FINITE IRREDUCIBLE GROUPS, GENERATED BY REFLECTIONS, ARE MONODROMY GROUPS OF SUITABLE SINGULARITIES

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1. Introduction

1. Groups, Generated by Reflections, and Singularities. The theory of singularities of smooth functions is closely connected with the theory of finite groups, generated by reflections. This connection appears in the following three assertions.

(1) The variety of orbits of the complexification of the action of a finite reflection group is biholomorphically equivalent with the base of a miniversal deformation of the corresponding singularity. Under this isomorphism the variety of nonregular orbits is mapped onto the bifurcation diagram.

(2) A reflection group is isomorphic with the monodromy group of the corresponding singularity.

(3) The isomorphism cited in the first assertion is defined by the period map, i.e., by integration of a holomorphic form defined on the total space of the bundle of hypersurfaces of level zero over the complement of the bifurcation diagram, with respect to a basis in the homology space of the fibre which depends continuously on a point of the base of the bundle.

Finite irreducible groups generated by reflections are classified and exhausted by the following list: A_μ, $B_\mu(=C_\mu)$, D_μ, E_6, E_7, E_8, F_4, G_2, $I_2(p)$, H_3, H_4. The majority of them are groups of symmetries of regular polyhedra: A_μ of a μ-dimensional simplex, B_μ of a μ-dimensional cube, G_2 of a hexagon, $I_2(p)$, $p > 6$, $p = 5$ of a p-gon, H_3 of an icosahedron, F_4, H_4 of the corresponding four-dimensional polyhedra [1, 2].

The singularities corresponding to the indicated groups are denoted by the same letters and are found in [3, 4, 5, 6]. In [4, 7, 8, 9], (1)-(3) are proved for the singularities A_μ, D_μ, E_6, E_7, E_8 of functions of an odd number of variables. In [5], (1) and (2) are proved for singularities B_μ, C_μ, F_4 of functions of an odd number of variables on a manifold with boundary. In [6], (1) is proved for the singularities G_2, $I_2(p)$, H_3 of functions on a manifold with singular boundary. Recently, O. P. Shcherbak produced a singularity which he called H_4, and proved (1) for it.

In this paper (3) is proved for the singularities B_μ, C_μ, F_4 of functions of an odd number of variables; the singularities cited, corresponding to the groups G_2, $I_2(p)$, H_3, are different from those cited in [6], but are closely connected with them; for these singularities (1)-(3) are proved. Analogs of (2) and (3) are unknown for the group H_4.

2. Symmetric Singularities. In [5] the following interpretation of singularities of functions on a manifold with boundary was used. After passage to the two-sheeted covering, functions on a manifold with boundary become functions which are symmetric with respect to the action of the group $\mathbf{Z}_2$, which changes the sign of one of the coordinates. In the present paper this analogy is extended. We consider singularities of functions, which are symmetric with respect to the cyclic group $\mathbf{Z}_p$, and their symmetric miniversal deformations. In this situation $\mathbf{Z}_p$ acts on the homology of nonsingular level hypersurfaces of the functions. This action commutes with the natural action of the fundamental group of the complement of the bifurcation diagram on the parameter space of the deformation. Thus, the homology splits into the direct sum of subspaces, which are invariant both with respect to the action of the group $\mathbf{Z}_p$, and with respect to the action of the fundamental group.

The groups G_2, $I_2(p)$, H_3 arise as images of the action of the fundamental group on a suitable invariant subspace in the homology of a suitable symmetric singularity. The period

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map is constructed as follows. On all level hypersurfaces of functions there is singled out uniquely a holomorphic form of highest degree. There is singled out a basis which is covariant constant in the Gauss–Manin connection, of a suitable invariant subspace of the homology. The period map relates a point of the complement of the bifurcation diagram to the vector of integrals of the form over the homology classes of the basis, defined up to the action of the monodromy group. One proves that this map extends holomorphically to a map of the parameter space of the deformation into the space of orbits of the complexification of the action of the corresponding group, generated by reflections, and has the properties cited in (1).

In this paper the concept of an equivalent vanishing vector in the homology of nonsingular level hypersurfaces of functions constituting a symmetric miniversal deformation is defined, generalizing the concept of a vanishing vector [10]. Equivariant vanishing vectors (with suitable degree of equivariance) for the singularities A_μ, D_μ, E_6, E_7, E_8, F_4, G_2 form a system of roots of the synonomous types; those for the singularities B_μ, C_μ do the same but for types C_μ, B_μ respectively. The collection of equivariant vanishing vectors for singularities $I_2(p)$, H_3 have properties analogous to the properties of systems of roots. Cf. Sec. 2.7 for more details.

The symmetric singularities corresponding to the groups G_2, $I_2(p)$, H_3 arise in the following way from the singularities G_2, $I_2(p)$, H_3 cited in [6] of functions on a manifold with singular boundary. In each case the pair manifold-boundary is isomorphic with the pair consisting of the space of orbits and the space of nonregular orbits of the group of symmetries of a suitable regular polyhedron. After passage to a suitable covering, the functions on the manifold with boundary turn into our functions, which are symmetric with respect to the group of symmetries of the polyhedron, in particular, with respect to the cyclic group of rotations of it.

Cf. [11, 12] also on symmetric singularities.

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2. Formulation of Results

1. Equivalent Monodromies, Vanishing Vectors, and Period Map. Let G be a finite group acting linearly on $\mathbf{C}^n$, $f: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be the germ of a holomorphic function at an isolated critical point, symmetric with respect to G. A deformation $F: (\mathbf{C}^n \times \mathbf{C}^\mu, 0 \times 0) \to (\mathbf{C}, 0)$ is said to be a *G-deformation*, if for any $\lambda \in \mathbf{C}^\mu$ the function $F(\cdot, \lambda)$ is G-invariant. A G-deformation of the germ f is said to be *versal*, if any other G-deformation of the germ is G-equivalent with a deformation induced from F (cf. [3, 13]) for more precision). A versal G-deformation with smallest number of parameters is called miniversal. As a miniversal G-deformation one can take $F(x, \lambda) = f(x) + \Sigma\lambda_j\varphi_j(x)$, where $\{\varphi_j\}$ generate a basis in $\mathbf{C}[[x_1, \ldots, x_n]]_G/(\partial f/\partial x)_G$, and by the index G we denote G-invariant series [13]. We always take $\varphi_1 \equiv 1$.

We choose a sufficiently small ball $B = \{x \in \mathbf{C}^n | \; ||x|| < \varepsilon\}$. Depending on ε we choose a sufficiently small ball $\Lambda = \{\lambda \in \mathbf{C}^\mu \; | \; ||\lambda|| < \delta\}$. We denote by V_λ the intersection of the zero level hypersurface of the function $F(\cdot, \lambda)$ with the ball B. By the *bifurcation diagram* of the deformation F is meant the subset $\Sigma \subset \Lambda$, consisting of those parameters λ, for which the hypersurface V_λ is singular. Over $\Lambda \setminus \Sigma$ the manifolds V_λ form a locally trivial bundle. With this bundle there are associated the cohomology bundle $H^{n-1} \to \Lambda \setminus \Sigma$ with fibre $H^{n-1}(V_\lambda, \mathbf{C})$ and the homology bundle $H_{n-1} \to \Lambda \setminus \Sigma$ with fibre $H_{n-1}(V_\lambda, \mathbf{C})$. The bundles H^{n-1} and H_{n-1} are provided with the Gauss–Manin connection.

On the fibres of these bundles the group G acts naturally. We consider the *canonical decomposition* (cf. [14]) of a representation of the group G on the space $H_{n-1}(V_\lambda, \mathbf{C})$: $H_{n-1}(V_\lambda, \mathbf{C}) = \bigoplus_\chi H_\chi(\lambda)$ (i.e., if $H_{n-1}(V_\lambda, \mathbf{C}) = U_1 \oplus \ldots \oplus U_k$ is the decomposition into the direct sum of irreducible representations, then $H_\chi(\lambda)$ is the direct sum of those irreducible representations U_j, which have character χ). Let $H^{n-1}(V_\lambda, C) = \bigoplus_\chi H^\chi(\lambda)$ be the canonical decomposition in cohomology. The spaces $H_{\bar\chi}(\lambda)$ and $H^\chi(\lambda)$ are naturally dual, where $\bar\chi$ is the character of the irreducible representation dual to the representation with character χ.

On the space $H_{n-1}(V_\lambda, C)$ the fundamental group $\pi_1(\Lambda \setminus \Sigma)$ of the complement of the bifurcation diagram acts. This action commutes with the action of the group G. Hence the spaces $H_\chi(\lambda)$ are invariant with respect to the action of the group $\pi_1(\Lambda \setminus \Sigma)$. The image of the natural representation ρ_χ (respectively ρ^χ) of the group $\pi_1(\Lambda\setminus\Sigma)$ on the space $H_\chi(\lambda)$ (respectively $H^\chi(\lambda)$) will be called the *χ-monodromy group* in homology (respectively cohomology). Notation: M_χ (respectively M^χ). It is obvious that $M_{\bar\chi} \simeq M^\chi$.

We define the period map. We denote by Ω^{n-1} the space of holomorphic $(n-1)$-forms on $B \times \Lambda$. By the *period map* of the form $\omega \in \Omega^{n-1}$ is meant the section $P_\omega: \lambda \mapsto [\omega]_\lambda \in H^{n-1}(V_\lambda, C)$ of the bundle H^{n-1}. For each integer $k \geqslant 0$ the section $P_\omega^k = (\nabla_{\partial/\partial\lambda_1})^k P_\omega$ is called the k-th adjoint period map of the form ω. (We recall that λ_1 is the free term of the deformation F). The section P_ω^k splits into a sum $\Sigma_\chi P_\omega^k$ of sections, where ${}_\chi P_\omega^k$ assumes values in H^χ. Obviously ${}_\chi P_\omega^k = (\nabla_{\partial/\partial\lambda_1})_\chi^k P_\omega$. The section ${}_\chi P_\omega^k$ is called the *k-th adjoint χ-period map* of the form ω. Let $\gamma_1(\lambda), \ldots, \gamma_m(\lambda)$ be a covariant constant basis in $H_{\bar\chi}(\lambda)$. The basis determines coordinates in $H^\chi(\lambda)$. In these coordinates

$$ {}_\chi P_\omega^k(\lambda) = \left(\frac{\partial}{\partial\lambda_1}\right)^k \left(\int_{\gamma_1(\lambda)} \omega, \ldots, \int_{\gamma_m(\lambda)} \omega \right). $$

We recall that a linear transformation of finite order of the space C^m is called a *pseudoreflection*, if exactly $m - 1$ of its eigenvalues are equal to 1. A pseudoreflection of order 2 is called a *reflection*. Let $M \subset GL(C^m)$ be a finite group, generated by pseudoreflections. Then the space C^m/M of its orbits is isomorphic with C^m [1, 15].

Let us assume that for some χ the group M^χ is a finite group generated by pseudoreflections. Then with the period map ${}_\chi P_\omega^k$ is associated the map ${}_\chi AP_\omega^k: \Lambda \setminus \Sigma \to H^\chi(\lambda_0)/M^\chi$, where $\lambda_0 \in \Lambda \setminus \Sigma$ is the distinguished point. The associated map is determined by the following rule: the point λ corresponds to the monodromy orbit on $H^\chi(\lambda_0)$, obtained by parallel transport of the value ${}_\chi P_\omega^k(\lambda)$ in the fibre over λ_0. It is easy to see that the map ${}_\chi AP_\omega^k$ is holomorphic.

Let us assume that for a general point $\lambda' \in \Sigma$ all singular points of the hypersurface $V_{\lambda'}$ are nondegenerate. For such a G-deformation F we define the concept of vanishing vector in $H_{n-1}(V_\lambda, Z)$, $\lambda \in \Lambda \setminus \Sigma$. Suppose given a path $\gamma(t)$, $t \in [0, 1]$ with initial point λ and end at a nonsingular point of the bifurcation diagram, not passing through other points of the diagram. To each singular point of the hypersurface $V_{\lambda(t)}$ in $H_{n-1}(V_{\lambda(t)}, Z)$ for t close to 1 there corresponds a standard vanishing cycle of Picard–Lefschetz [10]. Parallel transport over γ of Picard–Lefschetz cycles determines a collection of vectors in $H_{n-1}(V_\lambda, Z)$. Each of the vectors determined by this construction is called *vanishing* for the given deformation. It is easy to see that one can form a basis of $H_{n-1}(V_\lambda, Z)$ from vanishing vectors and the set of vanishing vectors is invariant with respect to monodromy (cf. [10]).

By the χ-*trace* of an integral lattice we mean the image of the projection of the lattice $H_{n-1}(V_\lambda, \mathbf{Z}) \subset H_{n-1}(V_\lambda, \mathbf{C})$ into $H_\chi(\lambda)$ along $\bigoplus_{\xi \neq \chi} H_\xi(\lambda)$. Notation: $\mathbf{Z}H_\chi(\lambda)$. We call a vector from $\mathbf{Z}H_\chi(\lambda)$ χ-*vanishing*, if it is the image of a vanishing vector. $\mathbf{Z}H_\chi(\lambda)$ has a natural structure as a module over the group ring $\mathbf{Z}[G]$ of the group G. The set of χ-vanishing vectors is invariant with respect to the action of the group G. The χ-trace and the set of χ-vanishing vectors are invariant with respect to χ-monodromy.

2. Symmetric Germs Corresponding to the Groups (cf. [5, 6], Sec. 1.1)

$$B_\mu:\ x_1^{2\mu} + x_2^2 + \ldots + x_n^2,\quad \mu \geqslant 2,$$
$$C_\mu:\ x_1^2 x_2 + x_2^\mu + x_3^2 + \ldots + x_n^2,\quad \mu \geqslant 2,$$
$$F_4:\ x_1^4 + x_2^3 + x_3^2 + \ldots + x_n^2,$$
$$G_2:\ x_1^3 + x_2^3 + x_3^2 + \ldots + x_n^2,$$
$$I_2(p):\ x_1^p + x_2^p + x_1^2 x_2^2 + x_3^2 + \ldots + x_n^2,\quad p = 5 \text{ or } p > 6,$$
$$H_3:\ x_1^5 + x_2^5 + x_3^2 + \ldots + x_n^2.$$

These germs have the following symmetry groups. $\mathbf{Z}_2$ is the group of symmetries of the germs B_μ, C_μ, F_4, acts by changes of sign of x_1. $\mathbf{Z}_3$, $\mathbf{Z}_p$, $\mathbf{Z}_5$ are the groups of symmetries of the germs G_2, $I_2(p)$, H_3, respectively. In these cases $\mathbf{Z}_p$ acts according to the rule (k): $(x_1, \ldots, x_n) \mapsto (\exp(2\pi ik/p)\, x_1, \exp(-2\pi ik/p)\, x_2, x_3, \ldots, x_n)$.

3. Canonical Decompositions of Homology. The group $\mathbf{Z}_p$ has p nonequivalent irreducible one-dimensional representations. Their characters are $\chi_s: (k) \mapsto \exp(2\pi iks/p)$, $s = 0, 1, \ldots, p-1$ In the following proposition we give the dimensions of the subspaces of the canonical decomposition $H_{n-1}(V_\lambda, \mathbf{C}) = \bigoplus_s H_{\chi_s}(\lambda)$.

Proposition 1. For the germs B_μ, C_μ, F_4, G_2, $I_2(p)$, H_3 the dimensions of the spaces $H_{\chi s}$ are given in the table

	B_μ	C_μ	F_4	G_2	$I_2(p)$	H_3
$s=0$	$\mu-1$	1	2	2	3	4
1	μ	μ	4	1	2	3
2	—	—	—	1	2	3
3,4	—	—	—	—	2	3
$4 < s < p$	—	—	—	—	2	—

The proposition is proved in Section 3.1.

4. Monodromy Groups. Until the end of the paper, if nothing is said to the contrary, we assume the *number n of variables is odd.*

In the following theorems the χ-monodromy groups of versal G-deformations of the germs B_μ, C_μ, F_4, G_2, $I_2(p)$, H_3 are described; cf. Section 2.2 for their symmetry groups.

THEOREM 1. For the germs B_μ, C_μ, F_4:

1. the χ_0-monodromy group is the group generated by reflections of type $A_{\mu-1}$, A_1, A_2, respectively.

2. the χ_1-monodromy group is the group generated by reflections of type B_μ, C_μ, F_4 respectively [5].

Remark. It is known that the groups B_μ, C_μ are isomorphic as linear groups, but not isomorphic as automorphism groups of the corresponding lattices. If the χ_1-monodromy groups are considered as automorphism groups of χ_1-traces of integral lattices, then the χ_1-monodromy groups of the germs B^μ, C_μ are groups of types C_μ, B_μ respectively, cf. Section 2.7.

THEOREM 2. For the germ G_2:

1. the χ_0-monodromy group is the group generated by reflections of type G_2.

2. the χ_s-monodromy group, s = 1, 2, is the group generated by reflections of type A_1.

THEOREM 3. For the germ $I_2(p)$:

1. the χ_0-monodromy group is infinite.

2. the χ_s-monodromy group s = 1, ..., p − 1, is the group generated by reflections of type $I_2(p_s)$, where $p_s = p/GCD(p, s)$.

3. If GCD (p, j) = GCD(p, l), then the kernels of the representations ρ_{χ_s}: $\pi_1(\Lambda \setminus \Sigma) \to \mathrm{Aut}\, H_{\chi_s}$ for s = j, l coincide.

4. The representations ρ_{χ_j} and ρ_{χ_l} are equivalent, if and only if $j + l = p$.

THEOREM 4. For the germ H_3:

1. the χ_0-monodromy group is infinite.

2. the χ_s-monodromy group, s = 1, 2, 3, 4, is the group generated by reflections of type H_3.

3. The kernels of the representations ρ_{χ_s} for s = 1, 2, 3, 4, coincide.

4. The representations ρ_{χ_j} and ρ_{χ_l} are equivalent if and only if $j + l = 5$.

Theorems 1-4 are proved in Sec. 3.3.

Remark. The structure of the χ_0-monodromy group for the germs $I_2(p)$, H_3 is unknown.

5. Period Maps. Let U be a vector space, $M \subset GL(U)$ be a finite group. By $\Sigma(U/M)$ we denote the set of nonregular orbits.

In the following theorems we describe the χ-periods in miniversal G-deformations of the germs B_μ, C_μ, F_4, G_2, $I_2(p)$, H_3; cf. Sec. 2.2 for their symmetry groups G.

THEOREM 1. Suppose given one of the germs B_μ, C_μ, F_4 with an odd number $n = 2k + 1$ of variables. Let us assume that $\omega \in \Omega^{n-1}$ is a sufficiently general form in the sense of Sec. 2.6. Then the map ${}_{\chi_1}AP^k_\omega$: $\Lambda \setminus \Sigma \to H^{\chi_1}(\lambda_0)/M^{\chi_1}$, where $\lambda_0 \in \Lambda \setminus \Sigma$, extends holomorphically to the bifurcation diagram Σ and gives an isomorphism of pairs $(\Lambda, \Sigma) \to (H^{\chi_1}(\lambda_0)/M^{\chi_1}, \Sigma(H^{\chi_1}(\lambda_0)/M^{\chi_1}))$.

THEOREM 2. For a germ G_2 with an odd number $n = 2k + 1$ of variables and forms $\omega \in \Omega^{n-1}$, sufficiently general in the sense of Sec. 2.6, the map ${}_{\chi_0}AP^k_\omega$: $\Lambda \setminus \Sigma \to H^{\chi_0}(\lambda_0)/M^{\chi_0}$, $\lambda_0 \in \Lambda \setminus \Sigma$, extends holomorphically to the bifurcation diagram Σ and gives an isomorphism of pairs $(\Lambda, \Sigma) \to (H^{\chi_0}(\lambda_0)/M^{\chi_0}, \Sigma(H^{\chi_0}(\lambda_0)/M^{\chi_0}))$.

For the germs $I_2(p)$, H_3 bifurcation diagram Σ consists of two (three for even p) irreducible components. We denote by Σ_1 the component consisting of those parameters $\lambda \in \Lambda$ for which the function $F(\cdot, \lambda)$ has $0 \in B$ as a critical point with the critical value zero; by Σ_2 the union of the remaining components.

THEOREM 3. Suppose given one of the germs $I_2(p)$, H_3 with an odd number $n = 2k + 1$ of variables. Let us assume that $\omega \in \Omega^{n-1}$ is a sufficiently general form in the sense of Sec. 2.6. Then the maps ${}_{\chi_1}AP^k_\omega$, ${}_{\chi_{p-1}}AP^k_\omega$ (for the germ H_3 p = 5) extend holomorphically to the bifurcation diagram Σ, the holomorphic extensions are maps of maximal rank, the preimages of the sets $\Sigma(H^{\chi_1}(\lambda_0)/M^{\chi_1})$, $\Sigma(H^{\chi_{p-1}}(\lambda_0)/M^{\chi_{p-1}})$ respectively coincide with Σ_2 (here $\lambda_0 \in \Lambda \setminus \Sigma$).

Theorems 1-3 are proved in Sec. 4.

Remark. We consider for miniversal G-deformations of germs B_μ, C_μ, F_4, G_2 with an odd number $n = 2k + 1$ of variables, the period map ${}_\chi AP^k_\omega$, where ω is a sufficiently general form, χ is a character, different from those cited in Theorems 1, 2. One proves analogously to

Theorems 1, 2 that the map of χ-periods extends holomorphically to the bifurcation diagram and is a map of maximal rank, where the preimage of the set of nonregular orbits of χ-monodromy coincides with one of the irreducible components of the bifurcation diagram. For miniversal G-deformations of the germs $I_2(p)$, H_3 and characters different from χ_1, χ_{p-1} we have been unable to construct good period maps.

6. A form $\omega \in \Omega^{n-1}$ in general position is defined by conditions on the jet at the point $0 \times 0 \in B \times \Lambda$ of the restriction to $B \times 0$ of the form $\sum a_{l_1,\ldots,l_n} x_1^{l_1} \ldots x_n^{l_n}\, dx_1 \wedge \ldots \wedge dx_n$.

The form $\omega \in \Omega^{n-1}$ is called a *form in general position* for the germs:

1) B_μ, C_μ, F_4, G_2, if $a_{0,\ldots,0} \neq 0$,

2) $I_2(p)$, H_3, if $a_{1,0,\ldots,0} \cdot a_{0,1,0,\ldots,0} \neq 0$.

7. System of Roots. The groups generated by reflections of types A_μ, B_μ, C_μ, D_μ, E_6, E_7, E_8, F_4, G_2 are crystallographic. With each crystallographic group there is associated a system of roots, cf. [1]. From the point of view of the theory of singularities, the system of roots is the set of vanishing vectors. For germs A_μ, D_μ, E_6, E_7, E_8 with odd n the set of vanishing vectors forms a one-dimensional root system in $H_{n-1}(V_\lambda, Z)$ [4].

Proposition 1. For germs B_μ, C_μ, F_4 with odd n and their versal Z_2-deformations, the sets:

1. of χ_0-vanishing vectors in H_{χ_0} form systems of roots of types $A_{\mu-1}$, A_1, A_2 respectively.

2. of χ_1-vanishing vectors in H_{χ_1} form systems of roots of types C_μ, B_μ, F_4 respectively (cf. [5]).

Proposition 2. For a germ G_2 with odd n and its versal Z_3-deformation, the set:

1. of χ_0-vanishing vectors in H_{χ_0} forms a system of roots of type G_2.

2. of χ_s-vanishing vectors in H_{χ_s}, s = 1, 2 forms a system of roots of type A_1.

As is known, a *system of roots* in a vector space means a finite subset R of it with the following properties:

1) R generates the space and does not contain the zero vector;

2) for any vector $a \in R$ there exists a reflection h_a with respect to a, carrying R into itself;

3) for any $a, b \in R$ $h_a(b) - b = ha$, where $h \in \mathbf{Z}$.

Generalization. Let χ be a character of a one-dimensional representation of the finite group G. We set $\mathbf{Z}[G]_\chi = \{z \in \mathbf{C} \mid z = \sum_{g \in G} k_g \chi(g)$, where $k_g \in \mathbf{Z}\}$. We call a finite subset R of a complex vector space a system of $Z[G]_\chi$-roots, if it has properties 1), 2), 3'), where

3') for any $a, b \in R$ $h_a(b) - b = za$, where $z \in \mathbf{Z}[G]_\chi$.

Proposition 3. For a germ of type $I_2(p)$ or H_3 with odd n and its versal Z_p-deformation (for the germ H_3, p = 5), the set:

1. of χ_0-vanishing vectors is infinite.

2. of χ_s-vanishing vectors in H_{χ_s}, s = 1, ..., p − 1 forms a system Z of $[Z_p]_{\chi_s}$-roots.

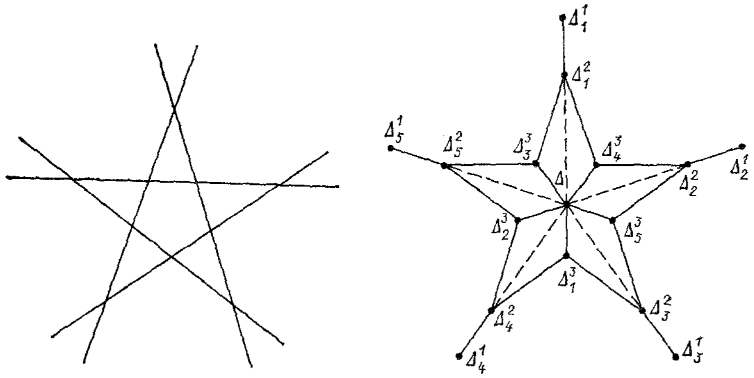
The proofs of the propositions are in Sec. 3.4.

Remark. With a system of roots there is naturally associated a Hermitian form, which is invariant with respect to reflections and is integer-valued on pairs of roots [1]. With each system of $Z[G]_\chi$-roots, corresponding to $I_2(p)$, H_3, there is associated a Hermitian form, invariant with respect to reflections and the action of the group G, assuming values in $Z[G]_\chi$ on pairs of roots. Such a form is the form $(a, \bar{b})$, where $(\cdot,\cdot)$ is the intersection form. We note that reflection in the vector a, preserving the form, can be described in the form $h_a(b) = b - 2\frac{(b, \bar{a})}{(a, \bar{a})}a$. In particular, condition 3') can be written in the form $\frac{2(b, \bar{a})}{(a, \bar{a})} \in \mathbf{Z}[G]_\chi$.

3. Calculation of Monodromy and Root Systems

In this section we prove the assertions of Paragraphs 2.3, 2.4, 2.7.

1. Proof of Proposition 2.3. For the germs B_μ, C_μ, F_4, the proposition is proved in [5]. For a germ H_3 the proposition follows from the following lemma.



LEMMA 1. For a germ H_3 in $H_{n-1}(V_\lambda, C)$ there exists a basis $\Delta, \Delta_1^1, \ldots, \Delta_5^1, \Delta_1^2, \ldots, \Delta_5^2, \Delta_1^3, \ldots, \Delta_5^3$, on which Z_5 acts by

$$(1): \Delta \mapsto \Delta, \quad \Delta_j^k \mapsto \Delta_{j+1}^k, \quad \Delta_5^k \mapsto \Delta_1^k \quad \text{for } k = 1, 2, 3;\ j = 1, 2, 3, 4,$$

and whose Dynkin diagram is pictured in the figure.

COROLLARY.

$$H_{\chi_0} = \langle \Delta; (\Delta_1^k + \ldots + \Delta_5^k)/5, \quad k = 1, 2, 3 \rangle, \tag{1}$$
$$H_{\chi_s} = \left\langle \frac{1}{5} \sum_{j=1}^{5} \varepsilon^{s(j-1)} \Delta_j^k; \quad k = 1, 2, 3 \right\rangle,$$

where $\varepsilon = \exp(2\pi i/5)$, $s = 1, 2, 3, 4$.

Proof of Lemma 1. It suffices to prove the lemma for $n = 2$, i.e., for the germ $l = x_1^5 + x_2^5$ (cf. [10, Sec. 2]). Let F be a deformation of the germ f, which in the coordinates $u = (x_1 + x_2)/2$, $v = (x_1 - x_2)/2i$ has the form

$$F(u, v; \lambda) = 32 \prod_{k=0}^{4} \left[\left(\cos\frac{2\pi k}{5}\right) u + \left(\sin\frac{2\pi k}{5}\right) v - \lambda \right],$$

$\lambda \in \mathbf{R}$.

The zero level manifold of the function $F(\cdot, \lambda)$ is pictured in the figure. The basis constructed from the figure by Gusein-Zade's method is the one sought.

The proof of the proposition for G_2 repeats the proof for H_3. In this case the zero level manifold of the corresponding function of two variables is three lines.

We prove the position for $I_2(p)$.

LEMMA 2. For a germ $I_2(p)$ in $H_{n-1}(V_\lambda, C)$ there exists a basis $\Delta, \Delta_1^1, \ldots, \Delta_p^1, \Delta_1^2, \ldots, \Delta_p^2$, on which Z_p acts by

$$(1): \Delta \mapsto \Delta, \quad \Delta_p^k \mapsto \Delta_1^k, \quad \Delta_j^k \mapsto \Delta_{j+1}^k \text{ for } k = 1, 2,\ j = 1, \ldots, p-1.$$

The classes $\Delta_1^k, \ldots, \Delta_p^k$, k = 1, 2, have the following intersection indices: $(\Delta_p^1, \Delta_p^2) = (\Delta_1^1, \Delta_1^2) = (\Delta_j^1, \Delta_j^2) = (\Delta_j^1, \Delta_{j+1}^2) = 1$ for j = 1, ..., p − 1; $(\Delta_j^k, \Delta_j^k) = (-1)^{n(n-1)/2}(1 + (-1)^{n-1})$ for k = 1, 2, j = 1, ..., p; the remaining intersection indices are equal to zero.

COROLLARY.

$$H_{\chi_0} = \left\langle \Delta; \frac{1}{p}\sum_{j=1}^{p} \Delta_j^k;\ k = 1, 2 \right\rangle,$$
$$H_{\chi_s} = \left\langle \frac{1}{p}\sum_{j=1}^{p} \varepsilon^{s(j-1)}\Delta_j^k;\ k = 1, 2 \right\rangle, \tag{2}$$

where $\varepsilon = \exp(2\pi i/p)$, s = 1, ..., p − 1.

Remark. The bases in (1), (2) serve as bases over Z[G] of the traces of the integral lattice.

2. Proof of Lemma 2. It suffices to prove the lemma for n = 2, i.e., for the germ $f = x_1^p + x_2^p + x_1^2x_2^2$, cf. [10, Sec. 2].

We set $F(x, \lambda) = f(x_1, x_2) + \lambda_2 x_1 x_2 + \lambda_1$; $u = x_1^p + x_2^p$, $v = x_1x_2$. The map $\varphi: (x_1, x_2) \mapsto (u, v)$ is a 2p-sheeted covering, which branches along $\Gamma = \{(u, v) \in \mathbf{C}^2 \mid v^p = 4u^2\}$. Consequently, the nonsingular manifold $V_\lambda = \{F(\cdot, \lambda) = 0\}$ is a 2p-sheeted covering of $W_\lambda = \{(u, v) \in \mathbf{C}^2 \mid u + v^2 + \lambda_2 v + \lambda_1 = 0\}$ with branching in $\Gamma \cap W_\lambda$. We shall produce "semicycles" on W_λ, from whose preimages we construct the basis sought in $H_1(V_\lambda, Z)$.

For small λ, W_λ has 4 points of intersection with Γ near zero. We fix a generic value $\lambda^0 = (\lambda_1^0, \lambda_2^0)$. The function $F(\cdot, \lambda^0)$ has three critical values. One of them is λ_1^0. Let λ_1^1, λ_1^2 be the other two. On the line $\mathbf{C}^1$ of values of the function, we draw two segments $\gamma^1(t), \gamma^2(t)$, $t \in [0, 1]$, with origin at the common point λ_1^* and ends at λ_1^1, λ_1^2, respectively. As t → 1 on $W_{(\gamma^j(t), \lambda_2^0)}$ the two points of intersection with Γ merge. By vanishing semicycle we mean the small curve on $W_{(\gamma^j(t), \lambda_2^0)}$ with ends at the merging points of intersection with Γ. For $t \in [0, 1]$ we single out a vanishing semicycle on $W_{(\gamma^j(t), \lambda_2^0)}$ which depends continuously on t. We denote it by $\delta_j(t)$. One can choose semicycles so that on $W_{(\lambda^*, \lambda_0^2)}$ they intersect in one point. The preimage under φ of each of the semicycles consists of p closed curves. We orient the curves. The homology classes defined by them will be denoted by $\Delta_1^j, \ldots, \Delta_p^j$. It is easy to see that one can choose the indices and the orientations so that all the intersection indices of the classes $\{\Delta_l^j\}$ are just as indicated in the lemma, and that group Z_p permutes $\Delta_1^j, \ldots, \Delta_p^j$ cyclically.

3. Proof of the Theorems of Sec. 2.4. Proof of Theorem 2.4.4. Let Δ, Δ_l^k be the classes indicated for H_3 in Paragraph 3.1. The monodromy group is generated by the following four transformations (generalized Picard–Lefschetz transformations):

$$h = h_\Delta,\ h_k = h_{\Delta_5^k} h_{\Delta_4^k} \ldots h_{\Delta_1^k} \quad (k = 1, 2, 3),$$

where $h_{\Delta_l^k}$ is the reflection defined by the Picard–Lefschetz formula:

$$h_{\Delta_l^k}(x) = x + (-1)^{n(n+1)/2}(x, \Delta_l^k)\,\Delta_l^k.$$

Point 1 of the theorem follows from the fact that the restriction of the transformation hh_3 to H_{χ_0} has infinite order.

Proof of Point 2. Let $j \neq 1$. It is easy to see that $h|H_{\chi_j} = \mathrm{Id}$. One can see, by direct calculation with the help of the Dynkin diagram, that on each H_{χ_j} the transformations h_1, h_2, h_3 satisfy the relations $(h_3h_2)^5 = (h_1h_3)^2 = (h_2h_1)^3 = h_k^2 = \mathrm{Id}$ for k = 1, 2, 3. As is known (cf. [2]), these relations define the group H_3 of symmetries of an icosahedron. Consequently, the χ_j-monodromy group is isomorphic with some quotient group of the group H_3. Since H_3 is isomorphic with the direct product $\mathbf{Z}_2 \times \mathfrak{A}_5$ of the group Z_2 and the group $\mathfrak{A}_5$ of even permutations of five elements, the χ_j-monodromy group is isomorphic with one of the groups Z_2, $\mathfrak{A}_5$, H_3 (since the group $\mathfrak{A}_5$ is simple). Obviously $M_{\chi_j} \neq \mathbf{Z}_2$. Calculating the value on h_1 of the character of the representation ρ_{χ_j}, we see that $M_{\chi_j} \neq \mathfrak{A}_5$.

Point 3 follows from the fact that the relations listed are the defining relations for the group M_{χ_j}.

Point 4 is proved by explicit calculation of the characteristics of the representations ρ_{χ_j}, considered as representations of the finite group $\pi_1(\Lambda \setminus \Sigma)/\mathrm{Ker}\,\rho_{\chi_j}$. Theorem 2.4.4 is proved.

The proof of Theorem 2.4.2 is analogous to the proof of Theorem 2.4.4. Point 1 of Theorem 2.4.3 follows from the fact that the order of the classical monodromy transformation is infinite; points 2-4 of Theorem 2.4.3 are proved analogously to the corresponding assertions of Theorem 2.4.4; cf. [5] for point 2 of Theorem 2.4.1; point 1 of Theorem 2.4.1 follows from the lemma.

Let $f: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be the germ of a holomorphic function, $\varphi: \mathbf{C}^n \to \mathbf{C}^n$ be the map defined by $\varphi(x_1, \ldots, x_n) = (x_1^2, x_2, \ldots, x_n)$. Let us assume that f and $f \circ \varphi$ have nonisolated critical points at the origin. The germ $f \circ \varphi$ is invariant with respect to the group Z_2 of sign changes of x_1.

LEMMA. The χ_0-monodromy of a versal Z_2-deformation of the germ $f \circ \varphi$ is isomorphic with the monodromy group of a versal deformation of the germ f.

Proof. Let F be a versal Z_2-deformation of the germ $f \circ \varphi$. F naturally defines a deformation $\varphi_* F$ of the germ f ($\varphi_* F \circ \varphi = F$). The lemma follows from two facts. The first is that $\varphi_* F$ is a versal deformation of the germ f. The second is that the space $H_{\chi_0}(\lambda)$ corresponding to the deformation F is canonically isomorphic with the fibre over λ of the homology bundle of the deformation $\varphi_* F$.

4. Proofs of the Propositions of Paragraph 2.7. The *proof of Proposition 2.7.3* is based on the following easy lemma.

LEMMA 1. Let χ be a character of the group Z_p ($p = 5$ for H_3); then for any χ-vanishing vector a there exist a vector e from the basis indicated in Paragraph 3.1 of H_χ, a transformation $h \in M_\chi$, and a number $j \in \mathbf{Z}$, such that $a = \varepsilon^j he$, where $\varepsilon = \exp(2\pi i/p)$. Conversely, any vector of the form $\varepsilon^j he$ is χ-vanishing.

Point 1 of Proposition 2.7.3 follows from the lemma and the fact that the group M_{χ_0} is infinite. To prove point 2, we verify conditions 1), 2), 3') of the definition of a system of $Z[G]_\chi$-roots. Condition 1) obviously holds. To prove condition 2) we consider the vector $e_k = \frac{1}{p} \sum_{j=1}^{p} \varepsilon^{s(j-1)} \Delta_j^k$, which belongs to the basis of H_{χ_s} indicated in Paragraph 3.1. We shall show that the transformation $h_k = h_{\Delta_p^k} \ldots h_{\Delta_1^k}$ (cf. Paragraph 3.3) is a reflection in H_{χ_s} with respect to the vector e_k. In fact, $h_k(e_k) = -e_k$. Moreover, the transformation h_k in

$H_{n-1}(V_\lambda, C)$ has a p-dimensional antiinvariant subspace, generated by the vectors Δ_j^k ($j = 1, \ldots, p$), and an invariant subspace of the complementary dimension. Since the group Z_p cyclically permutes the Δ_j^k, the intersection of the antiinvariant subspace with H_{χ_s} is one-dimensional (and consequently generated by e_k), and the intersection of the invariant subspace with H_{χ_s} is $(p-1)$-dimensional. Further, if α is an arbitrary χ_s-vanishing vector, then $\alpha = \varepsilon^j h e_k$ for some k, j, h. Then the transformation $h_\alpha = h^{-1} h_k h$ is reflection with respect to the vector α, belongs to the group M_{χ_s}, and hence carries χ_s-vanishing vectors into χ_s-vanishing vectors. Condition 3') follows from the following lemma.

LEMMA 2. Let $e_k = \frac{1}{p}\sum_{j=1}^{p} \varepsilon^{s(j-1)}\Delta_j^k$ be a vector from the basis in H_{χ_s}, indicated in Paragraph 3.1; then for any $b \in ZH_{\chi_s}$, we have $h_k(b) - b = ze_k$, where $z \in \mathbf{Z}[\mathbf{Z}_p]_{\chi_s}$.

Proof. Let $\tilde{b} \in H_{n-1}(V_\lambda, \mathbf{Z})$ project to b. Then $h_k(\tilde{b}) - \tilde{b}$ projects to $h_k(b) - b$. But $h_k(\tilde{b}) = \tilde{b} + \sum_{j=1}^{p} a_j\Delta_j^k$, where $a_j^k \in \mathbf{Z}$. Since Δ_j^k projects to $\varepsilon^{-s(j-1)}e_k$, one has $h_k(b) - b = \left(\sum_{j=1}^{p} a_j \varepsilon^{-s(j-1)}\right) e_k$, which proves the lemma. Proposition 2.7.3 is proved.

Proof of Proposition 2.7.1. One proves, analogously to Proposition 2.7.3, that for germs B_μ, C_μ, F_4, the set of χ-vanishing vectors in H_χ forms a system of roots. The type of the system of roots is determined by the Cartan matrix in the basis of the system of roots (cf. [16]). The determination of the type of a system of roots is based on the following obvious lemma.

LEMMA 3. Let us assume that R is a root system in $H_{n-1}(V_\lambda, C)$, S is its basis, $\pi_k: H_{n-1}(V_\lambda, \mathbf{C}) \to H_{\chi_k}$ is the projection along $\bigoplus_{j\neq k} H_{\chi_j}$, and $\pi_k(R)$ is a root system in H_{χ_k}. Then $\pi_k(S)$ is a basis of the system $\pi_k(R)$.

The proofs of points 1, 2 for the germs B_μ, C_μ, F_4 are analogous. We give the proof of point 1 for the germ B_μ. If one forgets about the group of symmetries Z_2, then a germ B_μ has type $A_{2\mu-1}$. For a germ $A_{2\mu-1}$ we consider in $H_{n-1}(V_\lambda, Z)$ the standard distinguished basis $\Delta, \Delta_1^1, \ldots, \Delta_1^{\mu-1}, \Delta_2^1, \ldots, \Delta_2^{\mu-1}$, for which the group Z_2 acts according to the formulas:

$$(1): \Delta \mapsto -\Delta;\ \Delta_1^k \to -\Delta_2^k \text{ for } k = 1, \ldots, \mu-1,$$

the intersection matrix has the form $(\Delta, \Delta) = (\Delta_s^k, \Delta_s^k) = \pm 2$ for $s = 1, 2$; $k = 1, \ldots, \mu-1$; $(\Delta, \Delta_1^1) = (\Delta, \Delta_2^1) = (\Delta_1^k, \Delta_1^{k-1}) = (\Delta_2^k, \Delta_2^{k-1}) = 1$ for $k = 2, \ldots, \mu-1$, the remaining intersection indices are equal to zero. This basis is a basis of a root system of type $A_{2\mu-1}$. According to Lemma 3, $\{(\Delta_1^k - \Delta_2^k)/2,\ k = 1, \ldots, \mu-1\}$ is a basis of a root system in H_{χ_0}. It is easy to calculate its Cartan matrix and see that the proposition is valid.

4. Proofs of Theorems on the Period Map

Theorems 2.5-1-2.5.3 are proved by Looijenga's scheme from [9]. For example, we prove Theorem 2.5.3 for a germ H_3 and character χ_1.

We fix a miniversal Z_5-deformation

$$F = x_1^5 + x_2^5 + x_3^2 + \ldots + x_n^2 + \lambda_1 + \lambda_2 x_1 x_2 + \lambda_3 x_1^2 x_2^2 + \lambda_4 x_1^3 x_2^3.$$

LEMMA 1. Let $\omega \in \Omega$ be a sufficiently general form, $\gamma_1(\lambda), \gamma_2(\lambda), \gamma_3(\lambda) \in H_{\chi_1}(\lambda)$ be a multivalued constant basis. Then the map

$$\chi_1 P_\omega^k: \lambda \mapsto \left(\frac{\partial}{\partial\lambda_1}\right)^k \left(\int_{\gamma_1(\lambda)} \omega, \int_{\gamma_2(\lambda)} \omega, \int_{\gamma_3(\lambda)} \omega\right),\quad \lambda \in \Lambda \setminus \Sigma,$$

is a multivalued mapping of maximal rank for λ sufficiently close to $0 \in \Lambda$.

Proof (cf. [17], Sec. 10], [18]). We shall prove that the minor J of the Jacobi matrix of the map ${}_{\chi_1}P^k_{\omega}$, corresponding to $\partial/\partial\lambda_1$, $\partial/\partial\lambda_2$, $\partial/\partial\lambda_3$, is different from zero on $\Lambda \setminus \Sigma$. This assertion is a direct consequence of the following four assertions (cf. [17, Sec. 10]).

I. J^2 is a meromorphic function on Λ, which is holomorphic on $\Lambda \setminus \Sigma$.

II. For generic values $\lambda_2 = \lambda_2^0$, $\lambda_3 = \lambda_3^0$, $\lambda_4 = \lambda_4^0$ the line λ_1 intersects Σ_1 in one point, Σ_2 in three points. We denote the corresponding values of the parameter λ_1 by $\lambda_1^1, \ldots, \lambda_1^4$.

III. For λ_1 tending to λ_1^j, $j = 2, 3, 4$, along the line indicated, $J = c(\lambda_1 - \lambda_1^j)^{-1/2} + O(1)$. For λ_1 tending to λ_1^1 along the line indicated $J = O(1)$.

IV. For λ_1 tending along the axis to the point $0 \in \Lambda$ $J = c\lambda_1^{-3/5} + o(\lambda_1^{-3/5})$, where $c \neq 0$.

I is a consequence of the regularity of the Gauss–Manin connection. II is proved by direct calculation (cf. with the proof of point 4 of Proposition 2.3).

We prove III. Let $2 \leqslant j \leqslant 4$. A circuit around the point $\lambda^j = (\lambda_1^j, \lambda_2^0, \lambda_3^0, \lambda_4^0)$ on the line λ_1 induces reflection in H_{χ_4}. We change the covariant constant basis linearly over R in a small neighborhood of the point λ^j so that the class $\gamma_1(\lambda)$ becomes antiinvariant with respect to reflections, classes $\gamma_2(\lambda)$, $\gamma_3(\lambda)$ become invariant (from this, J is multiplied by a constant). On the line λ_1, passing through λ_1^j, one has series expansions

$$\frac{\partial}{\partial\lambda_s}\left(\frac{\partial}{\partial\lambda_1}\right)^k \int\limits_{\gamma_1(\lambda)} \omega = (\lambda_1 - \lambda_1^j)^{-1/2} \sum_{r \geqslant 0} (\lambda_1 - \lambda_1^j)^r a_r(\lambda_2^0, \lambda_3^0, \lambda_4^0), \tag{1}$$

$$\frac{\partial}{\partial\lambda_s}\left(\frac{\partial}{\partial\lambda_1}\right)^k \int\limits_{\gamma_p(\lambda)} \omega = \sum_{r \geqslant 0} (\lambda_1 - \lambda_1^j)^r b_r^p(\lambda_2^0, \lambda_3^0, \lambda_4^0),$$

where p = 2, 3; s = 1, 2, 3, 4, the numbers a_r, b_r^p depend holomorphically on λ_2^0, λ_3^0, λ_4^0.

The first expansion is a consequence of the standard direct calculations in a neighborhood of a nondegenerate critical point. The second expansion is a consequence of the theorem on the boundedness of integrals over invariant cycles [19]. The existence of the expansions proves the first part of assertion III. One proves analogously that the χ_4-monodromy corresponding to a circuit along the line λ_1 about the point λ^1 is the identity transformation. Hence for $(\lambda_1, \lambda_2^0, \lambda_3^0, \lambda_4^0) \to \lambda^1$ all the elements of the Jacobi matrix are bounded. III is proved.

We prove IV. We expand ω in characters of the group Z_5: $\omega = \Sigma\omega^{\chi_j}$. For covariant constant $\gamma(\lambda) \in H_{\chi_4}(\lambda)$ we have

$$\frac{\partial}{\partial\lambda_j}\left(\frac{\partial}{\partial\lambda_1}\right)^k \int\limits_{\gamma(\lambda)} \omega = \frac{\partial}{\partial\lambda_j}\left(\frac{\partial}{\partial\lambda_1}\right)^k \int\limits_{\gamma(\lambda)} \omega^{\chi_4} = -\left(\frac{\partial}{\partial\lambda_1}\right)^k \int\limits_{\gamma(\lambda)} (x_1x_2)^{j-1}(ax_1 + O(x^2, \lambda))\, dx_1 \wedge \ldots \wedge dx_n/d_xF,$$

where $a \in C$ and $a \neq 0$ by virtue of the generality of the form ω (cf. [17, Sec. 10]).

According to [20], the forms $\omega_j = (x_1x_2)^{j-1}x_1dx_1 \wedge \ldots \wedge dx_n/d_xF$, $j = 1, 2, 3$, for λ belonging to the λ_1 axis, generate in $H^{n-1}(V_\lambda, C)$, linearly independent cohomology classes. It follows from the quasihomogeneity of the forms it follows that for λ belonging to the λ_1 axis,

$$\int\limits_{\gamma(\lambda)} \omega_j = \lambda_1^{\alpha_j}\,\text{const}, \quad \text{where} \quad \alpha_j = (2j+1)/5 + k - 1. \tag{2}$$

The form $(x_1x_2)^{j-1} O(x^2)\, dx_1 \wedge \ldots \wedge dx_n/d_xF$ has a large degree of quasihomogeneity compared with ω_j. Hence the preceding formulas prove IV.

Remark. The germ $I_2(p)$ is the only one of those listed in the theorems which is not quasihomogeneous. In case of the germ $I_2(p)$ to prove assertion IV, for calculating the weights of the forms ω_1, ω_2 (i.e., the numbers α_1, α_2) and for proof of the linear independence of the cohomology classes generated by the forms, it is necessary to use Theorem 4.3 of [17]. Only the calculation of the weight of the form $\omega_2 = x_1^2 x_2 dx/d\,(x_1^p + x_1^2 x_2^2 + x_2^p + x_3^2 + \ldots + x_n^2) = \smile dx_1 \wedge dx_3 \wedge dx_4 \wedge \ldots \wedge dx_n/2 - p x_2^{p-1}\,dx/2d\,(x_1^p + x_1^2 x_2^2 + x_2^p + x_3^2 + \ldots + x_n^2)$ proceeds directly with the help of total resolution of singularities.

LEMMA 2. The map ${}_{\chi_1}AP_{\omega}^{k}$ extends homomorphically to Σ, while Σ_2 into the set of nonregular orbits.

The proof proceeds with the help of formulas analogous to (1), cf. [9].

LEMMA 3. The restriction of the map ${}_{\chi_1}P_{\omega}^{k}$ to the λ_1 axis has the form ${}_{\chi_1}P_{\omega}^{k}(\lambda_1, 0, 0, 0) = (c_1, c_2, c_3)\,\lambda_1^{1/10} + o\,(\lambda_1^{1/10})$, where $(c_1, c_2, c_3) \neq 0$.

The *proof* follows from (2).

Remarks. 1. The number of reflections among the elements of the group H_3 is equal to 15.

2. Let $\Sigma(\mathbf{C}^3/H_3)$ be the discriminant of the natural projection $\pi\colon \mathbf{C}^3 \to \mathbf{C}^3/H_3$ (i.e., the set of nonregular orbits). Let D be the polynomial on $\mathbf{C}^3$ defining the union of all planes, each of which is fixed with respect to some reflection from H_3. Then D^2 is invariant with respect to H_3, and if I is the polynomial on $\mathbf{C}^3/H_3$ with the property $D^2 = \pi^* I$, then I determines $\Sigma(\mathbf{C}^3/H_3)$ (without multiplicities).

Property 1 is obvious, property 2 can be found in [1, Chap. V].

LEMMA 4. The manifold of the discrimant $\Sigma\ (H^{\chi_1}/H_3)$ for the map ${}_{\chi_1}AP_{\omega}^{k}$ coincides with Σ_2.

Lemma 4 is a consequence of the remarks and Lemmas 2, 3 (cf. [9]).

LEMMA 5. The map ${}_{\chi_1}AP_{\omega}^{k}|_{\lambda_4=0}\colon \Lambda \cap \{\lambda_4 = 0\} \to H^{\chi_1}/H_3$ is nondegenerate.

Proof. Loss of nondegeneracy contradicts Remark 2 (cf. Sec. 4 in [9]).

Lemmas 4, 5 imply Theorem 2.5.3 for H_3.

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MODELS OF REPRESENTATIONS OF CLASSICAL GROUPS AND THEIR HIDDEN SYMMETRIES

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0. The Main Results

Three important constructions lie behind the motivation of this paper:

1. The classical realization of irreducible representations of the group SO_3 acting on functions on the two-dimensional sphere. We may say that the space of functions on the two-dimensional sphere is a model of representations of SO_3 (meaning that its decomposition into irreducible components contains all the irreducible representations of SO_3, each appearing with multiplicity one).

2. A recent construction of Biederharn and Flath [1]: they built a model (in the sense indicated above) of finite-dimensional irreducible representations of the Lie algebra $sl(3, \mathbf{C})$; they also found that the action of $sl(3, \mathbf{C})$ on this model extends to an action of the larger Lie algebra $so(8, \mathbf{C})$.

3. The starting point of the R. Penrose's twistor program (see [2]): the complexification of the Minkowski space $\mathbf{R}^4$ followed by compactification leads to the Grassman manifold of 2-planes in $\mathbf{C}^4$.

We show here that these constructions are different aspects of a unifying construction of models of representations which is carried out below for all classical groups. A fourth important aspect of this construction is a remarkable parallelism between exterior and symmetric algebras; one of its consequences is that Lie supergroups and superalgebras arise naturally in the "purely even" problem.

Let us give a systematic description of the content of this paper, beginning with the results concerning the first construction.

Let G be a reductive algebraic group over $\mathbf{C}$. A model of representations of the group G is defined as a representation of G which decomposes into the direct sum of all (finite-dimensional) irreducible algebraic representations in which each such representation appears with multiplicity one.* One of the most natural ways of realizing a model is to express it as an induced representation $\mathrm{Ind}_M^G \tau$. This realization is the most convenient when $\tau = 1$: in this case the model is realized in the space of regular functions on the homogeneous space G/M. A

*H. Weyl's "unitary trick" shows that constructing such a model is equivalent to constructing a model of representations of a compact form of G. In this paper we use the language of complex groups; henceforth, by group we shall always mean, without further mention, an algebaic group over $\mathbf{C}$.

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