BEHAVIOR OF HAMILTONIAN SYSTEMS CLOSE TO INTEGRABLE

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In this note we consider the behavior of action variables I in a system of canonical equations of Hamilton with the Hamiltonian

$$H = H_0(I) + \varepsilon H_1(I, \varphi), \qquad \varepsilon \ll 1,$$

during an interval of time large in comparison with an arbitrary power of $1/\varepsilon$. Here the "action variable" I belongs to a domain of the Euclidean space E^n, dim $E^n = n$, and the angle variable φ to an n-dimensional torus; the unperturbed Hamiltonian H_0 and the "perturbation" εH_1 are analytic functions, and H_1 is periodic of period 2π in φ.

As is well known ([1, 2]), there exists in the phase space of the system, subject to the satisfaction of certain conditions, a closed nowhere dense set, consisting of invariant tori close to the tori defined by the equation I = const (we shall refer to this invariant set as a Kolmogorov set). If the number of degrees of freedom is equal to 2, then when specific conditions ([2]) are satisfied, the Kolmogorov two-dimensional tori "divide" the level surface of the Hamiltonian function, and then for all initial conditions I(0), $\varphi(0)$ the quantity $\|I(t)-I(0)\|$ will be small for all t, $-\infty < t < +\infty$. If, however, n > 2, the so-called Arnol'd diffusion is observed in the complement of the Kolmogorov set; the point I(t) departs, although very slowly, from its initial position I(0) (see [3]).

In Theorem 1, stated below, the mean speed of departure I(t) from the initial point is bounded from above, at one stroke, for all initial conditions. It turns out that if the function H_0 satisfies certain conditions, which we shall refer to as steepness conditions (defined below), then after a long interval of time of order $\exp(1/\varepsilon^a)$ the point I(t) does not depart from I(0) by more than a small distance of order ε^b, where $0 < a < 1$, $0 < b < 1$. Functions of a general position will be steep (see Theorem 2). Thus, in the general case the mean speed of departure I(t) decreases exponentially with a linear decrease in the perturbation, i.e., as in the examples of instability constructed by V. I. Arnol'd.

The motion of the point I(t) can be decomposed into a rapid "vibration" (of order ε) about a mean position and a "drift," i.e., a displacement of this mean position. The drift may be compared with movement into "a dense forest with meadows:" "on the meadows" the drift I(t) can be rapid, but in order to depart from I(0) by a distance greater than the dimensions of the "meadow" it is necessary to "get through the dense forest," where the drift speed is very small. When the function $H_0(I)$ is not steep, the "meadows" may degenerate into "clearings" and the point I(t) may depart from I(0) with a speed of order ε. In general, knowing the steepness coefficients of $H_0(I)$ (see the definition below), one can estimate the dimensions of the "meadows" and, by the same token, it appears, one can give a fairly detailed estimate of the deviation of I(t) from I(0) for large t. The theorem formulated below is a first step in this direction.

If A is a set in E^n, then $A-\varepsilon$ will denote a set of points, which is contained in A together with its ε-neighborhood.

THEOREM 1. Let the function $H = H_0(I) + H_1(I, \varphi)$ be periodic of period 2π in φ and analytic in the domain $F: \operatorname{Re} I \in G$, $|\operatorname{Im} I| < \sigma$, $|\operatorname{Im} \varphi| < \rho$, where $\sigma > 0$, $\rho > 0$, and $G \subset E^n$. Let $H_0(I)$ be a function with characteristic $\L$, which is steep* in G, and let $M = \sup H_1(I, \varphi)$ for I, $\varphi \in F$. Then positive numbers M_0,

*See the definition of steepness given below.

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C, and K exist, depending only on H_0, G, ρ, and σ, such that if $0 < M < M_0$, then for an arbitrary solution of the system with Hamiltonian H and initial conditions I(0), $\varphi(0) \in F-d$, and all $t \in [0, T]$,

$$\| I(t) - I(0) \| < d,$$

where $T = \exp(CM^{-a})$, $d = M^b$, $a = 3/(12/\zeta) + 3n + 14$, $b = Ka$.

Definition. Let the function $f(x_1, \ldots, x_n)$ be defined in the domain $G \subset E^n$, and let us assume that $\operatorname{grad} f|_x \neq 0$ for all $x \subset G$. Let $\{\Lambda^r(x)\}$ be a set of affine planes $\Lambda^r(x) \subset E^n$, $\dim \Lambda^r(x) = r$, passing through the point x and perpendicular to $\operatorname{grad} f|_x$. We shall say that the function $f(x)$ is steep in G if for each $r = 1, \ldots, n-1$, numbers $K_r > 0$, $\delta_r > 0$, and $\alpha_r \geq 1$ exist (which we shall term steepness coefficients) such that for all $x \in G$ the inequalities

$$\max_{0 \leqslant \eta \leqslant \xi} \min_{y \in \Lambda^r(x) \cap G, \|y-x\| = \eta} \| \operatorname{grad} (f|_{\Lambda^r(x)})|_y \| > K_r \xi^{\alpha_r}$$

are satisfied for all $\Lambda^r(x) \in \{\Lambda^r(x)\}$ and all ξ, such that $0 < \xi \leq \delta_r$. By a characteristic of steepness we shall mean a number ζ such that

$$1/\zeta = (\alpha_1(\alpha_2 \ldots (\alpha_{n-2}(\alpha_{n-1}(1/\alpha_n) + n - 1) + n - 2) + \ldots + 2) + 1).$$

THEOREM 2. A function $f(x)$ is steep in a neighborhood of a point x if $\operatorname{grad} f|_x$ is nonzero, and if the coefficients in the Taylor's series expansion of $f(x)$ at the point x do not satisfy any one of an infinite number of specific algebraic equations.

By way of application we consider the problem of m bodies: m points attracted to one another according to Newton's law. Let the mass of one body (the sun) be much larger than the mass of the remaining bodies (the planets) and let us assume that the initial conditions are such as to stipulate at the initial instant: a) the motion of the planets, close to circular, b) the smallness of the inclinations of the planes of the orbits to one another and the same direction of revolution of the planets, c) the distances between the orbits of the planets being not too close.

The equations describing the variation of the major semi-axes a_i of the orbits, and the motion of the planets along their orbits, may be written in a system of action-angle coordinates $I = I_1, \ldots, I_{m-1}$, $\varphi = \varphi_1, \ldots, \varphi_{m-1}$ (where $I_i \sim \sqrt{a_i}$) in Hamiltonian form with Hamiltonian $H = H_0(I) + H_1(I, \varphi, t)$ (the variation of the eccentricities and inclinations may be taken into account through the dependence of H_1 on t). The assertions of Theorem 1 may be generalized to a system with a Hamiltonian of this kind providing that the function H_0 is steep and that the partial derivatives of H_1 stay small during the time interval considered. In our case, $H_0(I) = -(c_1/I_1^2) - \ldots - (c_{m-1}/I_{m-1}^2)$ is a steep function with the maximum steepness characteristic. But the partial derivatives of H_1 do not remain small for a close approach of even two of the planets. However, using conditions a), b), and c) we can show that such an approach does not take place over a long period of time. As a result we find that over a long time interval, estimateable by our theorem, the motion of the planets will satsify conditions a), b), and c) and, in addition, the lengths a_i of the orbits of the semi-axes are almost invariable.

LITERATURE CITED

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