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THREE PROBLEMS OF ARONSZAJN IN MEASURE THEORY

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In this paper we solve three problems posed in [1]. En route we prove an assertion (the corollary of Theorem 1) which may be regarded as the answer to a well-known question raised by Gel'fand [2]. The approach to these problems is based on the theory of differentiable measures developed in [3-6].

1. Notations and Terminology. Henceforth X and $\mathscr{B}(X)$ denote, respectively, a separable Banach space and the σ-algebra of Borel subsets of X. By a measure on X we mean a countably additive function (not necessarily nonnegative) on $\mathscr{B}(X)$ with values in R. The symbol $|\mu|$ denotes the total variation of the measure μ [7]. A measure μ on X is said to be continuous in the direction of the vector $h \in X$ if $\lim_{t\to 0} \mu(A + th) = \mu(A)$ for all $A \in \mathscr{B}(X)$ [8]. Measure μ is said to be differentiable in the direction of h (in the sense of Skorohod [5]) if for any continuous bounded function f: X → R the limit $\lim_{t\to 0}\frac{1}{t}\int (f(x+th)-f(x))\mu(dx)$ exists. In this case a measure $d_h\mu$ exists, called the derivative of μ in the direction of h, such that the indicated limit equals $\int f(x)d_h\mu(dx)$ [6]. The infinite differentiability in the direction of h is defined naturally. A measure will be said to be densely differentiable if it is differentiable in the direction of all vectors of some sequence with dense linear span. A measure is quasi-invariant if it is equivalent to its translates by the elements belonging to a dense linear subspace. For each sequence $\{a_n\} \subset X$ we denote by $\mathscr{U}\{a_n\}$ the collection of all sets $B \in \mathscr{B}(X)$ such that $B = \bigcup_n B_n$, where $B_n \in \mathscr{B}(X)$ and $\operatorname{mes}((B_n + x) \cap R^1 a_n) = 0$. $\forall n, \forall x$ (mes denotes the standard Lebesgue measure on the line $R^1 a_n$); in other words, every section of the set B_n by a line parallel with a_n has measure zero. Let $\mathscr{U} = \cap\, \mathscr{U}\{a_n\}$, where the intersection is taken over all sequences $\{a_n\}$ with dense linear span. The sets in collection $\mathscr{U}$ are referred to as exceptional. Measure μ is said to be absolutely continuous with respect to $\mathscr{U}\{a_n\}$ if $\mu(A) = 0$ $\forall A \in \mathscr{U}\{a_n\}$, and singular with respect to $\mathscr{U}\{a_n\}$ if there is an $A \in \mathscr{U}\{a_n\}$ such that $|\mu|(A) = |\mu|(X)$ [1]. A nonzero measure is said to be exceptional if it is singular with respect to all classes $\mathscr{U}\{a_n\}$, where $\{a_n\}$ has a dense linear span, whereas μ(A) = 0 for all $A \in \mathscr{U}$ [1]. Therefore, an exceptional measure is "concentrated" on each of the collections $\mathscr{U}\{a_n\}$, but vanishes on their intersections. The class $\mathscr{U}$ was introduced in [1], where it was shown that for X finite-dimensional, $\mathscr{U}$ coincides with the σ-algebra of Borel subsets with zero Lebesgue measure. In the general case $\mathscr{U}$ retains some features of this σ-algebra. For example, every Lipschitz function f: X → R is differentiable everywhere except for the points of an exceptional set [1].

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2. Problems. Aronszajn has formulated the following problems:

(1) describe all measures which are absolutely continuous with respect to $\mathcal{U}\{a_n\}$ (for a fixed sequence $\{a_n\}$);

(2) find if exceptional measures actually exist;

(3) indicate the class of measures μ for which the integration by parts formula: $(x)(h)\mu(dx) = \int f(x)\nu_h(dx)$ holds for all bounded Lipschitz functions f, where $h \in X, \nu_h$ is some measure and f'(x)(h) denotes the derivative of f in the direction of h evaluated at the point x. In particular, f'(x)(h) must be defined μ-almost everywhere.

If X is finite-dimensional, m is the Lebesgue measure on X, and $\{a_n\}$ is a basis in X, then the class of measures indicated in problem (1) is the class of m-absolutely continuous measures; the answer to (2) is negative [1]; and the solution to problem (3) (under the requirement that the integration by parts formula should hold for all h) is given by the class of measures $\mu \ll m$ with the property that the generalized derivative of the function $\rho = d\mu/dm$ in any direction is a bounded measure (this follows from [4] and Theorem 4 below).

3. Formulation of Results

THEOREM 1. Let $\{a_n\} \subset X$. A measure is continuous in all directions a_n if and only if it is absolutely continuous with respect to some measure which is infinitely differentiable in all directions a_n.

COROLLARY. Every quasiinvariant measure is absolutely continuous with respect to some densely differentiable quasiinvariant measure.

Remark 1. Theorem 1 and its corollary hold true for Radon measures in an arbitrary sequentially complete locally convex space.

THEOREM 2 (Solution to Problem (1)). A measure is absolutely continuous with respect to $\mathcal{U}\{a_n\}$ if and only if it is continuous in the directions $\{a_n\}$ or, equivalently, if and only if it admits a density with respect to some measure which is infinitely differentiable in the directions $\{a_n\}$.

THEOREM 3 (Solution to Problem (2)). In any infinite-dimensional separable Banach space exceptional measures exist. Moreover, an exceptional measure exists which is not continuous in any direction and whose finite-dimensional projections have infinitely differentiable densities with respect to the Lebesgue measure.

THEOREM 4 (Solution to Problem (3)). Let f: X → R be a bounded Lipschitz function and let μ be a densely differentiable measure for which $d_h\mu$ exists. Then

$$\int f'(x)(h)\mu(dx) = \int f(x)\, d_h\mu(dx).$$

Conversely, if for every element h belonging to some linear subspace there is a measure ν_n such that

$$\int f'(x)(h)\mu(dx) = \int f(x)\nu_h(dx)$$

for all bounded Lipschitz function f, then there exists $d_h\mu = \nu_h$.

Remark 2. From Theorem 4 it follows that in an infinite-dimensional space there are no nonzero measures for which the integration by parts formula is valid in all directions.

The definitions of negligible sets given in [1] and [9] can be generalized to include, in the finite-dimensional case, all (and not only Borel) null sets in the sense of Lebesgue. We denote by $\mathscr{A}$ the class of sets which belong to the Lebesgue extension of $\mathscr{B}$(X) relative to any densely differentiable measure.

Definition 1. The set $A \in \mathscr{A}$ is called exceptional if for every sequence $\{a_n\}$ with dense linear space one can find sets $B_n \in \mathscr{A}$ such that $A = \bigcup_n B_n$ and $\operatorname{mes}((B_n + x) \cap R^1 a_n) = 0\ \forall n, \forall x \in X$.

Definition 2. The set $A \in \mathscr{A}$ is called negligible if $\mu(A) = 0$ for any densely differentiable measure μ.

THEOREM 5. The two properties described in Definitions 1 and 2 are equivalent.

THEOREM 6. A measure vanishes on all negligible sets if and only if it belongs to the closure of the linear space spanned by all densely differentiable measures taken with respect to the topology of convergence on sets of $\mathscr{A}$.

4. Elements of the Proofs. Theorem 2 is proved using Theorem 1 and results of [9, 10]. We sketch the proof of Theorem 3. It suffices to examine the case where X is a Hilbert space. Let A and K be nuclear self-adjoint operators with dense ranges and the property $A(X) \cap K(X) = 0$ ([11]), and let $h \notin A(X) + K(X)$. Consider the Gaussian measure $\gamma(t)$ with Fourier transform $x \to \exp(i(th, x) - (\exp tA \cdot K^4 \exp tAx, x))$. The measure defined by the formula $C \mapsto \int_0^1 \gamma(t)(C)\,dt$ has the desired properties.

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