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DETERMINANTS OF CAUCHY–RIEMANN OPERATORS OVER A RIEMANN SURFACE

Daniel Quillen

UDC 517.43

1. Let M be a compact one-dimensional complex manifold of genus g, and let E be a C^∞ vector bundle over M of rank r and degree d. Let $\Omega^{p,q}(E)$ be the vector space of smooth forms on M of type (p, q) with values in E. By a Cauchy–Riemann or $\bar\partial$-operator on E we mean a differential operator $D:\Omega^{0,0}(E) \to \Omega^{0,1}(E)$ which locally, in terms of a local coordinate z and a local frame in E, has the form $D = d\bar z(\partial_{\bar z} + \alpha(z))$, where $\alpha(z)$ is a smooth matrix function. Such operators are in one-to-one correspondence with holomorphic structures on the vector bundle E. We denote by $\mathcal{A}$ the space of these operators; it is an affine space relative to the complex vector space $\mathcal{B} = \Omega^{0,1}(\mathrm{End}\, E)$.

The purpose of this paper is to present a construction of determinants for such $\bar\partial$-operators based on the concept of determinant line bundle and the theory of zeta function determinants for positive elliptic operators.

Since $\bar\partial$-operators go from one vector space to another, we explain what is meant by determinants in this case. Consider first the family of operators $T:V^0 \to V^1$, where V^0 and V^1 are vector spaces of the same finite dimension. Each T induces a map from $\lambda(V^0)$ to $\lambda(V^1)$, where $\lambda(V)$ denotes the highest exterior power of V. Hence T determines an element σ_T of the line $\lambda(V^0)^* \otimes \lambda(V^1)$, where the asterisk denotes dual vector space. Upon choosing a generator for this line, σ_T can be identified with a function det(T), which is holomorphic in T and is nonzero exactly where the operator T is invertible.

In the infinite-dimensional case of $\bar\partial$-operators the above line is replaced by the line $\mathcal{L}_D = \lambda(\mathrm{Ker} D)^* \otimes \lambda(\mathrm{Coker}\, D)$, which depends on the operator D. The family of $\mathcal{L}_D$ forms a holomorphic line bundle $\mathcal{L}$ over the space $\mathcal{A}$, called the determinant line bundle. The analogue of the assumption that V^0 and V^1 have the same dimension is the condition that the index of the $\bar\partial$-operators be zero, that is, $d = r(g - 1)$. In this case there is a canonical section σ of $\mathcal{L}$ which is holomorphic and such that $\sigma_D \neq 0$ if and only if D is invertible. If we

Institut des Hautes Etudes Scientifiques, 35, route de Chartres, 91440 Bures-sur-Yvette, France. Published in Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 19, No. 1, pp. 37-41, January-March, 1985. Original article submitted June 18, 1984.

construct a trivialization of $\mathcal{L}$ as a holomorphic line bundle, then the canonical section σ can be identified with a holomorphic function det(D) on $\mathcal{A}$, which we call the determinant since it is nonzero exactly where the operator D is invertible.

In order to construct this trivialization, we define a hermitian inner product on $\mathcal{L}$, using the zeta function determinant of the Laplacian D*D. This is essentially the idea of "analytic torsion" (see [1]). The inner product and the holomorphic structure determine a connection on $\mathcal{L}$, whose curvature turns out to be remarkably simple and is described in Sec. 4. By a simple modification of the inner product we obtain a flat connection, and integrating the connection furnishes the desired trivialization of the determinant line bundle.

2. In this section we describe the determinant line bundle in more detail. Let $\mathcal{F}$ be the space of Fredholm operators T from one Hilbert space $\mathcal{H}^0$ to another $\mathcal{H}^1$. As an open subset of the Banach space of bounded operators, $\mathcal{F}$ is a complex Banach manifold. Over $\mathcal{F}$ is a morphic line bundle $\mathcal{L}$ with the fibres $\mathcal{L}_T = \lambda(\mathrm{Ker}\,T)^* \otimes \lambda(\mathrm{Coker}\,T)$ defined as follows.

For any finite-dimensional subspace F of $\mathcal{H}^1$, let U_F be the set of T which are transversal to F in the sense that Im T + F = $\mathcal{H}^1$. For such T one has an exact sequence $0 \to \mathrm{Ker}\,T \to T^{-1}F \to F \to \mathrm{Coker}\,T \to 0$ and an associated canonical isomorphism $\mathcal{L}_T = \lambda(\mathrm{Ker}\,T)^* \otimes \lambda(\mathrm{Coker}\,T) \simeq \lambda(T^{-1}F)^* \otimes \lambda(F)$. The set U_F is open, and the family of subspaces $T^{-1}F$ form a holomorphic vector bundle over U_F, whence the family of lines on the right side of this isomorphism forms a holomorphic line bundle over U_F. The holomorphic structure on $\mathcal{L}$ is determined by requiring the above isomorphism to be an isomorphism of holomorphic line bundles over U_F for any F.

Over the connected component of $\mathcal{F}$ consisting of operators of index zero we define a section σ of $\mathcal{L}$ by setting $\sigma_T = 0$ if T is not invertible, and setting $\sigma_T = 1$ under the canonical isomorphism $\mathcal{L}_T = \mathbf{C}$, when T is invertible. It can be shown that this section is holomorphic [this would not be true if we took $\mathcal{L}$ to have the fibres $\lambda(\mathrm{Ker}\,T) \otimes \lambda(\mathrm{Coker}\,T)^*$].

Given a $\bar{\partial}$-operator on E we associate the induced Fredholm operator from the space of square integrable sections of E to the Sobolev space of (0, 1)-forms with values in E having square integrable first derivatives. This gives a linear map from the affine space $\mathcal{A}$ to a coset of $\mathcal{F}$ modulo compact operators; hence pulling back the above line bundle one obtains a determinant line bundle over the space $\mathcal{A}$, which is holomorphic and has a canonical section when the index is zero.

3. We next define an inner product on the determinant line bundle, supposing we are given an inner product on E and a Riemannian metric on M compatible with its complex structure. The spaces $\Omega^{0,q}(E)$ then have inner products allowing one to associate to a $\bar{\partial}$-operator D its adjoint D* and Laplacian $\Delta = D^*D$. The vector spaces Ker D and Coker D $\simeq$ Ker D* inherit inner products from the ones on $\Omega^{0,q}(E)$.

Let $\zeta(s)$ be the zeta function of the elliptic operator Δ; it is a meromorphic function of s equal for Re(s) > 1 to $\Sigma\lambda^{-s}$, where λ runs over the nonzero eigenvalues of Δ, which is regular at s = 0 and depends smoothly on the operator Δ. The number $\exp(-\zeta'(0))$ has a well-known interpretation as the determinant of Δ acting on the orthogonal complement of Ker D.

We now define an inner product on $\mathcal{L}_D = \lambda(\mathrm{Ker}\,D)^* \otimes \lambda(\mathrm{Ker}\,D^*)$ by taking the inner product induced by the ones on Ker D and Ker D* and multiplying it by the zeta determinant $\exp(-\zeta'(0))$. More precisely, by choosing orthonormal bases for these kernels and taking the Grassman product of the basis of elements, we obtain a nonzero element v in $\mathcal{L}_D$ which is unique up to a scalar of absolute value 1. The inner product is given by setting $\|v\|^2 = \exp(-\zeta'(0))$.

<u>Proposition.</u> The inner products on the family of lines $\mathcal{L}_D$ determine a smooth inner product on the determinant line bundle.

To see this, let $a \geqslant 0$, and let F_a^0 (resp. F_a^1) be the subspace spanned by the eigenvectors of D*D (resp. DD*) having eigenvalues $\leqslant a$. One has a canonical isomorphism $\mathcal{L}_D = \lambda(F_a^0)^* \otimes \lambda(F_a^1)$, and it is easy to see that relative to this isomorphism the inner product on $\mathcal{L}_D$ coincides with the one induced by the inner product on F_a^q multiplied by $\exp(-\zeta'_{>a}(0))$, where $\zeta_{>a}(s) = \Sigma\lambda^{-s}$ with λ running over the eigenvalues of Δ greater than a. The subspace F_a^q and the function $\zeta_{>a}$ depend smoothly on the operator D provided a is not an eigenvalue of Δ. Since a is arbitrary, we see that the inner product is smooth.

In the case where the $\bar{\partial}$-operators have index zero the metric on the determinant line bundle is given by

$$\|\sigma_D\|^2 = \det_\zeta(D^*D),$$

where σ is the canonical section and $\det_\zeta(\Delta)$ is defined to be $\exp(-\zeta'(0))$ when $\operatorname{Ker}\Delta = 0$ and 0 otherwise.

4. A holomorphic line bundle equipped with an inner product has a canonical connection compatible with the two structures, whose curvature is $\bar\partial\partial \log \|s\|^2$, where s is any local holomorphic section. When the underlying manifold is simply connected, the curvature form determines the line bundle and inner product up to isomorphism. We now identify the curvature form of the determinant line bundle.

The inner product on E induces one on $\mathscr{B} = \Omega^{0,1}(\operatorname{End} E)$ as follows. Given B in $\mathscr{B}$, say $B = \alpha(z)d\bar z$ relative to a local orthonormal framing of E, let $B^+ = \alpha(z)^* dz$ in $\Omega^{1,0}(\operatorname{End} E)$. Then $\operatorname{tr}_E(B^+B)$ is a form of type (1, 1) which can be integrated:

$$\| B \|^2 = \int_M \frac{i}{2\pi} \operatorname{tr}_E(B^+B).$$

As the space $\mathscr{A}$ of $\bar\partial$-operators is an affine space relative to $\mathscr{B}$, this inner product determines a Kahler form on $\mathscr{A}$. Specifically, the Kahler form is $\partial\bar\partial q$, where q is the quadratic function $q(D) = \|D - D_0\|^2$, and D_0 is a basepoint in $\mathscr{A}$. The Kahler form is independent of the basepoint.

<u>THEOREM 1.</u> The curvature of the determinant line bundle is equal to the Kahler form on $\mathscr{A}$.

If we multiply the inner product on $\mathscr{L}$ by the function e^q, then the connection corresponding to the new inner product is flat according to this theorem. Hence we obtain a trivialization of $\mathscr{L}$ by taking an everywhere flat section (which exists as $\mathscr{A}$ is contractible). In the case where the $\bar\partial$-operators have index zero the image of the canonical section σ under this trivialization is a holomorphic function on $\mathscr{A}$, and we have the following.

<u>COROLLARY.</u> Given a basepoint D_0, there exists a holomorphic function $\det(D; D_0)$ on $\mathscr{A}$, which is unique up to a scalar of absolute value one, such that

$$\det_\zeta(D^*D) = e^{-\|D-D_0\|^2} |\det(D; D_0)|^2.$$

Because of the dependence on the basepoint, the determinant $\det(D; D_0)$ is not invariant under gauge transformations. The case of line bundles over an elliptic curve shows that it is not possible to produce determinants which are both holomorphic and gauge-invariant.

The following sections describe the proof of the theorem.

5. Given a $\bar\partial$-operator D, we construct a parametrix $G_0(z, z')$ for it in the following way. Let ∇ be the unique connection on E compatible with the inner product and the operator D. Let $F(z, z'): E_{z'} \to E_z$ be the parallel transport with respect to ∇ along the geodesic from z' to z. Let $r^2(z, z')$ be the distance squared between z' and z. Put $G_0(z', z) = (1/2\pi i) \times [dz' \partial_{z'} \log r^2(z, z')] F(z', z)$. This is well defined in a neighborhood of the diagonal in $M \times M$.

Suppose now that D is invertible and let $G(z, z') = \langle z | D^{-1} | z' \rangle$ denote the Schwartz kernel of the operator D^{-1}. We define the finite part of G along the diagonal to be the element J of $\Omega^{1,0}(\operatorname{End} E)$ given by

$$J(z') = \lim_{z \to z'} (G(z, z') - G_0(z, z')).$$

In terms of a local orthonormal framing for E we have the following local formulas:

$$ds^2 = \rho(z) |dz|^2;$$
$$D = d\bar z(\partial_{\bar z} + \alpha);$$
$$\nabla = dz(\partial_z - \alpha^*) + d\bar z(\partial_{\bar z} + \alpha);$$
$$F(z, z') = 1 + (z - z')\alpha^*(z') - (\bar z - \bar z')\alpha(z') + \ldots;$$
$$G(z, z') = \frac{i}{2\pi} \frac{dz'}{z - z'} \{1 + (z - z')\beta(z') - (\bar z - \bar z')\beta(z') + \ldots\}.$$

Here $\beta(z)$ is a smooth matrix function determined globally by the operator D, which depends holomorphically on D. One calculates

$$J = \frac{i}{2\pi} dz\left(\beta - \alpha^* - \frac{1}{2}\partial_z \log \rho\right). \qquad (*)$$

THEOREM 2. One has

$$\lim_{t\to 0} \langle z \mid e^{-t\Delta}G \mid z\rangle = J(z)$$

uniformly in z, and consequently for any B in $\mathscr{B}$

$$\lim_{t\to 0} \mathrm{Tr}\,(e^{-t\Delta}D^{-1}B) = \int_M \mathrm{tr}\,(JB).$$

This follows from the continuity of $G - G_0$ along the diagonal and the formula

$$\lim_{t\to 0} \langle z \mid e^{-t\Delta}\, G_0 \mid z\rangle = 0,$$

which is derived by calculating the asymptotic expansion of the heat kernel.

6. We now describe the proof of Theorem 1. By the device of adding to E a vector bundle of the opposite index, we may assume that the index is zero. It suffices to check that the curvature and Kahler forms coincide over a one-parameter family $D = D_w$ of invertible $\bar\partial$-operators depending holomorphically on the complex variable w. The curvature form is then $\bar\partial\partial \log\|\sigma\|^2 = dw\, d\bar w\, \partial^2_{\bar w w}\zeta'(0)$, where we recall $\zeta(s) = \mathrm{Tr}(\Delta^{-s})$, $\Delta = D^*D$. One has

$$-\partial_w\zeta(s) = s\,\mathrm{Tr}\,(\Delta^{-s-1}\partial_w\Delta) = s\,\mathrm{Tr}\,(\Delta^{-s}D^{-1}\partial_w D) = \frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\,(e^{-t\Delta}D^{-1}\partial_w D)\,t^{s-1}\,dt = s\left\{\int \mathrm{tr}\,(J\partial_w D) + O(s)\right\} \text{as } s\to 0,$$

where the last step uses Theorem 2. It follows that $\partial_w\zeta(0) = 0$ and that $-\partial_w\zeta'(0) = \int \mathrm{tr}\,(J\partial_w D)$.

In the formula (*) for J only the α^* term is not holomorphic with respect to w. Thus

$$\partial_{\bar w}J = -\frac{i}{2\pi}\,dz\,\partial_{\bar w}\alpha^* = -\frac{i}{2\pi}(\partial_w D)^+,$$
$$\partial^2_{\bar w w}\zeta'(0) = \int \frac{i}{2\pi}\,\mathrm{tr}\,(\partial_w D)^+\,\partial_w D,$$

which proves the theorem.

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STRUCTURE OF THE SPECTRUM OF THE SCHRÖDINGER OPERATOR WITH ALMOST-PERIODIC POTENTIAL IN THE VICINITY OF ITS LEFT EDGE

Ya. G. Sinai

UDC 517.43+517.9

1. Statement of Problem and Formulation of the Results

Paper [1] dealt with the structure of the spectrum of the one-dimensional Schrödinger operator with an almost-periodic potential for large values of E; it was shown that the spectrum contains a Cantor set such that the measure of its complement decreases as E grows. Close results were obtained in [2]. Subsequently, the results of [1, 2] were improved in [3], and then were extended to the difference Schrödinger operator in [4]. The present paper deals with the structure of the spectrum of the difference Schrödinger operator in the vicinity of its left edge. Judging by the methods it uses, it is a direct continuation of [1]. Since many arguments and estimates actually repeat those of [1], we omit them here.

First, let us describe the operators of interest to us. We confine ourselves to the main example and then indicate its straightforward generalizations. The example we have in mind

L. D. Landau Theoretical Physics Institute, Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 19, No. 1, pp. 42-48, January-March, 1985. Original article submitted February 7, 1984.

 0016-2663/85/1901-0034$09.50