existence of the algebra $H_3^3(R)$. Such an algebra would be the simplest algebra for solving the problem posed in § 1.

We shall find the trace of the left side of (12). We get:

$$\sum_{i=1}^{3}\mathrm{Sp}\,[\gamma_{ks}^{i}]^2=(\gamma_{22}^{2}-\gamma_{23}^{3})^2+(\gamma_{23}^{2}+\gamma_{22}^{3})+1=0. \tag{13}$$

Equation (13) has no solutions with values in R. Whence follows the following theorem.

THEOREM 1. There does not exist an algebra u_i in (1). We shall seek structural constants of an algebra $H_3^3(C)$. If (1) is considered as the Cauchy–Riemann conditions for a right potential, then from (1) and (3) we get:

$$\sum_{i=1}^{3}\left[\gamma_{11}^{i}-\frac{1}{2}(\gamma_{22}^{i}+\gamma_{33}^{i})\right]\varphi_i=0,\ \ \sum_{i=1}^{3}(\gamma_{23}^{i}-\gamma_{32}^{i})\,\varphi_i=0,$$
$$\sum_{i=1}^{3}\left(\gamma_{21}^{i}+\frac{1}{2}\gamma_{12}^{i}\right)\varphi_i=0,\ \ \sum_{i=1}^{3}\left(\gamma_{31}^{i}+\frac{1}{2}\gamma_{13}^{i}\right)\varphi_i=0. \tag{14}$$

Equation (14) should be satisfied for any harmonic mapping which is a right potential. Obviously the linear mapping

$$u_i=\sum_{j=1}^{3}a_j^i x_j,\ \ a_j^i\in R\ \ \ (i=1, 2, 3)$$

satisfies (1), i.e., is harmonic. It is easy to show that for any fixed a_1^1, a_1^2, a_1^3, the remaining coefficients in the last formula can be determined in such a way as to realize (3). That is, there exist analytic linear mappings with arbitrarily given a_1^1, a_1^2, a_1^3. Whence it is easy to conclude that φ_i in (14) can assume arbitrary values, which is equivalent to all the coefficients of φ_i being equal to zero. Referring (14) here to a commutative algebra with principal unit e_1, we will have:

$$\gamma_{21}^{2}=\gamma_{31}^{3}=-\frac{1}{2},\ \gamma_{22}^{2}+\gamma_{33}^{2}=0,\ \gamma_{23}^{2}=\gamma_{32}^{2},$$
$$\gamma_{31}^{2}=\gamma_{21}^{3}=0,\ \gamma_{22}^{3}+\gamma_{33}^{3}=0,\ \gamma_{23}^{3}=\gamma_{32}^{3}. \tag{15}$$

Since according to the lemma a monogenic function is a right potential, (15) should be considered along with (12), whence we get

$$\gamma_{22}^{2}=-\gamma_{23}^{3},\ \gamma_{23}^{2}=\gamma_{22}^{3},\ (\gamma_{23}^{3})^{2}+(\gamma_{22}^{3})^{2}=-\frac{1}{4}.$$

Thus, the subdefinite system (12) along with (15) reduces to three equations with four unknowns. In the last equation, setting $\gamma_{23}^{3}=(i/2)\sin\omega$ and $\gamma_{22}^{3}=(i/2)\cos\omega$, where $\omega\in C$, we get for a basis of the image α of the algebra A,

$$E_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad E_2=\begin{bmatrix}0&-\frac{1}{2}&0\\1&-\frac{i}{2}\sin\omega&\frac{i}{2}\cos\omega\\0&\frac{i}{2}\cos\omega&\frac{i}{2}\sin\omega\end{bmatrix},\quad E_3=\begin{bmatrix}0&0&-\frac{1}{2}\\0&\frac{i}{2}\cos\omega&\frac{i}{2}\sin\omega\\1&\frac{i}{2}\sin\omega&-\frac{i}{2}\cos\omega\end{bmatrix}. \tag{16}$$

Thus, the algebra $H_3^3(C)$ exists. Multiplying the matrices of (16) we get a Kelly table for $H_3^3(C)$

$$\begin{array}{c|lll} & e_1 & e_2 & e_3\\ \hline e_1 & e_1 & e_2 & e_3\\ e_2 & e_2 & -\frac{e_1}{2}-\frac{i}{2}(\sin\omega e_2-\cos\omega e_3) & \frac{i}{2}(\cos\omega e_2+\sin\omega e_3)\\ e_3 & e_3 & \frac{i}{2}(\cos\omega e_2+\sin\omega e_3) & -\frac{e_1}{2}+\frac{i}{2}(\sin\omega e_2-\cos\omega e_3)\end{array} \tag{17}$$

Equation (17) is the Kelly table of a one-parameter family of algebras. The parameter ω, which takes values in the complex field, will be written as an index in the notation of the algebra in the following way: $H_n^m(F)_\omega$. For example, the algebra given in [5] would be denoted in the present system by $\mathbf{H}_3^3(\mathbf{C})_{\pi/4}$.

§ 5. Analytic Functions on the Algebras $\mathbf{H}_3^3(\mathbf{C})_\omega$

We shall find conditions under which hypercomplex functions on such structures

$$f(\zeta)=\sum_{i=1}^{3}U_i e_i,\quad \zeta=\sum_{i=1}^{3}x_i e_i,\quad x_i\in R,\ U_i(x_1,x_2,x_3)\in C$$

are right potentials on $H_3^3(C)_\omega$.

From (3) and (16) we have

$$\left[\frac{\partial U_i}{\partial x_k}\right]_1^3=\begin{bmatrix}\varphi_1&-\frac{1}{2}\varphi_2&-\frac{1}{2}\varphi_3\\ \varphi_2&\varphi_1-\frac{i}{2}(\sin\omega\,\varphi_2-\cos\omega\,\varphi_3)&\frac{i}{2}(\cos\omega\,\varphi_2+\sin\omega\varphi_3)\\ \varphi_3&\frac{i}{2}(\cos\omega\varphi_2+\sin\omega\,\varphi_3)&\varphi_1+\frac{i}{2}(\sin\omega\varphi_2-\cos\omega\varphi_3)\end{bmatrix}$$

This matrix equation is equivalent to the system of equations

$$\frac{\partial U_1}{\partial x_2}=-\frac{1}{2}\frac{\partial U_2}{\partial x_1},\quad \frac{\partial U_1}{\partial x_3}=-\frac{1}{2}\frac{\partial U_3}{\partial x_1},$$
$$\frac{\partial U_3}{\partial x_2}=\frac{\partial U_2}{\partial x_3}=\frac{i}{2}\left(\cos\omega\frac{\partial U_2}{\partial x_1}+\sin\omega\frac{\partial U_3}{\partial x_1}\right),$$
$$\frac{\partial U_2}{\partial x_2}=\frac{\partial U_1}{\partial x_1}-\frac{i}{2}\left(\sin\omega\frac{\partial U_2}{\partial x_1}-\cos\omega\frac{\partial U_3}{\partial x_1}\right), \tag{18}$$

$$\frac{\partial U_3}{\partial x_3} = \frac{\partial U_1}{\partial x_1} + \frac{i}{2}\left(\sin\omega \frac{\partial U_2}{\partial x_1} - \cos\omega \frac{\partial U_3}{\partial x_1} \right),$$

which will also be called Cauchy – Riemann conditions for a right potential on the harmonic algebras $H_3^3(C)_\omega$. These conditions are written as in the theory of functions of a complex variable in the form of a system of linear partial differential equations.

Adding the last two equations in (18), we get

$$\frac{\partial U_1}{\partial x_1} = \frac{1}{2}\left(\frac{\partial U_2}{\partial x_2} + \frac{\partial U_3}{\partial x_3} \right).$$

From this equation and from the first three equations of (18) it follows that the imaginary and real parts of the vectors

$$U = \left(U_1, -\frac{1}{2}U_2, -\frac{1}{2}U_3\right) = \left(u_1 + iv_1, -\frac{1}{2}(u_2 + iv_2), -\frac{1}{2}(u_3 + iv_3)\right)$$

are harmonic vectors. Thus, (18) connects two conjugate harmonic vectors in a sense analogous to that imbedded in the concept of "conjugate" for two harmonic scalar functions connected by the usual Cauchy – Riemann conditions.

For functions of a hypercomplex variable on the algebras $H_3^3(C)_\omega$ one has the following theorem.

THEOREM 2. The function $f(\zeta) = \sum_{k=1}^{3} U_k e_k$, $\zeta = \sum_{k=1}^{3} x_k e_k$ defined in some domain $D \subseteq H_3^3(C)_\omega$ is monogenic in this domain if and only if it has differentiable components $U_1(x_1, x_2, x_3)$, $U_2(x_1, x_2, x_3)$, $U_3(x_1, x_2, x_3)$ at each point of this domain (as functions of three real variables) and if the Cauchy – Riemann conditions (18) are satisfied.

This theorem is completely analogous to the theorem on differentiability of functions of a complex variable.

According to the lemma, a monogenic function is a right potential. It was shown at the beginning of § 5, that if $f(\zeta)$ is a right potential, then (18) holds. Whence follows the necessity of the conditions of the theorem.

We shall prove the sufficiency of the conditions of the theorem. Suppose these conditions are satisfied. Then for finite increments we will have:

$$\Delta U_1 = \frac{\partial U_1}{\partial x_1}\Delta x_1 + \frac{\partial U_1}{\partial x_2}\Delta x_2 + \frac{\partial U_1}{\partial x_3}\Delta x_3 + L_1,$$

$$\Delta U_2 = \frac{\partial U_2}{\partial x_1}\Delta x_1 + \frac{\partial U_2}{\partial x_2}\Delta x_2 + \frac{\partial U_2}{\partial x_3}\Delta x_3 + L_2,$$

$$\Delta U_3 = \frac{\partial U_3}{\partial x_1}\Delta x_1 + \frac{\partial U_3}{\partial x_2}\Delta x_2 + \frac{\partial U_3}{\partial x_3}\Delta x_3 + L_3,$$

where L_1, L_2, L_3 are quantities which are infinitesimal compared with the increment of the argument (x_1, x_2, x_3). The finite increment $\Delta f(\zeta)$, using (18) will be:

$$\Delta f(\zeta) = \Delta U_1 e_1 + \Delta U_2 e_2 + \Delta U_3 e_3 + e_1\left(\frac{\partial U_1}{\partial x_1}\Delta x_1 + \frac{1}{2}\frac{\partial U_2}{\partial x_1}\Delta x_2 + \frac{1}{2}\frac{\partial U_3}{\partial x_1}\Delta x_3\right)$$
$$+ e_2\left\{\frac{\partial U_2}{\partial x_1}\Delta x_1 + \left[\frac{\partial U_1}{\partial x_1} - \frac{i}{2}\left(\sin\omega\frac{\partial U_2}{\partial x_1} - \cos\omega\frac{\partial U_3}{\partial x_1}\right)\right]\Delta x_2 + \right.$$
$$\left. + \frac{i}{2}\left(\cos\omega\frac{\partial U_2}{\partial x_1} + \sin\omega\frac{\partial U_3}{\partial x_1}\right)\Delta x_3\right\} + e_3\left\{\frac{\partial U_3}{\partial x_1}\Delta x_1 + \frac{i}{2}\left(\cos\omega\frac{\partial U_2}{\partial x_1}\right.\right.$$
$$\left.\left. + \sin\omega\frac{\partial U_3}{\partial x_1}\right)\Delta x_2 + \left[\frac{\partial U_1}{\partial x_1} + \frac{i}{2}\left(\sin\omega\frac{\partial U_2}{\partial x_1} - \cos\omega\frac{\partial U_3}{\partial x_1}\right)\right]\Delta x_3\right\}$$
$$+ L(\Delta\zeta) = (e_1\Delta x_1 + e_2\Delta x_2 + e_3\Delta x_3)\left(e_1\frac{\partial U_1}{\partial x_1} + e_2\frac{\partial U_2}{\partial x_1} + e_3\frac{\partial U_3}{\partial x_1}\right) + L(\Delta\zeta),$$

where $L(\Delta\zeta)$ is a hypercomplex quantity of higher order of smallness than $\Delta\zeta$ [with respect to the norms in $H_3^3(C)_\omega$].

Fixing h, setting $\Delta\zeta = \varepsilon h$ with real ε and dividing the last equation by ε, passage to the limit as $\varepsilon \to 0$ shows the monogenicity of $f(\zeta)$.

It is easy to prove that the sum and product of a finite number of monogenic functions on the algebras $H_3^3(C)_\omega$ are monogenic functions. Whence it follows that the class of monogenic functions on $H_3^3(C)_\omega$ contains at least all polynomials.

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