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RELATIVISTICALLY INVARIANT QUASICLASSICAL LIMITS OF INTEGRABLE TWO-DIMENSIONAL QUANTUM MODELS

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Two-dimensional quantum integrable models whose quasiclassical limits are principal chiral fields with symmetric and nonsymmetric Lagrangians are proposed.

1. In this paper, we propose the analog of the Hamiltonian structure considered in [1] for the principal chiral field equation

$$2g_{\xi\eta}=g_\xi g^{-1}g_\eta+g_\eta g^{-1}g_\xi, \tag{1}$$

where g is a function of the cone variables ξ and η with values in invertible matrices. It is shown that with respect to this structure (which is not canonical) Eq. (1) is the quasiclassical limit of an integrable quantum model, which is constructed on the basis of the Yang identity [2]. This provides one further way of proving the integrability of (1), which was established by Zakharov and Mikhailov [3], and, in particular, to calculate in a very simple manner the Poisson brackets (in the introduced structure) of the coefficients of the S matrix associated with (1) (cf. [1]).

The same device can be applied to Baxter's identity [4] (found in the integration of the XYZ model) instead of the Yang identity. In a limiting case, we obtain a special case of the equation of the nonsymmetric principal chiral field,

$$Jg_{\xi\eta}g^{-1}+g_{\xi\eta}g^{-1}J=g_\eta g^{-1}g_\xi g^{-1}J+g_\xi g^{-1}g_\eta g^{-1}J, \tag{2}$$

corresponding to the Lagrangian

$$-\frac{1}{2}\int \mathrm{Sp}(g_\xi g^{-1}Jg_\eta g^{-1}),$$

for g with values in O(3) (J is a constant diagonal matrix). In this special case, (2) has a representation of zero curvature of elliptic type and can in principle be integrated by the inverse scattering method for arbitrary J.

The present paper has been written under the influence of the quantum inverse method created by Faddeev and others (see [5-7]) and also the paper [1], in which Sklyanin obtained the Landau–Lifshitz equation of the theory of ferromagnetism as the classical limit of the XYZ model. We draw attention to two features of the models we consider, namely, the use of nonlocal Hamiltonians and cone variables. Note that the construction of a systematic quantum theory of chiral fields with values on the two-dimensional sphere is realized in the paper of Takhtadzhyan and Faddeev on the quantum inverse scattering method and the O(3)

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nonlinear σ model.

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2. We give briefly some results from the theory of the scattering equation

$$E_\xi=\alpha^{-1}UE, \tag{3}$$

where U and E are matrix-valued functions of ξ, and α is a parameter. For purposes of simplification, we shall assume that the coefficients u_{ij} of the matrix $U = (u_{ij})$ decrease fairly rapidly as $\xi \to \pm\infty$, for example, that they have compact support. We denote by $E_\pm$ the solutions of (3) normalized by the conditions $E_\pm \to 1$ as $\xi\to\pm\infty$ $(\mathbf{1}=(\delta_{ij}))$. We set $T(\alpha)=(t_{ij})=E_+^{-1}E_-$. Then (cf. for example, [9])

$$\delta t_{pq}/\delta U^t=\alpha^{-1}E_-\mathbf{1}^{qp}E_+^{-1}, \tag{4}$$

where $\mathbf{1}^{qp}=(\delta_{iq}\delta_{jp})$, and t denotes the transpose.

We postulate the Poisson brackets between u_{pq}: $\{u_{pq}(\xi),\ u_{rs}(\xi')\}=i(\delta_{qr}u_{ps}(\xi)-\delta_{ps}u_{rq}(\xi))\delta(\xi-\xi')$. We calculate $\{t_{pq}(\alpha),\ t_{rs}(\beta)\}$ for $\alpha \neq \beta$. Since $(\alpha-\beta)\cdot(\alpha\beta)^{-1}\ \mathrm{Sp}(UV_\beta^{rs}V_\alpha^{pq}-UV_\alpha^{pq}V_\beta^{rs})=\mathrm{Sp}(V_\alpha^{pq}V_\beta^{rs})_\xi$, where $V_\alpha^{pq}=\alpha^{-1}E_-\mathbf{1}^{qp}E_+^{-1}$, and Sp is the trace,

$$\{t_{pq}(\alpha),t_{rs}(\beta)\}=i\int_{-\infty}^{+\infty}\mathrm{Sp}(UV_\beta^{rs}V_\alpha^{pq}-UV_\alpha^{pq}V_\beta^{rs})\,d\xi=i\alpha\beta(\alpha-\beta)^{-1}\,\mathrm{Sp}(V_\alpha^{pq}V_\beta^{rs})\,\Big|_{\xi=-\infty}^{\xi=+\infty}=$$

$$i(\alpha-\beta)^{-1}(t_{rq}(\alpha)t_{ps}(\beta)-t_{ps}(\alpha)t_{rq}(\beta)).$$

3. We denote by I_n^{ij} the matrix $\mathbf{1}^{ij}$ concentrated at site n of a one-dimensional lattice with infinitesimal step $\varkappa$. We regard I_n^{ij} as operators with the commutation relations $[I_n^{ij}, I_m^{kl}]=(\delta_{jk}I_n^{il}-\delta_{il}I_n^{kj})\cdot\delta_{nm}$. We set

$$\mathscr{L}_n(\alpha)=\alpha(\alpha+\varkappa)^{-1}I_n\mathbf{1}+\varkappa(\alpha+\varkappa)^{-1}\sum_{l,m}I_n^{lm}\mathbf{1}^{lm},$$

where $I_n=\sum_i I_n^{ii}$ ($\mathscr{L}_n$ is a matrix with coefficients in the algebra generated by I_n^{ij}),

$$\mathscr{R}(\alpha)=\alpha(\alpha-\varkappa)^{-1}\mathbf{1}\otimes\mathbf{1}-\varkappa(\alpha-\varkappa)^{-1}\sum_{l,m}\mathbf{1}^{lm}\otimes\mathbf{1}^{ml},$$

where $\otimes$ is the tensor product of matrices, and $\tilde{\mathscr{L}}_n=\mathscr{L}_n\otimes\mathbf{1}$, $\tilde{\tilde{\mathscr{L}}}_n=\mathbf{1}\otimes\mathscr{L}_n$. The factorization relation for the S matrix of [2] can be rewritten (cf. [1]) in the form of the identity

$$\tilde{\mathscr{L}}_n(\alpha)\tilde{\tilde{\mathscr{L}}}_n(\beta)\mathscr{R}(\beta-\alpha)=\mathscr{R}(\beta-\alpha)\tilde{\tilde{\mathscr{L}}}_n(\beta)\tilde{\mathscr{L}}_n(\alpha), \tag{5}$$

where $\alpha,\ \beta\in\mathbf{C}$, $\alpha,\ \beta,\ \alpha-\beta\neq-\varkappa$. For integral $M \leq N$, we define the matrix $\mathscr{T}=\mathscr{T}_M{}^N(\alpha)=\mathscr{L}_N\mathscr{L}_{N-1}\ldots\mathscr{L}_{M+1}\mathscr{L}_M(\alpha)$ [1].* Then $\mathscr{T}$ satisfies the same relation (5) as $\mathscr{L}_n$.

We consider a quantum model dependent on some fixed $\alpha\in\mathbf{C}$ with Hamiltonian $\mathscr{H}_\alpha=\mathrm{Sp}\,\mathscr{T}(\alpha)$ and equations of motion $\dot{\mathscr{X}}=i\varkappa^{-1}[\mathscr{H}_\alpha,\ \mathscr{X}]$. To represent the equations of motion in the form of a closed system of relations, we introduce the operators

$$\mathscr{U}_n=\sum_{i,j}I_n^{ij}\mathbf{1}^{ij},\ \mathscr{V}_n(\alpha)=\mathscr{T}_M{}^n(\alpha)\mathscr{T}_{n+1}^N(\alpha)\ (M\leq n\leq N),\ \mathscr{V}_{M-1}=\mathscr{V}_N=\mathscr{T}.$$

For integral m, n $(M \leq m \leq n \leq N)$, we obtain from (5) the identity

$$\dot{\mathscr{T}}_m{}^n(\beta)=i(\alpha-\beta)^{-1}(\mathscr{V}_n(\alpha)\mathscr{T}_m{}^n(\beta)-\mathscr{T}_m{}^n(\beta)\mathscr{V}_{m-1}(\alpha)). \tag{6}$$

For $m = n$, $\beta = 0$ we arrive at the equation

$$\dot{\mathscr{U}}_n=i\alpha^{-1}(\mathscr{V}_n(\alpha)\mathscr{U}_n-\mathscr{U}_n\mathscr{V}_{n-1}(\alpha)), \tag{7a}$$

which together with the trivially verified relation

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* The coefficients of $\mathscr{T}$ can be expressed in terms of the operators I_n^{ij} for all n $(M \leq n \leq N)$.

$$\mathscr{V}_n-\mathscr{V}_{n-1}=\varkappa\alpha^{-1}(\mathscr{U}_n\mathscr{V}_{n-1}(\alpha)-\mathscr{V}_n(\alpha)\mathscr{U}_n) \tag{7b}$$

forms a closed system of equations of motion for the proposed model.

Some quasiclassical variants of the identity (6) can be found in [10].

The system (7) can be transformed by setting

$$i\mathscr{V}_n(\alpha)=(\alpha/2)\dot{\mathscr{G}}_n\mathscr{G}_n^{-1},\quad \mathscr{U}_n=\varkappa^{-1}(\alpha/2)\Big(\mathscr{G}_n\mathscr{G}_{n-1}^{-1}-\prod_{k=M}^{N}I_k\mathbf{1}\Big)$$

for some (α-dependent) operator-matrix function $\mathscr{G}_n$ of n. As a result, we obtain the quantum analog of Eq. (1) (cf. [11]):

$$\dot{\mathscr{G}}_n\mathscr{G}_n^{-1}(\mathscr{G}_n+\mathscr{G}_{n-1})=(\mathscr{G}_n+\mathscr{G}_{n-1})\mathscr{G}_{n-1}^{-1}\dot{\mathscr{G}}_{n-1}\quad (M\leqslant n\leqslant N),\quad \dot{\mathscr{G}}_{M-1}\mathscr{G}_{M-1}^{-1}=\dot{\mathscr{G}}_N\mathscr{G}_N^{-1}. \tag{8}$$

Substituting m = M in (6), we readily find a zero-curvature representation for (8) and (7). If in the limit we set $[I_n^{ij}, I_m^{kl}]=0$, then (8) goes over into an integrable discrete system (a chain) of equations for the matrix-valued functions $\{g_n(t)\}$ of the continuous parameter t ($\dot{\mathscr{G}}_n\mapsto dg_n(t)/dt$).

The considered model can be integrated by the quantum inverse scattering technique. In particular, from (5) for $\mathscr{T}$ there follows the equation $[\mathscr{H}_\alpha, \mathscr{H}_\beta]=0$, and, expanding $\ln\mathscr{H}_\beta$ in a series in β in the neighborhood of $\beta = 0$, we obtain an infinite series of local quantum conservation laws. Models with Hamiltonians, the coefficients of the expansion $\ln\mathscr{H}_\beta$, have been investigated in a number of papers (see [12, 13]). In [13], one can also find the calculation of the spectrum of $\mathscr{H}_\beta$.

4. We calculate the quasiclassical limit of the constructed model as $\varkappa\to 0$ ($\hbar=1$), $\varkappa N\to+\infty$, $\varkappa M\to-\infty$. We introduce the continuous variables ξ and η and make the substitution $I_n^{lm}-\delta_{lm}I_n\mapsto u_{lm}(\xi)$, $\mathscr{X}\mapsto\partial X/\partial\eta$, $i\varkappa^{-1}[\ ,\ ]\mapsto\{\ ,\ \}$, $\delta_{nn'}\mapsto\varkappa\delta(\xi-\xi')$. Ignoring the terms of order $\varkappa^2$, we arrive at the equation of motion $U_\eta=\{H_\alpha, U\}$, where $U=(u_{ij})$, $H_\alpha=\mathrm{Sp}\,T(\alpha)$, and the Poisson brackets between u_{ij} are the same as in Sec. 2. Using (4), we find that $U_\eta=i\alpha^{-1}[V, U]$, where $V=E_-E_+^{-1}$. Since $V_\xi=\alpha^{-1}[U, V]$, setting $i\alpha^{-1}V={}^1/_2 g_\eta g^{-1}$, $\alpha^{-1}U={}^1/_2 g_\xi g^{-1}$, we arrive at Eq. (1).

In the same limit, the identity (5) for $\mathscr{T}$ can be rewritten as the relation

$$\{\tilde{T}(\alpha),\tilde{\tilde{T}}(\beta)\}=i[R(\beta-\alpha),\tilde{T}(\alpha)\tilde{\tilde{T}}(\beta)], \tag{9}$$

where $\tilde{T}=T\otimes\mathbf{1}$, $\tilde{\tilde{T}}=\mathbf{1}\otimes T$, $R=\partial\mathscr{R}/\partial\varkappa|_{\varkappa=0}$. Making calculations, we obtain the formulas of Sec. 2 (cf. [1]).

5. We now argue similarly on the basis of Baxter's identity [4] in the form of [1] (see also [6]). We denote by S_n^i the operator representing the Pauli matrix σ^i concentrated at site n. We recall that $\sigma^0=\mathbf{1}$, $\sigma^1=\mathbf{1}^{12}+\mathbf{1}^{21}$, $\sigma^2=i\mathbf{1}^{12}-i\mathbf{1}^{21}$, $\sigma^3=\mathbf{1}^{11}-\mathbf{1}^{22}$. We introduce the functions $W_0=1$, $W_1(\alpha)=\mathrm{sn}(\varkappa)\mathrm{sn}(\alpha+\varkappa)^{-1}$, $W_2(\alpha)=\mathrm{dn}\,\mathrm{sn}^{-1}(\alpha+\varkappa)\,\mathrm{sn}\,\mathrm{dn}^{-1}(\varkappa)$, $W_3(\alpha)=\mathrm{cn}\,\mathrm{sn}^{-1}(\alpha+\varkappa)\mathrm{sn}\,\mathrm{cn}^{-1}(\varkappa)$, where sn, cn, dn are elliptic functions of modulus k. We set

$$\mathscr{L}_n(\alpha)=\sum_{i=0}^{3}W_i(\alpha)S_n^i\sigma^i,\quad \mathscr{R}(\alpha)=\sum_{i=0}^{3}W_i(\alpha)\sigma^i\otimes\sigma^i.$$

Then the identity (5) holds for $\mathscr{L}$ and $\mathscr{R}$. As in Sec. 3, we introduce $\mathscr{T}(\alpha)$, $\mathscr{H}_\alpha=\mathrm{Sp}\,\mathscr{T}(\alpha)$ and consider the model with the Hamiltonian $\mathscr{H}_\alpha$. We have $S_n^p=i\varkappa^{-1}[\mathscr{H}_\alpha, S_n^p]$, where

$$[S_n^p, S_m^q]=2i\sum_r\varepsilon^{pqr}S_n^r\delta_{nm}.$$

This model can also be integrated (see [4, 6]). For it, there is an analog of the identity (6).

6. We calculate the limit of the considered model as $\varkappa\to 0$. We replace S_n^i by $s_i(\xi)$. Then $\mathscr{H}_\alpha$ goes over into $H_\alpha = \mathrm{Sp}\,T(\alpha)$, where $T(\alpha)$ is determined (see Sec. 2) for the equation

$$E_\xi=\sum_{i=1}^{3}w_i(\alpha)s_i(\xi)\sigma^iE,\quad w_1(\alpha)=\mathrm{sn}^{-1}(\alpha),\ w_2(\alpha)=\mathrm{dn}\,\mathrm{sn}^{-1}(\alpha),\ w_3(\alpha)=\mathrm{cn}\,\mathrm{sn}^{-1}(\alpha)$$

(see [1]). The equation of motion can be rewritten in the form $s_\eta = 2(\delta H_\alpha/\delta s)\times s$, where $s = (s_1, s_2, s_3)^t$, and $\times$ is the vector product. It follows from formula (4) that $J^{-1}(\delta H_\alpha/\delta s)_\xi=2iJs\times J^{-1}\delta H_\alpha/\delta s$, where $J=\mathrm{diag}(w_1(\alpha),$

$w_2(\alpha)$, $w_3(\alpha)$). Making the substitution $2is=u$, $2J^{-1}\delta H_\alpha/\delta s=v$, we arrive at the system of equations

$$u_\eta=(Jv)\times u, \quad v_\xi=(Ju)\times v, \tag{10}$$

which is the required limit. Using the analog of the identity (6), we can write down the quantum variant of the system (10). Note that for T from this section (9) is also satisfied (see [1]).

7. We transform (10) to the form (2). It can be verified that there exists a function $g(\xi, \eta)$ that takes values in the orthogonal (complex) 3 × 3 matrices and for which $g_\xi g^{-1}=\tilde{U}$, $g_\eta g^{-1}=\tilde{V}$, where

$$\tilde{U}=\sum_{i,j,k}\varepsilon^{ijk}c_iu_i\mathbf{1}^{jk},\quad \tilde{V}=\sum_{i,j,k}\varepsilon^{ijk}c_iu_i\mathbf{1}^{jk}, u=(u_1,u_2,u_3)^t, v=(v_1,v_2,v_3)^t,\quad c_1^2=j_{12}j_{23},\quad c_2^2=j_{23}j_{21},\quad c_3^2=j_{31}j_{32},\quad j_{kl}=w_k(\alpha)+w_l(\alpha).$$

Then g satisfies Eq. (2). The converse is also true. Note that the condition of orthogonality of g is compatible with Eq. (2).

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