ASYMPTOTIC OF THE LARGEST AND THE TYPICAL DIMENSIONS OF IRREDUCIBLE REPRESENTATIONS OF A SYMMETRIC GROUP

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In this article we give order-sharp two-sided bounds of the largest and the typical dimensions of irreducible representations of the symmetric group $\mathfrak{S}_N$ for $N \to \infty$. Both problems are solved simultaneously and are connected with the earlier-proved theorem about the limit form of a typical Young diagram. Some applications of the obtained results are indicated.

This article is devoted to the memory of Vladimir Abramovich Rokhlin.

1. FORMULATION OF THE RESULTS

Let $\mathfrak{S}_N$ be the symmetric group of degree N and $\widehat{\mathfrak{S}}_N$ be the set of equivalence classes of its complex irreducible representations. If $\Lambda \in \widehat{\mathfrak{S}}_N$, then we denote the dimension of the representation Λ by $\dim \Lambda$. In this article we prove the following theorem.

THEOREM 1. There exist positive constants c_0 and c_1 such that

$$e^{-\frac{c_1}{2}\sqrt{N}}\sqrt{N!} \leqslant \max_{\Lambda \in \widehat{\mathfrak{S}}_N} \dim \Lambda \leqslant e^{-\frac{c_0}{2}\sqrt{N}}\sqrt{N!}$$

for all N = 1, 2,....

The problem on the computation of the largest dimension was posed long back (see [1]). Let us recall that by the Burnside formula we have

$$\sum_{\Lambda \in \widehat{\mathfrak{S}}_N} \dim^2 \Lambda = N!,$$

Therefore, $\dim \Lambda < \sqrt{N!}$ for all $\Lambda \in \widehat{\mathfrak{S}}_N$, and the natural normalization of the dimension is $\dim \Lambda/\sqrt{N!}$. The hypothesis about the existence of the large representations whose dimensions satisfy the inequality $\dim \Lambda/\sqrt{N!} \geqslant 1/N$ has been advanced in [2]. This hypothesis has been justified by numerical data, obtained (see [3]) for $N \leqslant 75$ (!). Later on it was elucidated that it is corroborated for N = 81 (see [2]). However, one could think that $\max \dim \Lambda/\sqrt{N!} \geqslant P(N)^{-1}$, where P is a polynomial. It follows from Theorem 1 that this is not so, and the ratio $\max_{\Lambda} \dim \Lambda/\sqrt{N!}$ decreases with the rate $\exp(-c\sqrt{N})$, i.e., quicker than any polynomial.

The problem about the largest dimension turns out to be closely connected with another problem – that about the typical dimension. For $\Lambda \in \widehat{\mathfrak{S}}_N$, let us set $\mu_N(\Lambda) = \dim^2 \Lambda/N!$. It follows from the Burnside formula that μ_N is a probability measure on $\widehat{\mathfrak{S}}_N$. It should be called the Plancherel measure (see [4]). Let us observe that the Plancherel measure is naturally selected; $\mu_N(\Lambda)$ is the relative dimension of the isotypic component of the representation $\Lambda \in \widehat{\mathfrak{S}}_N$ in the regular representation of the group $\mathfrak{S}_N$, since the multiplicity of a representation is equal to its dimension [the subspace of all the vectors in $\mathbf{C}(\mathfrak{S}_N)$ that are transformed with respect to a given representation is called an isotypic component]. Namely, we should study the statistics and the asymptotic of the characters with respect to this measure. It turns out that the asymptotic of the typical (with respect to the Plancherel measure) dimension coincides, in order, with the asymptotic of the largest dimension.

THEOREM 2. There exist positive constants c_0' and c_1' such that

$$\lim_{N\to\infty} \mu_N\left\{\Lambda:\ c_0' < -\frac{2}{\sqrt{N}} \ln \frac{\dim \Lambda}{\sqrt{N!}} < c_1'\right\} = 1.$$

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In other words,

$$\sum_{\Lambda\in\mathcal{F}_N}\frac{\dim^2\Lambda}{N!}=1-o(1),$$

where

$$\mathcal{F}_N=\{\Lambda\in\hat{\mathfrak{S}}_N:\ \sqrt{N!}\,e^{-\frac{c_1'}{2}\sqrt{N}}<\dim\Lambda<\sqrt{N!}\,e^{-\frac{c_0'}{2}\sqrt{N}}\}.$$

The interaction between Theorems 1 and 2 is noteworthy. A lower bound for the typical, and, by the same token, for the largest, dimension has been obtained in [4] (see also [6]). Let us observe that a somewhat rougher lower bound of the largest dimension is given in McKay's article [3] (see Sec. 3). The limit form Ω of the typical Young diagram* is also found in [4, 6] (see the theorem of Sec. 3). It was natural to suppose that the diagram of representations of the largest dimension also converge in appropriate scale to the same curve. Here we show that this is actually so. Moreover, we obtain an upper bound of the largest and, consequently, of the typical dimension. It is unexpected that the logarithmic orders of the largest and the typical dimensions are the same (the difference is perhaps only in the constant), but, namely, this is the situation and both problems are connected.

The constants in Theorem 2 do not pretend exactness. We prove only that we can take $c_0' = c_0 = 0.2313$ and $c_1' = c_1 = 2.5651$. However, Theorem 2 most readily admits the following strengthening: There exists a constant c (the entropy of the Plancherel measure) such that for each $\varepsilon > 0$

$$\lim_{N\to\infty}\mu_N\left\{\Lambda:\ \left|\frac{2}{\sqrt{N}}\ln\frac{\dim\Lambda}{\sqrt{N!}}-c\right|<\varepsilon\right\}=1$$

and even that

$$\lim_{N\to\infty}\frac{2}{\sqrt{N}}\ln\frac{\dim\Lambda}{\sqrt{N!}}=c$$

for almost all infinite Young tableaux, i.e., increasing sequences $\{\Lambda_N\}_1^\infty$, in the sense of the Plancherel measure on tableaux (see [4] for its definition). These statements would imply the asymptotic equidistribution property of the Plancherel measure and would serve as the analogue of the Shannon—Macmillan—Breiman theorem, and the constant c is the "specific entropy of representation." We are concerned with the not entirely ordinary Shannon theorem: A Markov chain with quickly growing number of states is considered and it is required to prove the asymptotic equidistribution property of the states at the moment N for $N \to \infty$. It follows from numerical experiments, carried out in colloboration with Gribov (see [5]) that this is probable and that $c > 1.8$.

As far as the constants c_0 and c_1 in Theorem 1 are concerned, their refinement requires new methods. It is even possible (this can be explained by some numerical data) that the limit

$$\lim_{N\to\infty}\frac{2}{\sqrt{N}}\ln\max_{\Lambda}\frac{\dim\Lambda}{\sqrt{N!}}$$

does not exist, and the partial limits depend on the arithmetic of the sequence N. Nevertheless, it is more probable that the arithmetical properties of N manifest themselves in the subsequent terms of the asymptotic. The properties of the diagram of the largest dimension and the character of its convergence to Ω are also not known. It would be interesting to elucidate whether this diagram is the best approximation to Ω in some natural metric. The deviations of the typical diagram from Ω must be subject, in the underlying scale, to a limit theorem. This conjecture has also been put forward in [6].

A theorem about the limit form has been proved by the authors in [4] and, independently, by Logan and Shepp in [6]. In the present article, we give its detailed proof for completeness of treatment. In main features, it coincides with the preliminary one (and with the proof in [6]) and follows the following plan: transformation of the formula for the dimension, solution of a variational problem, and proof of the uniqueness of its solution. To

*In [4] different terminology has been used: The Young diagrams are called the Young tableaux and vice versa. The terminology, used in the later works, including the present article, can be assumed to become standard; it coincides with the classical one (see [13]).

this end, the norm $\|f\|^2 = -\iint \ln 2|s-t|\cdot f'(s)\cdot f'(t)\,ds\,dt$ has been used for the hook integral in [4]; it is written in the form $\|f\|^2 = \int |\omega|\cdot|\hat{f}(\omega)|^2\,d\omega$ in [6].

These arguments enable us to investigate the asymptotic of the form and the dimension, typical with respect to the Plancherel measure of the diagram, but do not give information about the largest dimension. The following step is essentially new: With the help of a known lemma (see, e.g., [10]), this norm can be rewritten in the form

$$\|f\|^2 = \frac{1}{2}\iint\left(\frac{f(s)-f(t)}{s-t}\right)^2 ds\,dt,$$

namely, it gives a basis for obtaining the upper bound in Theorem 1. Appearance of the Sobolev norm and the Hilbert integral in this combinatorial problem seems astonishing.

As one of the applications of the theorem on the limit form, a proof of the Ulm hypothesis was announced in [4]. It requires new arguments in comparison with Theorem 3; namely, an additional upper bound of the length of the first row of a random Young diagram. We give here a detailed proof. There are other applications of the obtained results. A. B. Gribov has found the limit form of the typical Young tableau. S. V. Kerov and G. Gitel'son have shown that the problem about the typical symmetries of tensors, i.e., about the typical diagrams in the sense of representations of the group U(N), has the same solution as in [4]. Other classical series most readily lead to the same limit form of the diagram. Since $\dim\Lambda = \chi_\Lambda(e)$, where χ_Λ is the character of the representation Λ, Theorem 2 gives the asymptotic of the values of the typical character at the identity. The problem about the asymptotic of the values of $\chi_\Lambda(\sigma)$ for $\sigma \neq e$ for the typical, with respect to the Plancherel measure, representations of Λ has been investigated by S. V. Kerov.

The order of treatment in our article is as follows. The whole of Sec. 2 is devoted to the study of the hook formula for dimension. In part 1 an integral representation of this formula in a continuous limit is obtained. The change of variable from part 2 turns the hook integral into a quadratic form. In part 3 the curve Ω, realizing the minimum of this form, is described. In part 4, it is established that the hook integral coincides with the square of the Sobolev norm $W_2^{1/2}$ of deviation from Ω. In Sec. 3 we give proofs of Theorems 1 and 2, and also state and prove the main theorem of [4] (Theorem 3). In this section, we prove the Ulm hypothesis about the length of a monotonic subsequence of a random sequence and establish the necessary (for this) refinements of Theorem 3. Here we use the RSK (Robinson–Shannon–Knuth) combinatorial algorithm and its properties (Corollaries 1 and 2). In Sec. 4, some unsolved problems are discussed and the statphysical formulation of the problem about the limit form of a diagram is given. In particular, the limit form for a uniform distribution, found by A. Vershik and M. Salai, is described.

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2. THE HOOK INTEGRAL

1. Hook Formula in Continuous Limit. The Hook Integral

It is well known that the classes of irreducible representations of the symmetric group $\mathfrak{S}_N$ are parametrized by Young diagrams [13]. A Young diagram with N cells can be defined in various ways: As a finite N-element order ideal of the lattice $\mathbf{Z}_+ \oplus \mathbf{Z}_+$, as a partition of the natural number set N, etc. Here it will be convenient for us to represent a Young diagram with length of sides $\lambda_1, \lambda_2, \ldots, \lambda_m$ as a subset of the plane $\mathbf{R}^2$:

$$\Lambda = \bigcup_{k=1}^{m}\{(x, y) \in \mathbf{R}^2 : k-1 \leqslant x \leqslant k,\ 0 \leqslant y \leqslant \lambda_k\}.$$

Let $\mathcal{Y}_N\ (\simeq \hat{\mathfrak{S}}_N)$ denote the set of diagrams with N cells and let $\dim\Lambda$ be the dimension of the representation, corresponding to the diagram $\Lambda \in \mathcal{Y}_N$. The remarkable Frame–Robinson–Thrall formula [8, 13] (the hook formula) expresses $\dim\Lambda$ in terms of the data of the diagram $\Lambda \in \mathcal{Y}_N$:

$$\dim\Lambda = \frac{N!}{\Pi h_{ij}}. \tag{1}$$

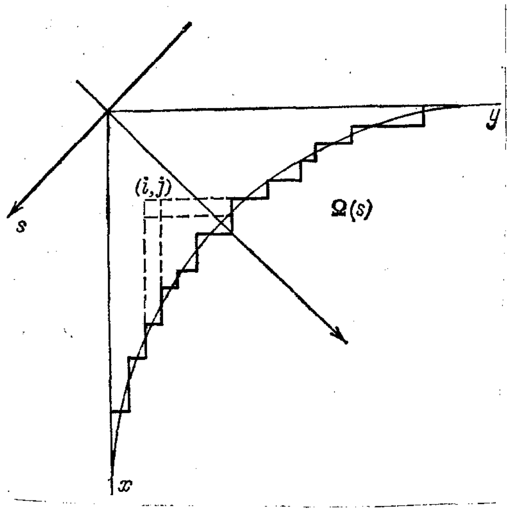

Fig. 1

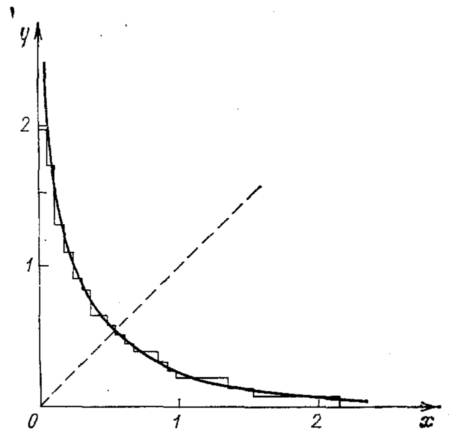

Fig. 2

Here the product is taken over all the cells (i, j) of the diagram Λ and $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$ (λ'_j is the length of the j-th column) is the number of cells of the hook with the vertex (i, j), see Fig. 1. This formula is equivalent to the Frobenius formula, but is much more convenient on account of its symmetry and multiplicativity. From it, we get the following formula for the Plancherel measure:

$$\mu_N(\Lambda) = \frac{\dim^2 \Lambda}{N!} = \frac{N!}{\Pi h_{ij}^2}. \tag{2}$$

Taking the logarithm and dividing by $\sqrt{N}$, with the help of the Stirling formula we get

$$-\frac{\ln \mu_N(\Lambda)}{\sqrt{N}} = J_\Lambda \sqrt{N} - \varepsilon_N, \tag{3}$$

where $J_\Lambda = 1 + \frac{2}{N}\sum_{i,j} \ln \frac{h_{ij}}{\sqrt{N}}$, and

$$\varepsilon_N = \frac{1}{\sqrt{N}}\left(\ln \sqrt{2\pi N} + \frac{1}{12N} + \ldots\right) = o\left(\frac{\ln N}{\sqrt{N}}\right);$$

ε_N depends only on N.

Let $y = F(x)$ be a finite bounded nonincreasing function, defined on $[0, \infty)$. Let us set $F^{-1}(y) = \inf\{x: F(x) \leqslant y\}$ and $h_F(x, y) = F(x) + F^{-1}(y) - x - y$ [the hook of the point (x, y)] and let $s_F = \{(x, y): 0 \leqslant y < F(x),\ 0 \leqslant x < \infty\}$ be a subgraph of F.

We call the double integral

$$\theta_F = 1 + 2\iint_{s_F} \ln h_F(x, y)\, dx\, dy$$

the *hook integral* of the function F.

For a Young diagram Λ with N cells, let $\tilde{\Lambda}$ denote the subset of the plane $\mathbf{R}^2$ obtained by contracting Λ by $\sqrt{N}$ times. We call $\tilde{\Lambda}$ the normalized diagram; its area is equal to one. Let $F = F_\Lambda$ be a function such that $S_F = \tilde{\Lambda}$. It is clear that J_Λ is the integral sum for the hook integral θ_F. Below we prove that if $F_{\Lambda_N} \to F$ for a sequence of diagrams $\Lambda_1, \Lambda_2, \ldots$, then

$$-\lim_{N\to\infty} \frac{1}{N} \ln \mu_N(\Lambda_N) = \theta_F. \tag{4}$$

Example. Let Λ_N be a triangular diagram and $\tilde{\Lambda}_N$ converge to a subgraph of the function $F(x) = \sqrt{2} - x$, $0 \leqslant x \leqslant \sqrt{2}$. Then $\dim \Lambda_N = \sqrt{N!}\exp\left(-N\frac{2\sqrt{2}}{3} + o(N)\right)$ and $\mu_N(\Lambda_N)$ decreases exponentially.

In the same manner, we can compute the asymptotic of the dimension by Eq. (4) also for other curves, except one: In Sec. 3 we show that there exists only one curve Ω [to which the typical as well as the largest (with respect to dimension) normalized diagrams converge], for which the right-hand side of (4) vanishes: $\theta_\Omega = 0$. It turns out that we must pass to another normalization to study the asymptotic of the dimension in this case: $-\ln \mu_N(\Lambda)/\sqrt{N}$, and $\mu_N(\Lambda_N)$ here decreases subexponentially.

In order to estimate the deviation of the integral sum J_Λ from the hook integral, let us consider the quantity $\tilde{\theta}(\Lambda) = (J_\Lambda - \theta_\Lambda)\sqrt{N}$, for which

$$-\frac{\ln \mu_N(\Lambda)}{\sqrt{N}} = \theta_\Lambda \sqrt{N} + \tilde{\theta}(\Lambda) - \varepsilon_N. \tag{5}$$

LEMMA 1.

$$\tilde{\theta}(\Lambda) = \frac{1}{\sqrt{N}} \sum_{i,j} c(h_{ij}),$$

where $c(x) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+1)x^{2k}}$. In particular, $\tilde{\theta}(\Lambda) > 0$ for all $\Lambda \in \mathcal{Y}_N$.

Proof. Let $\square_{ij} \subset \tilde{\Lambda}$ be the image of the cell (i, j) under the contraction of the diagram Λ. If $(x, y) \in \square_{ij}$, then

$$h(x, y) = \frac{h_{ij}}{\sqrt{N}} - (x - x_i) - (y - y_j).$$

Therefore, $2 \iint_{\square_{ij}} \ln h(x, y)\, dx\, dy = \frac{2}{N} \ln \frac{h_{ij}}{\sqrt{N}} - c(h_{ij}) \frac{1}{N}$ and $J_\Lambda - \theta_\Lambda = \frac{1}{N} \sum_{i,j} c(h_{ij})$.

We can show that for the diagram of the largest dimension and for typical diagrams the value of $\tilde{\theta}(\Lambda)$ is bounded: $\tilde{\theta}(\Lambda) < 0.6762$. Thus, the difference

$$-\frac{\ln \mu_N(\Lambda)}{\sqrt{N}} - \theta_\Lambda \sqrt{N}$$

is uniformly bounded with respect to N.

2. Reduction of the Hook Integral to Quadratic Form

We introduce the coordinates $X = (x - y)/2$ and $Y = (x + y)/2$ (see Fig. 1). In these coordinates, the boundary of the normalized (contracted) diagram is the graph of a function, which we denote by L_Λ. This function is piecewise linear and continuous. Moreover,

1) $L'_\Lambda(X) = \pm 1$;

2) $L_\Lambda(X) \geqslant |X|$ and $L_\Lambda(X) = |X|$ for sufficiently large $|X|$. In addition, $\tilde{\Lambda} = \{(X, Y): |X| \leqslant Y \leqslant L(X)\}$.

For each piecewise smooth function $Y = L(X)$ we set

$$\theta(L) = 1 + 2 \iint_{t<s} [\ln 2(s - t)](1 - L'(s))(1 + L'(t))\, ds\, dt. \tag{6}$$

LEMMA 2. $\theta_\Lambda = \theta(L_\Lambda)$ for all $\Lambda \in \mathcal{Y}_N$.

Proof. Setting $x = L_\Lambda(t) - t$ and $y = L_\Lambda(s) + s$, we can easily verify that $h(x, y) = 2(s - t)$ and the Jacobian of the substitution gives the desired integral. Let us observe that the Jacobian is nonzero only in a finite interval $[a, b]$ by virtue of the property 2) of L_Λ.

3. Critical Point of the Hook Integral and Its First Variation at This Point

The following function plays a fundamental role in the study of the asymptotic of Young diagrams:

$$\Omega(X) = \begin{cases} \frac{2}{\pi}(X \arcsin X + \sqrt{1 - X^2}) & \text{for } |X| \leqslant 1 \\ |X| & \text{for } |X| \geqslant 1 \end{cases} \tag{7}$$

Let $f(X) = L(X) - \Omega(X)$. If the property 2) is fulfilled, then f is a finite function.

LEMMA 3. The hook integral can be expressed in the form

$$\theta(L) = -\iint \ln 2|s-t| f'(s) f'(t)\, ds\, dt + 4 \int_{|s|>1} f(s) \operatorname{arch}|s|\, ds. \tag{8}$$

COROLLARY. $\theta(\Omega) = 0$.

Proof of the Lemma. We fix numbers a and b such that f(s) = 0 outside the interval (a, b). For brevity, we set

$$\Phi_0(x) = -\ln 2|x|, \quad \Phi_n(x) = \int_0^x \Phi_{n-1}(y)\, dy$$

for n = 1, 2,... and $H(x) = x \operatorname{arch}|x| \mp \sqrt{x^2-1}$ for $\pm x \geqslant 1$ and H(x) = 0 for $|x| \leqslant 1$. Direct computation with the use of the integrals

$$\frac{1}{\pi}\int_{-1}^{1} \frac{dt}{(s-t)\sqrt{1-t^2}} = \begin{cases} \pm\dfrac{1}{\sqrt{s^2-1}} & \text{for } \pm s > 1 \\ 0 & \text{for } |s| \leqslant 1 \end{cases}$$

and $-\frac{1}{\pi}\int_{-1}^{1} \frac{\ln|t|\, dt}{\sqrt{1-t^2}} = \ln 2$ (Dwight, 863.41) shows that

$$I(s) = \int_a^b \Phi_0(s-t)\, \Omega'(t)\, dt = \Phi_1(a-s) + \Phi_1(b-s) - 2H(s).$$

On the other hand, the double integral on the right-hand side of (8) can be rewritten in the form

$$\int_a^b I(s)\, \Omega'(s)\, ds - 2\int_a^b I(s)\, L'(s)\, ds + \int_a^b\int_a^b \Phi_0(s-t)\, L'(s)\, L'(t)\, ds\, dt,$$

and the hook integral can be rewritten as

$$\theta(L) = 1 - \int_a^b\int_a^b \Phi_0(s-t)\, ds\, dt - 2\int_a^b \Phi_1(a-t)\, L'(t)\, dt - 2\int_a^b \Phi_1(b-t)\, L'(t)\, dt + \int_a^b\int_a^b \Phi_0(s-t)\, L'(s)\, L'(t)\, ds\, dt.$$

One more computation, using the integral $\int_{-1}^{1} \Phi_2(s)\, \Omega''(s)\, ds = 1/2$ (see Dwight, 863.41, 42), gives the identity

$$\int_a^b I(s)\, \Omega'(s)\, ds = 1 - 2\Phi_2(b-a) - 4\int_a^b H(s)\, \Omega'(s)\, ds.$$

Since $\int_a^b\int_a^b \Phi_1(s-t)\, ds\, dt = 2\Phi_2(b-a)$, the proof of the lemma is complete.

4. Hook Integral and the Sobolev Norm

We show that the quadratic part of the hook integral coincides with the Sobolev norm

$$\|f\|_\theta^2 = \iint \left(\frac{f(s)-f(t)}{s-t}\right)^2 ds\, dt \tag{9}$$

in the space of piecewise-smooth functions.

LEMMA 4.

$$\theta(\Omega + f) = \frac{1}{2}\|f\|_\theta^2 + 4\int_{|s|>1} f(s) \operatorname{arch}|s|\, ds.$$

Proof. Let

$$(Gf)(s) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)\, dt}{s-t} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \ln|s-t|\, f'(t)\, dt$$

be the Hilbert transform for f. The Fourier transform for (Gf) (s) is equal to $(\widehat{Gf})(\omega) = i \operatorname{sign} \omega \cdot \hat{f}(\omega)$.

By the Plancherel formula, the double integral on the right-hand side of (8) can be rewritten in the form

$$\pi \int_{-\infty}^{\infty} (Gf)(s)\overline{f'(s)}\,ds = \frac{\pi}{2\pi} \int_{-\infty}^{\infty} i \operatorname{sign}\omega \cdot \hat{f}(\omega)\,\overline{i\omega\hat{f}(\omega)}\,d\omega = \frac{1}{2} \int_{-\infty}^{\infty} |\omega|\,|\hat{f}(\omega)|^2\,d\omega,$$

and the last integral coincides with $\|f\|_\theta^2/2$ (see [10, p. 83]).

COROLLARY. If $L(s) \geqslant s$ and $L \neq \Omega$, then $\theta(L) > 0$. By the same token, Ω is the unique strict minimum for $\theta(\cdot)$ on the set of admissible functions.

3. PROOF OF THEOREMS

1. Lower Bound of the Hook Integral

LEMMA 5. If $\Lambda \in \mathcal{Y}_N$ and $f_\Lambda(s) = L_\Lambda(s) - \Omega(s)$, then for each $\varepsilon > 0$

$$\sqrt{N}\cdot\theta(L_\Lambda) \geqslant \frac{1}{4} \int_{-1}^{1} |f'_\Lambda(s)|^2\,ds - \varepsilon$$

for sufficiently large N.

Proof. The points $s_i = i/2\sqrt{N}$, $i \in \mathbf{Z}$, divide the line into intervals of length $\Delta s = 1/2\sqrt{N}$, on which the function L_Λ is linear. Let s_i^* realize the minimum value of $[f'_\Lambda(s)]^2$ for $s_i \leqslant s \leqslant s_{i+1}$. Then

$$\left(\frac{f_\Lambda(s) - f_\Lambda(t)}{s-t}\right)^2 \geqslant [f'_\Lambda(s_i^*)]^2$$

for all $s, t \in [s_i, s_{i+1}]$ and, integrating in (9) only over the union of the squares $s, t \in [s_i, s_{i+1}]$, $i \in \mathbf{Z}$, we get

$$\|f_\Lambda\|^2 \geqslant \sum_i [f'_\Lambda(s_i^*)]^2 (\Delta s)^2 = \frac{1}{2\sqrt{N}} \sum_i [f'_\Lambda(s_i^*)]^2 \Delta s.$$

Replacing the integral sum on the right-hand side by the corresponding integral and using Lemma 4, we get

$$\theta(\Lambda)\sqrt{N} \geqslant \frac{1}{2}\|f_\Lambda\|^2\sqrt{N} \geqslant \frac{1}{4}\int_{-\infty}^{\infty} [f'_\Lambda(s)]^2\,ds - \varepsilon$$

for arbitrary $\varepsilon > 0$.

COROLLARY. (Lower Bound in Theorem 1) — $\frac{\ln \mu(\Lambda)}{\sqrt{N}} \geqslant c_0$ or $\max \dim \Lambda \leqslant \sqrt{N!}\, e^{-c_0\sqrt{N}}$.

For c_0 we can give the estimate

$$c_0 = {}^1/_4 \int_{-1}^{1} |\operatorname{sign} s - \Omega'(s)|^2\,ds = \frac{2}{\pi^2}(\pi - 2) \approx 0.2313.$$

2. Lower Bound for the Dimensions of Typical Diagrams

Let us set $M_N = \left\{\Lambda \in \mathcal{Y}_N\colon \theta(L_\Lambda) < \frac{2\pi}{\sqrt{6N}}\right\}$.

LEMMA 6. $\lim\limits_{N\to\infty} \mu_N(M_N) = 1$.

Proof. The total number of diagrams with N cells is given by the function p(N) and by the Euler–Hardy–Ramanujan formula

$$p(N) \sim \frac{1}{4N\sqrt{3}}\, e^{\frac{2\pi}{\sqrt{6}}\sqrt{N}}.$$

Therefore, if $\Lambda \in \mathcal{Y}_N \setminus M_N$, then $\mu_N(\Lambda) \leqslant e^{-\frac{2\pi}{\sqrt{6}}\sqrt{N}}$ by (5), and $\mu_N(\mathcal{Y}_N \setminus M_N) \leqslant p(N)\cdot e^{-\frac{2\pi}{\sqrt{6}}\sqrt{N}} \to 0.$

COROLLARY.

$$\lim_{N\to\infty} \mu_N\{\Lambda\colon \dim\Lambda > \sqrt{N!}\, e^{-\frac{\pi}{\sqrt{6}}\sqrt{N}}\} = 1.$$

Thus, both bounds and, with them, Theorems 1 and 2 are proved.

Let us observe that McKay [3] has used analogous arguments to obtain a lower bound for the largest dimension. In place of the Burnside formula, he has used the identity $\sum_{\Lambda}\dim\Lambda = t_N$, where $t_N \sim \text{const}\cdot\left(\frac{N}{e}\right)^{N/2}\cdot e^{\sqrt{N}}$ is the number of the involutions in $\mathfrak{S}_N$ (see [8]). He gets $\frac{1}{\sqrt{N!}}\max\dim\Lambda \geqslant \frac{t_N}{p_N\sqrt{N}} \sim \text{const}\cdot e^{-\left(\frac{2\pi}{\sqrt{6}}-1\right)\sqrt{N}}$. Our bound is somewhat more precise: $\pi/\sqrt{6} = 1.2825 < 1.5651 = (2\pi/\sqrt{6}) - 1$ (see part 4 of Sec. 3).

We call the diagrams from M_N essential.

Now, we prove a theorem on the limit form of essential diagrams.

THEOREM 3 (cf. [4, 6]). If $\Lambda \in M_N$, then

$$\sup_s |L_\Lambda(s) - \Omega(s)| < C\cdot N^{-1/6}.$$

Proof. First of all, let us find a bound for the L_2-norm of the difference $f_\Lambda(s) = L_\Lambda(s) - \Omega(s)$. Selecting, as in part 2.3, an interval $[-a, a]$, containing the support of f_Λ, we can divide the integral (9) into two parts:

$$\| f_\Lambda \|_\theta^2 = \int_{-a}^{a}\int_{-a}^{a}\left(\frac{f(s)-f(t)}{s-t}\right)^2 ds\,dt + 8\int_{-a}^{a}\frac{f(s)\,ds}{a^2-s^2}. \tag{10}$$

Consequently, for an essential diagram $\Lambda \in M_N$

$$\left\| \frac{f_\Lambda(s)}{\sqrt{a^2-s^2}} \right\|_{L_2}^2 \leqslant \frac{1}{4}\cdot\frac{2\pi}{\sqrt{6N}} \text{ and } \| f_\Lambda(s) \|_{L_2}^2 \leqslant \frac{\pi a^2}{2\sqrt{6N}}.$$

Using the fact that $|f'_\Lambda(s)| \leqslant 2$ for each Young diagram Λ, we get a bound for the uniform norm:

$$\| f_\Lambda \|_{L_\infty}^3 \leqslant 6\cdot\| f_\Lambda \|_{L_2}^2 \leqslant \frac{3\pi a^2}{\sqrt{6N}},$$

and the theorem is proved: $\|f_\Lambda\|_{L_\infty} \leqslant C\cdot N^{-1/6}$.

3. A Bound for the Length of the First Row

We show that the length of the first row $r_1(\Lambda)$ of an essential diagram $\Lambda \in \mathcal{Y}_N$ has the asymptotic $r_1(\Lambda) \sim 2\sqrt{N}$. It follows from Theorem 3 that $r_1(\Lambda)/2\sqrt{N} \geqslant 1 - C\cdot N^{-1/6}$, i.e., only a lower bound is obtained. In order to obtain an upper bound for $r_1(\Lambda)$, we find the asymptotic of the mean value of $r_1(\Lambda)$ with respect to the Plancherel measure.

LEMMA 6.

$$\sum_{\Lambda\in\mathcal{Y}_N} r_1(\Lambda)\mu_N(\Lambda) < 2\sqrt{N}.$$

Proof. Let T be the space of (infinite) Young tableaux and μ be the Plancherel measure on T (see [11]). The measure μ is a Markov measure; if the Young diagram $\Lambda \in \mathcal{Y}_k$ is obtained from a diagram $\lambda \in \mathcal{Y}_{k-1}$ by the addition of one cell, then the transition probability $p_{\lambda,\Lambda} = \dim\Lambda/k\cdot\dim\lambda$. We use the following identity, characteristic for the Plancherel measure:

$$k\cdot p_{\lambda,\Lambda}^2 = \frac{\mu_k(\Lambda)}{\mu_{k-1}(\lambda)}. \tag{11}$$

Let $\lambda' \in \mathcal{Y}_k$ denote the Young diagram obtained from $\lambda \in \mathcal{Y}_{k-1}$ by the addition of a cell in the first row and let ψ_k denote the indicator function of the set of the tableaux $t = (\lambda_1, \lambda_2, \ldots) \in T$, for which $\lambda_k = \lambda'_{k-1}$. The mean-value of the function ψ_k with respect to the Plancherel measure, denoted by $\langle\psi_k\rangle$, is given by $\langle\psi_k\rangle = \sum_{\lambda\in\mathcal{Y}_{k-1}} \mu_{k-1}(\lambda)p_{\lambda,\lambda'}$; whence, by virtue of (11), we get

$$\langle\psi_k\rangle^2 \leqslant \sum_{\lambda\in\mathcal{Y}_{k-1}} \mu_{k-1}(\lambda)\,p_{\lambda,\lambda'}^2 = \frac{1}{k}\sum_{\lambda\in\mathcal{Y}_{k-1}} \mu_k(\lambda') < \frac{1}{k},$$

i.e., $\langle\psi_k\rangle < \frac{1}{\sqrt{k}}$. Since $r_1(\lambda_N) = \sum_{k=1}^{N} \psi_k(t)$, we finally get

$$\langle r_1\rangle = \sum_{k=1}^{N} \langle\psi_k\rangle < \sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 2\sqrt{N}.$$

By Lemma 6, $\langle r_1/\sqrt{N}\rangle < 2$. On the other hand, by Theorem 3, we have $r_1/\sqrt{N} > 2 - \varepsilon$ almost surely with respect to the measure μ_N for arbitrary $\varepsilon > 0$ as $N \to \infty$, and the following theorem is proved.

THEOREM 4.

$$\lim_{N\to\infty} \mu_N \left\{\Lambda \in \mathcal{Y}_N : \frac{r_1(\Lambda)}{\sqrt{N}} < 2 + \varepsilon\right\} = 1$$

for arbitrary $\varepsilon > 0$.

It is clear that the number of columns $c_1(\Lambda)$ of the typical diagram $\Lambda \in \mathcal{Y}_N$ also has the asymptotic $c_1(\Lambda) \sim 2\sqrt{N}$.

The solution of the well-known Ulm probability problem follows from Lemma 6. Let $\xi = \{\xi_k\}_1^N$ be a sequence of independent random variables with common continuous distribution. Let $R_1(\xi)$ denote the length of the largest increasing subsequence of the sequence ξ.

COROLLARY 1 [4]. For arbitrary $\varepsilon > 0$

$$\lim_{N\to\infty} P\left\{\left|\frac{R_1(\xi)}{2\sqrt{N}} - 1\right| < \varepsilon\right\} = 1.$$

Proof. The Robinson–Shensted–Knuth (RSK) algorithm gives (see [8, 9]) a mapping of the sequence space into the space of tableaux T. In this connection, the product measure with a continuous factor transforms into the Plancherel measure. Moreover, by a well-known property of the RSK algorithm, the length of the largest increasing subsequence coincides with the length of the first row of the corresponding Young diagram (the Shensted theorem, see [8]). The corollary now follows from Theorem 4.

The Shensted theorem admits the following generalization: The number of cells in the first k rows of a Young diagram is equal to the largest number of elements in the union of k monotonically increasing subsequences. Starting from this, we can obtain the following corollary of Theorem 3.

COROLLARY 2. Let $\xi = \{\xi_k\}_1^N$ be a sequence of independent random variables with common continuous distribution. Let $R_k(\xi)$ denote the maximum possible length of the union of k monotonically increasing subsequences. Then for all $\varepsilon > 0$

$$\lim_{N\to\infty} P\left\{\left|\frac{R_{\alpha\sqrt{N}}(\xi)}{N} - \int_0^{\alpha} F_\Omega(t)\,dt\right| < \varepsilon\right\} = 1,$$

where F_Ω is defined in Sec. 2.

We can strengthen these corollaries and give them the statements of individual theorems. For example, we have the following corollary.

COROLLARY 3. Almost surely

$$\lim_{N\to\infty} \frac{R_1(\xi^N(\omega))}{2\sqrt{N}} = 1.$$

Finally, the replacement of rows by columns in these statements leads to the same facts about monotonically decreasing sequences. We can also consider the union of the increasing and the decreasing sequences.

The proofs of the individual statements require, besides Theorems 3 and 4, application of the martingale technique and are not connected directly with the material of the present article [14]. The above-mentioned generalization of the Shensted theorem was obtained by the authors in 1977 [12] with the help of properties of the Knuth transform. A still more general theorem has been proved by Green and Kleitman [16] and Fomin [17].

4. PROBLEMS AND REMARKS

1) The upper bound of dimension in Theorem 1 is connected only with the first term in Eq. (5), i.e., with the hook integral. For refining the value of the constant in this bound, it is necessary to consider the second term also, i.e., $\tilde{\theta}(\Lambda)$. It is obvious from Lemma 1 that $\tilde{\theta}(\Lambda)$ can be decreased on account of the strengthening of the teeth of the diagram Λ. In this connection however, the deviation of L_Λ from Ω increases and the hook integral grows. By the same token, it remains unclear as to what additional properties of maximal diagrams isolate them among other essential Young diagrams.

2) The relative numbers of 1-hooks $\rho_1(\Lambda)N^{-1/2}$ is a good approximation to $\tilde{\theta}(\Lambda)$ (it is of independent interest). The authors do not know the asymptotic behavior of ρ_1. It is easily deduced from Theorem 3 that $\rho_1(\Lambda) \leqslant 4\sqrt{N}/\pi$ for $\Lambda \in \mathcal{Y}_N$. There are weighty indications that the mean (with respect to the Plancherel measure) value of $\rho_1(\Lambda)$ is greater than $\sqrt{N}$.

3) As also observed in [6], it would be interesting to obtain the analogue of the central limit theorem for the Plancherel measure.

4) We give one more, a more general, formulation of the problem, characteristic for statistical physics. Let $\beta \geqslant 0$ and let $\mu_{\beta,N}$ denote the following measure on $\hat{\mathfrak{S}}_N$:

$$\mu_{\beta,N}(\Lambda) = \frac{(\dim\Lambda)^\beta}{\Xi_{\beta,N}}, \text{where } \Xi_{\beta,N} = \sum_{\Lambda\in\hat{\mathfrak{S}}_N} (\dim\Lambda)^\beta.$$

For $\beta = 0$ we have $\Xi_{0,N} = p(N)$, $\mu_{0,N}(\Lambda) = 1/p(N)$ and for $\beta = 1$ it follows that $\Xi_{1,N}$ is the number of tableaux with N cells or the number of involutions in $\mathfrak{S}_N$, equal to it. It has the asymptotic $C\sqrt{N!}e^{\sqrt{N}}$ (see [8]). For $\beta = 2$, we have $\Xi_{2,N} = N!$ and $\mu_{2,N}$ is the Plancherel measure. The asymptotic of $\Xi_{\beta,N}$ for other β is not known.

It is easily deduced from the results of this article that the limit form of Young diagrams for $N \to \infty$ with respect to all the measures $\mu_{\beta,N}$ for $\beta \geqslant 1$ is the same as for $\beta = 2$, i.e., the curve Ω. The problem about the limit form for $\beta = 0$, i.e., relative to uniform distribution on diagrams, was posed by A. M. Vershik and solved by him by using bounds from [18]. This result was then obtained and refined (and a central limit theorem was proved) by M. Salai. Here is the formulation of the solution.

The limit form of Young diagrams for $N \to \infty$ with respect to a uniform distribution (of the measure $\mu_{0,N}$) in the scale $1:\sqrt{N}$ in the natural coordination is given by the curve, described by the equation (see Fig. 2)

$$e^{-\frac{\pi}{\sqrt{6}}x} + e^{-\frac{\pi}{\sqrt{6}}y} = 1.$$

The limit form for $0 < \beta < 1$ is not known.

5) It is appropriate also to recall other statistics on the diagrams and partitions: Haar [19] (the weight of a partition is proportional to the cardinality of the conjugacy class in $\mathfrak{S}_N$ with given cyclic structure), Borelian, etc. These problems will be considered separately.

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DETERMINANTS OF CAUCHY—RIEMANN OPERATORS OVER A RIEMANN SURFACE

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1. Let M be a compact one-dimensional complex manifold of genus g, and let E be a C^∞ vector bundle over M of rank r and degree d. Let $\Omega^{p,q}(E)$ be the vector space of smooth forms on M of type (p, q) with values in E. By a Cauchy—Riemann or $\bar\partial$-operator on E we mean a differential operator $D:\Omega^{0,0}(E) \to \Omega^{0,1}(E)$ which locally, in terms of a local coordinate z and a local frame in E, has the form $D = d\bar z(\partial_{\bar z} + \alpha(z))$, where $\alpha(z)$ is a smooth matrix function. Such operators are in one-to-one correspondence with holomorphic structures on the vector bundle E. We denote by $\mathcal{A}$ the space of these operators; it is an affine space relative to the complex vector space $\mathcal{B} = \Omega^{0,1}(\mathrm{End}\, E)$.

The purpose of this paper is to present a construction of determinants for such $\bar\partial$-operators based on the concept of determinant line bundle and the theory of zeta function determinants for positive elliptic operators.

Since $\bar\partial$-operators go from one vector space to another, we explain what is meant by determinants in this case. Consider first the family of operators $T:V^0 \to V^1$, where V^0 and V^1 are vector spaces of the same finite dimension. Each T induces a map from $\lambda(V^0)$ to $\lambda(V^1)$, where $\lambda(V)$ denotes the highest exterior power of V. Hence T determines an element σ_T of the line $\lambda(V^0)^* \otimes \lambda(V^1)$, where the asterisk denotes dual vector space. Upon choosing a generator for this line, σ_T can be identified with a function det(T), which is holomorphic in T and is nonzero exactly where the operator T is invertible.

In the infinite-dimensional case of $\bar\partial$-operators the above line is replaced by the line $\mathcal{L}_D = \lambda(\mathrm{Ker} D)^* \otimes \lambda(\mathrm{Coker}\, D)$, which depends on the operator D. The family of $\mathcal{L}_D$ forms a holomorphic line bundle $\mathcal{L}$ over the space $\mathcal{A}$, called the determinant line bundle. The analogue of the assumption that V^0 and V^1 have the same dimension is the condition that the index of the $\bar\partial$-operators be zero, that is, $d = r(g - 1)$. In this case there is a canonical section σ of $\mathcal{L}$ which is holomorphic and such that $\sigma_D \neq 0$ if and only if D is invertible. If we

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