# SOME QUASILINEAR SYSTEMS OCCURRING IN THE STUDY OF THE MOTION OF VISCOUS FLUIDS

A. P. Oskolkov

UDC 517.994

We prove the existence and uniqueness of generalized solutions of initial- and boundary-value problems for the modified equations of motion of a viscous fluid, for the modified equations of heat convection, and for the modified equations of magnetohydrodynamics containing linear terms with derivatives of third order which are models in the description of the flow of some classes of fluids possessing relaxational properties (including the presence of heat and electromagnetic fields).

In [1, 2], for the description of the flow of a class of non-Newtonian fluids, viz., that of weakly concentrated aqueous solutions of polymers, a quasilinear system of third order has been suggested, generalizing the Navier–Stokes system of equations,

$$\frac{\partial v_i}{\partial t}-\nu\Delta v_i+v_k\frac{\partial v_i}{\partial x_k}+\frac{\partial p}{\partial x_i}-\varkappa\left\{\frac{\partial\Delta v_i}{\partial t}+\frac{\partial}{\partial x_j}\left[v_k\frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right]\right\}=f_i,\ \operatorname{div}\vec{v}=0, \tag{1}$$

where $\varkappa\equiv\mathrm{const}>0$ is the relaxational viscosity coefficient. In [3-5], for the solution of the first initial- and boundary-value (IBV) problem relative to a simpler quasilinear system* of the third order:

$$\left.\begin{array}{l}\dfrac{\partial\vec{v}}{\partial t}-\nu\Delta\vec{v}+v_k\dfrac{\partial\vec{v}}{\partial x_k}+\operatorname{grad}p-\varkappa\left(\dfrac{\partial\Delta\vec{v}}{\partial t}+v_k\dfrac{\partial\Delta\vec{v}}{\partial x_k}\right)=\vec{f},\ \operatorname{div}\vec{v}=0,\ (x,t)\in Q_T\\ \vec{v}\Big|_{t=0}=\vec{v}_0(x),\quad \vec{v}\Big|_{\partial Q_T}=0\end{array}\right\} \tag{2}$$

under the additional boundary condition

$$(\Delta\vec{v})_n\Big|_{\partial Q_T}=0 \tag{3}$$

one has the a priori estimates:

$$\max_{0\le t\le T}\int_\Omega\left[\vec{v}^2(x,t)+\varkappa\vec{v}_x^2+\varkappa^2\vec{v}_{xx}^2\right]dx+\nu\varkappa\iint_{Q_T}\vec{v}_{xx}^2\,dQ\le C_1\left(\|\vec{v}_0\|_{W_2^2(\Omega)},\|\vec{f}\|_{L_{2,1}(Q_T)}\right), \tag{4}$$

$$\iint_{Q_T}(\vec{v}_t^2+\vec{v}_{xt}^2)\,dQ\le C_2\left(\frac{1}{\varkappa},\|\vec{v}_0\|_{W_2^2(\Omega)},\|\vec{f}\|_{L_2(Q_T)}\right) \tag{5}$$

*Obtained from system (1) for $\dfrac{\partial v_k}{\partial x_j}\dfrac{\partial}{\partial x_k}\left(\dfrac{\partial v_i}{\partial x_j}+\dfrac{\partial v_j}{\partial x_i}\right)\equiv 0,\ i=1,2,3.$

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR, Vol. 52, pp. 128-157, 1975.

0096-4104/78/0905-0765$07.50  

and on the basis of these estimates it has been attempted to prove, with the aid of a modified Galerkin method, the existence of generalized solutions of problem (2), (3) for which the integral (4), respectively, integrals (4) and (5) are finite. This attempt is unsuccessful: in order to carry out the modified Galerkin method applied in [3-5], it is necessary to construct a system of functions $\{\vec{\psi}_k(x)\}$, complete in $W_2^2(\Omega)\cap H(\Omega)$ and such that the solutions of the boundary-value problems $\varkappa\Delta\vec{\varphi}_k(x)-\vec{\varphi}_k(x)=\vec{\psi}_k(x), x\in\Omega;\ \vec{\varphi}_k|_{\partial\Omega}=0$, $k=1,2,\dots$, should also form a complete system in $W_2^2(\Omega)\cap H(\Omega)$, but this, as a detailed analysis has proved, is not in general, possible in the case $\Omega\neq E_n$.* Thus, the question of the solvability of the IBV problems (2), (3) and (2), and of similar boundary-value problems for the system (1), remains open. An exception is the Cauchy problem for system (2),† for which estimates (4), (5) allow us to prove, with the aid of the above-described modified Galerkin method, the existence of the generalized solution with finite integrals (4) and (5) under the condition that the initial function $\vec{v}|_{t=0}=\vec{v}_0(x), x\in E_n$ tends sufficiently fast to zero for $|x|\to\infty$. In addition, the estimates obtained in [6] for the solutions of the Cauchy problem for Eqs. (2), allow us to obtain the solvability of this problem in the small with respect to $t\,(0\leq t\leq t^*)$ and in better classes of functions, possessing finite integrals

$$\sum_{j=0}^{j}\Big[\max_{0\leq t\leq t^*}\int_{E_n}\big(|D_x^{\alpha}\Delta\vec{v}|^2+|\mathcal{D}_x^{\alpha+1}\vec{v}|^2\big)dx+\max_{E_n\times[0,t^*]}|\mathcal{D}_x^{\alpha}\vec{v}|\Big]+\sum_{\alpha=0}^{j}\iint_{E_n\times(0,t^*)}|D_x^{\alpha+1}D_t\vec{v}|^2dQ,\quad j\geq 1$$

under the condition that $\vec{v}_0(x)\in W_2^{j+3}(E_n)\cap J(E_n)$, while $\vec{f}(x,t)\in W_2^{j,0}(E_n), 0\leq t\leq T$, and $\int_0^T\|\vec{f}\|^2_{W_2^{j,0}(E_n)}dt<\infty$. We note that the uniqueness theorems of the different "good" generalized solutions of the initial- and boundary-value problems for system (1) and (2) proved in [4, 5] (see also [6]), allow us to hope that the boundary condition $\vec{v}|_{\partial Q_T}=0$ is sufficient for the correct formulation of these IBV problems.

In the present paper we investigate, first of all, the first IBV problem for the quasilinear system obtained from (1) by the linearization of the terms with third-order derivatives with respect to x:

$$L_i(\vec{v},\vec{V};p)\equiv\frac{\partial v_i}{\partial t}-\nu\Delta v_i+v_k\frac{\partial v_i}{\partial x_k}+\frac{\partial p}{\partial x_i}-\varkappa\Big\{\frac{\partial\Delta v_i}{\partial t}+\frac{\partial}{\partial x_j}\Big[V_k\frac{\partial}{\partial x_k}\Big(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\Big)\Big]\Big\}=f_i,\quad i=1,2,3 \tag{6}$$

$$\operatorname{div}\vec{v}=0,\ (x,t)\in Q_T;\quad \vec{v}\big|_{t=0}=\vec{v}_0(x),\quad \vec{v}\big|_{\partial Q_T}=0,$$

and we assume that the known vector $\vec{V}(x,t)\in\overset{\circ}{C}{}^{1,0}(\bar{Q}_T)\cap\overset{\circ}{J}(Q_T)$. For problem (6) we prove a uniqueness theorem of a "good" generalized solution, possessing, in particular, derivatives $\vec{v}_{xx}\in L_2(Q_T)$ and, depending on the occurrence of the argument $t$ in $\vec{V}(x,t)$, existence theorems for various

*If $\Omega\neq E_n$, then from the fact that the free term $\vec{\psi}_k(x)$ is solenoidal it does not follow, in general, that the solution $\vec{\varphi}_k(x)$ is solenoidal.

†Also the boundary-value problem for system (1) with $\frac{\partial}{\partial x_j}\big(v_k\frac{\partial^2 v_i}{\partial x_i\partial x_k}\big)\equiv 0$ (p. 779).

(p. 779)

generalized solutions of problem (6) in the large: if $\vec{V}\equiv\vec{V}(x)\equiv\vec{v}_0(x)\in \overset{\circ}{C}{}^1(\Omega)\cap \overset{\circ}{J}(\Omega)$, then the existence of "strong" solutions possessing derivatives $\vec{v}_x, \vec{v}_t, \vec{v}_{xt}\in L_2(\Omega)$, $0\leq t\leq T$ (solutions of the Ladyzhenskaya type for the Navier—Stokes equations [7]) and if $\vec{V}\equiv\vec{V}(x,t)\in \overset{\circ}{C}{}^{1,0}(\bar{Q}_T)\cap\overset{\circ}{J}(Q_T)$, $\vec{V}|_{t=0}=\vec{v}_0(x)$, then the existence of "weak" solutions possessing derivatives $\vec{v}_x\in L_2(\Omega_t)$, $0\leq t\leq T$ and weakly continuous with respect to $t\in[0,T]$ in $W_2^1(\Omega)$ (solutions of the Hopf type [7]).*

Similar results are obtained for the modified equations of heat convection

$$\left.\begin{aligned}
&L_i(\vec{v},\vec{V};p)=f_i+gS\gamma_i,\ i=1,2,3,\ \operatorname{div}\vec{v}=0,\ (x,t)\in Q_T,\ \vec{\gamma}=(0,0,1)\\
&\frac{\partial S}{\partial t}-\chi\Delta S+v_k\frac{\partial S}{\partial x_k}=0,\ (x,t)\in Q_T,\quad \chi\equiv\mathrm{Const}>0,\\
&\vec{v}|_{t=0}=\vec{v}_0(x),\ \ S|_{t=0}=S_0(x),\ \ x\in\Omega;\ \ \vec{v}|_{\partial Q_T}=0,\ \ S|_{\partial Q_T}=0,
\end{aligned}\right\}\qquad(7)$$

and, in the two-dimensional case, for the modified equations of magnetohydrodynamics:

$$\left.\begin{aligned}
&L_i(\vec{v},\vec{V};p)-H_k\frac{\partial H_i}{\partial x_k}=f_i-\frac{\partial}{\partial x_i}\frac{\vec{H}^2}{2},\ i=1,2,\ \operatorname{div}\vec{v}=0,\ (x,t)\in Q_T,\\
&\frac{\partial\vec{H}}{\partial t}+\operatorname{rot}^2\vec{H}-\operatorname{rot}[\vec{v}\times\vec{H}]=0,\ \ \operatorname{div}\vec{H}=0,\ \ (x,t)\in Q_T,\\
&\vec{v}|_{t=0}=\vec{v}_0(x),\ \ \vec{H}|_{t=0}=\vec{H}_0(x),\ \ x\in\Omega;\ \vec{v}|_{\partial Q_T}=0,\ H_n|_{\partial Q_T}=0,\ (\operatorname{rot}\vec{H})_\tau|_{\partial Q_T}=0.
\end{aligned}\right\}\qquad(8)$$

In addition, we obtain a series of a priori estimates for the solutions of IBV problems

$$\left.\begin{aligned}
&l(\psi)\equiv\frac{\partial}{\partial t}(D\psi-æD^2\psi)-\nu D^2\psi-\left\{\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\frac{D\psi-æD^2\psi}{r}-\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\frac{D\psi-æD^2\psi}{r}\right\}=F(r,z,t),\\
&D\psi\equiv\frac{\partial^2\psi}{\partial r^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{r}\frac{\partial\psi}{\partial r},\ \ (r,z,t)\in Q_T,\\
&\psi|_{t=0}=\psi_0(r,z),\ \ (r,z)\in\Omega;\ \psi|_{\partial Q_T}=\frac{\partial\psi}{\partial n}\Big|_{\partial Q_T}=0,
\end{aligned}\right\}\qquad(9)$$

and of their corresponding stationary boundary-value problem

$$\left.\begin{aligned}
&l_0(\psi)\equiv\nu D^2\psi+\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\frac{D\psi-æD^2\psi}{r}-\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\frac{D\psi-æD^2\psi}{r}=F(r,z),(r,z)\in\Omega,\\
&\psi|_{\partial\Omega}=\frac{\partial\psi}{\partial n}\Big|_{\partial\Omega}=0.
\end{aligned}\right\}\qquad(10)$$

Equation (9) is obtained from system (2) by investigating the axisymmetric flows having the angular component of the velocity equal to zero: $v^\theta=0$ in the cylindrical domains obtained

*The additional smoothness of $\vec{V}(x,t)$ with respect to the variable $t$ does not improve the situation.

from the revolution of the plane bounded domain $\Omega=\{r,z: r\geq\delta>0\}$ about the $z=0$ axis, by introducing the flow function $\psi(r,z,t)$: $v^r=\frac{1}{r}\psi_z$, $v^z=-\frac{1}{r}\psi_r$, and by deleting the strongly nonlinear terms containing the product of the second and fourth derivatives of the function $\psi$ with respect to $r, z$ (see [5, 6, 8]).

In [5] (see also [9]), together with the IBV problem (2), (3), one has investigated the IBV problem

$$\left.\begin{aligned}&\frac{\partial\vec{v}}{\partial t}-\nu\Delta\vec{v}+v_k\frac{\partial\vec{v}}{\partial x_k}-\varkappa\frac{\partial\Delta\vec{v}}{\partial t}+\operatorname{grad}p=\vec{f},\ \operatorname{div}\vec{v}=0,\ (x,t)\in Q_T,\\ &\vec{v}\big|_{t=0}=\vec{v}_0(x),\ x\in\Omega;\ \vec{v}\big|_{\partial Q_T}=0,\end{aligned}\right\}\tag{11}$$

which has a unique weak solution (solution in the sense of Hopf [7]) and for which with the aid of the standard Galerkin method one has proved the existence in the large of a generalized solution with finite integrals

$$\max_{0\leq t\leq T}\int_\Omega[\vec{v}^2(x,t)+\vec{v}_x^2]\,dx+\iint_{Q_T}(\vec{v}_t^2+\vec{v}_{xt}^2)\,dQ,\tag{12}$$

if $\vec{v}_0(x)\in W_2^2(\Omega)\cap H(\Omega)$, $\vec{f}(x,t)\in L_2(Q_T)$, and of a generalized solution with a finite integral

$$\max_{0\leq t\leq T}\int_\Omega(\vec{v}^2+\vec{v}_x^2+\vec{v}_t^2+\vec{v}_{xt}^2)\,dx,\tag{13}$$

if, in addition, $\vec{f}_t\in L_2(Q_T)$. Similar results hold also for the IBV problem

$$\begin{gathered}\frac{\partial v_i}{\partial t}-\nu\Delta v_i+v_k\frac{\partial v_i}{\partial x_k}-\varkappa\left\{\frac{\partial\Delta v_i}{\partial t}+\frac{\partial V_k(x,t)}{\partial x_j}\frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right\}+\frac{\partial p}{\partial x_i}=f_i,\ i=1,2,3\\ \operatorname{div}\vec{v}=0,\ (x,t)\in Q_T;\ \vec{v}\big|_{t=0}=\vec{v}_0(x),\quad \vec{v}\big|_{\partial Q_T}=0\,.\end{gathered}\tag{14}$$

Namely, if the linearizing vector $\vec{V}(x,t)\in\overset{\circ}{C}{}^{1,0}(\bar{Q}_T)\cap\overset{\circ}{J}(Q_T)$, then problem (14) has a unique weak solution (solution with a finite $\max_{0\leq t\leq T}\|\vec{v}_x\|_{L_2(\Omega)}$); if $\vec{V}(x,t)\in\overset{\circ}{C}{}^{1,0}(\bar{Q}_T)\cap\overset{\circ}{J}(Q_T)$, $\vec{v}_0(x)\in W_2^2(\Omega)\cap H(\Omega)$, $\vec{f}(x,t)\in L_2(Q_T)$, then problem (14) has a generalized solution with finite integrals (12); if, in addition, in $\bar{Q}_T$ the mixed derivatives $\vec{V}_{xt}(x,t)$ and $\vec{f}_t\in L_2(Q_T)$ are bounded, then problem (14) has a generalized solution with a finite integral (13). These results are proved in the same manner as their analogs for problem (11).

The paper consists of an introduction and of five small sections, in each of which one has adopted its own (binary) numerotation of the formulas and the constants. The author expresses his sincere thanks to O. A. Ladyzhenskaya for constructive discussions, in the course of which the above-mentioned deficiencies of the papers [3-5] have been revealed, necessitating the writing of the present paper.

## 1. Uniqueness of a "Good" Solution of the IBV Problem (6)

A "good" generalized solution of the IBV problem (6), where the known vector $\vec{V}(x,t)\in\overset{\circ}{C}{}^{1,0}(\bar{Q}_T)\cap\overset{\circ}{J}(Q_T)$, $\vec{V}\big|_{t=0}=\vec{v}_0(x)$, is a function $\vec{v}(x,t)\in H(\Omega)$, $0\leq t\leq T$, having a finite $\max_{0\leq t\leq T}\|\vec{v}(x,t)\|_{H(\Omega)}$, possessing derivatives $\vec{v}_{xx} \in L_2(Q_T)$ and satisfying the integral identity

$$\iint\limits_{Q_t} \{\vec{v}\vec{\Phi}_t + \varkappa\vec{v}_x\vec{\Phi}_{xt} - \nu\vec{v}_x\vec{\Phi}_x - v_k\vec{v}_{x_k}\vec{\Phi} - \varkappa V_k(x,t)\frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\Phi_{ix_j} + \vec{f}\vec{\Phi}\}dQ -$$
$$- \int\limits_{\Omega} \{\vec{v}(x,t)\vec{\Phi}(x,t) + \varkappa\vec{v}_x\vec{\Phi}_x\}dx + \int\limits_{\Omega} \{\vec{v}_0(x)\vec{\Phi}(x,0) + \varkappa\vec{v}_{0x}\vec{\Phi}_x(x,0)\}dx = 0, \quad 0 < t \leq T, \tag{1.1}$$

for any $\vec{\Phi}(x,t) \in H(\Omega)$, $0 \leq t \leq T$ and such that $\vec{\Phi}_t, \vec{\Phi}_{xt} \in L_2(\Omega)$, $0 \leq t \leq T$. We show that we have

<u>THEOREM 1.1</u>. The IBV problem (6) has at most one "good" generalized solution.

Indeed, assume that problem (6) has two "good" generalized solutions $\vec{v}_1$ and $\vec{v}_2$. Then, their difference $\vec{\omega}(x,t)$ satisfies the integral identity

$$\iint\limits_{Q_t} \{\vec{\omega}\vec{\Phi}_t + \varkappa\vec{\omega}_x\vec{\Phi}_{xt} - \nu\vec{\omega}_x\vec{\Phi}_x - \omega_k\vec{v}_{1x_k}\vec{\Phi} - v_{2k}\vec{\omega}_{x_k}\vec{\Phi} - \varkappa V_k\frac{\partial}{\partial x_k}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right)\Phi_{ix_j}\}dQ -$$
$$- \int\limits_{\Omega} \{\vec{\omega}\vec{\Phi} + \varkappa\vec{\omega}_x\vec{\Phi}_x\}dx = 0, \quad 0 < t \leq T. \tag{1.2}$$

We set in (1.2) $\vec{\Phi}(x,t) = \int\limits_0^t \frac{\vec{\omega}(x,\tau+\rho) - \vec{\omega}(x,\tau-\rho)}{2\rho}d\tau$, assuming that $\vec{\omega}(x,t) \equiv 0$ for $t<0$ and then we make $\rho$ tend to zero. Then we obtain for $\vec{\omega}(x,t)$ the equality (see [7])

$$\frac{1}{2}\int\limits_{\Omega} \{\vec{\omega}^2(x,t) + \varkappa\vec{\omega}_x^2\}dx + \nu\iint\limits_{Q_t} \vec{\omega}_x^2 dQ = -\iint\limits_{Q_t} \omega_k\vec{\omega}\frac{\partial \vec{v}_1}{\partial x_k}dQ - \varkappa\iint\limits_{Q_t} V_k\omega_{ix_j}\frac{\partial}{\partial x_k}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right)dQ \tag{1.3}$$

It is easy to see, integrating by parts, that

$$\iint\limits_{Q_t} V_k\omega_{ix_j}\frac{\partial}{\partial x_k}\omega_{ix_j}dQ = 0, \tag{1.4}$$

$$\iint\limits_{Q_t} V_k\omega_{ix_j}\frac{\partial}{\partial x_k}\omega_{jx_i}dQ = -\iint \frac{\partial V_k}{\partial x_i}\omega_{ix_j}\omega_{jx_k}dQ\ .^* \tag{1.5}$$

**Applying**, for the estimate of the first integral in the right-hand side of (1.3), the Hölder inequality and the maximization operation, and making use of the inequality (see, e.g., [7])

$$\|\omega\|_{L_4(\Omega)} \leq C_\Omega \|\omega_x\|_{L_2(\Omega)}, \tag{1.6}$$

as well as that of inequalities (1.4) and (1.5), we obtain from (1.3) the inequality

---

*In the process of the derivation of (1.5) one requires the second derivatives of $\vec{V}(x,t)$ with respect to $x$, but since in the final result they do not occur, in order to obtain (1.5) it is sufficient to make use of the averages of $\vec{V}$ over $\Omega$ with the subsequent limiting process of making the averaging radius tend to zero (see, e.g., [10]).

$$\int_\Omega(\vec{\omega}^2(x,t)+\varkappa\vec{\omega}_x^2)dx\leq C_1(\max_{0\leq t\leq T}\|\vec{v}_1\|_{H(\Omega)},\max_{\bar{Q}_T}|\vec{V}_x|)\iint_{Q_t}\vec{\omega}_x^2dQ,\ 0<t\leq T, \tag{1.7}$$

from which, as is known, it follows that $\vec{\omega}(x,t)\equiv 0$ in $Q_T$.

We consider the stationary boundary-value problem corresponding to the IBV problem (6)

$$L_{0i}(\vec{v},\vec{V}(x);p)\equiv -\nu\Delta v_i+v_k\frac{\partial v_i}{\partial x_k}+\frac{\partial p}{\partial x_i}-\varkappa\frac{\partial}{\partial x_j}\left[V_k(x)\frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right]=f_i(x),\ i=1,2,3, \tag{1.8}$$

$$\operatorname{div}\vec{v}=0,\ x\in\Omega;\ \vec{v}\big|_{\partial\Omega}=0;\ \vec{V}(x)\in\overset{\circ}{C}{}^1(\bar{\Omega})\cap\overset{\circ}{J}(\Omega),\ \vec{f}(x)\in L_2(\Omega),$$

and by a "good" generalized solution of the boundary-value problem (1.8) we shall mean a function $\vec{v}(x)\in H(\Omega)\cap W_2^2(\Omega)$ satisfying for $\forall\vec{\Phi}(x)\in H(\Omega)$ the integral identity

$$\int_\Omega\left(\nu\vec{v}_x\vec{\Phi}_x+v_k\vec{v}_{x_k}\vec{\Phi}+\varkappa V_k(x)\frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\Phi_{ix_j}-\vec{f}\vec{\Phi}\right)dx=0. \tag{1.9}$$

Setting in (1.9) $\vec{\Phi}=\vec{v}$, making use of the fact that $\int_\Omega v_k\vec{v}\vec{v}_{x_k}dx=0$, making use of equalities (1.4) and (1.5), applying Hölder's inequality, and making use of Freidrich's inequality [7]

$$\|\vec{v}\|_{L_2(\Omega)}\leq\overset{*}{C}_\Omega\|\vec{v}\|_{H(\Omega)},\quad\forall\vec{v}(x)\in H(\Omega), \tag{1.10}$$

we obtain for any "good" generalized solution of problem (1.8) the a priori estimate:

$$\|\vec{v}\|_{H(\Omega)}\leq\frac{C_\Omega^*}{\nu-\varkappa\max_{\bar{\Omega}}|\vec{V}_x|}\|\vec{f}\|_{L_2(\Omega)}\quad\left(\nu-\varkappa\max_{\bar{\Omega}}|\vec{V}_x|>0\right). \tag{1.11}$$

With the aid of estimate (1.11), similar to Theorem 1.1 one proves

THEOREM 1.2. Boundary-value problem (1.8), under the condition

$$(\nu-\varkappa\max_{\bar{\Omega}}|V_x|)^2-\overset{2}{C}_\Omega C_\Omega^*\|\vec{f}\|_{L_2(\Omega)}>0, \tag{1.12}$$

has at most one "good" generalized solution.

## 2. Existence in the Large of a "Strong" Generalized Solution of the Initial- and Boundary-Value Problem (6)

Assume $\Omega$ is a two-dimensional or a three-dimensional bounded domain. By a "strong" generalized solution of the IBV problem (6), where the known vector $\vec{V}\equiv\vec{V}(x)=\vec{v}_0(x)\in W_2^3(\Omega)\cap H(\Omega)$, we mean a function $\vec{v}(x,t)\in H(\Omega),\ 0\leq t\leq T$, possessing derivatives $\vec{v}_t,\vec{v}_{xt}\in L_2(\Omega), 0\leq t\leq T$, having a finite $\max_{0\leq t\leq T}\int_\Omega(\vec{v}_x^2(x,t)+\vec{v}_t^2+\vec{v}_{xt}^2)dx$, satisfying the integral identity

$$\iint_{Q_t}\left\{\vec{v}_t\vec{\Phi}+\varkappa\vec{v}_{xt}\vec{\Phi}_x+\nu\vec{v}_x\vec{\Phi}_x+v_k\frac{\partial\vec{v}}{\partial x_k}\vec{\Phi}-\varkappa v_{0k}(x)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\frac{\partial^2\Phi_i}{\partial x_j\partial x_k}\right\}dQ=\iint_{Q_t}\vec{f}\vec{\Phi}dQ,\ 0<t\leq T, \tag{2.1}$$

for $\forall\vec{\Phi}(x,t)\in\overset{\circ}{W}{}_2^{2,0}(Q_T)\cap\overset{\circ}{J}(Q_T)$ , and subject to the initial condition $\vec{v}|_{t=0}=\vec{v}_0(x),\ x\in\Omega$ .

The fundamental result of the present section is

THEOREM 2.1. Let $\vec{f}(x,t),\ \vec{f}_t\in L_2(Q_T),\ \vec{v}_0(x)\in W_2^3(\Omega)\cap H(\Omega)$. Then the IBV problem (6) has at least one "strong" generalized solution which can be obtained by the Galerkin method and for any such solution one has two series of estimates: the energy inequality

$$\max_{0\le t\le T}\int_\Omega\{|\vec{v}(x,t)|^2+æ|\vec{v}_x|^2\}dx+\nu\iint_{Q_T}\vec{v}_x^{\,2}dQ\le C_1(\|\vec{f}\|_{L_2(Q_T)},\max_{\bar{\Omega}}|\vec{v}_{0x}|);\tag{2.2}$$

in the three-dimensional case the estimate

$$\max_{0\le t\le T}\int_\Omega(\vec{v}_t^{\,2}+\vec{v}_{xt}^{\,2})dx\le C_2(\|\vec{f},\vec{f}_t\|_{L_2(Q_T)},\ \|\vec{v}_0\|_{W_2^3(\Omega)};\ æ),\tag{2.3}$$

where $C_2\to\infty$ for $æ\to 0$ ; in the two-dimensional case the uniform estimate with respect to $æ\in[0,1]$

$$\max_{0\le t\le T}\int(\vec{v}_t^{\,2}+æ\vec{v}_{xt}^{\,2})dx+\iint_{Q_T}\vec{v}_{xt}^{\,2}dQ\le C_3(\|\vec{f},\vec{f}_t\|_{L_2(Q_T)},\|\vec{v}_0\|_{W_2^3(\Omega)};\nu)\tag{2.4}$$

Proceeding with the proof of Theorem 2.1, we note first of all that for the solutions of problem (6) the derivative $\vec{v}_t(x,0)$ for $\vec{v}_0(x)\in W_2^3(\Omega)\cap H(\Omega)$ and $\vec{f}(x,0)\in L_2(\Omega)$ is uniquely defined as an element of the space $W_2^2(\Omega)\cap H(\Omega)$ . Indeed, by virtue of system (6)

$$v_{it}(x,0)-æ\Delta v_{it}(x,0)+\frac{\partial p}{\partial x_i}(x,0)=f_i(x,0)+\nu\Delta v_i(x,0)-v_{0k}\frac{\partial v_{0i}}{\partial x_k}+æ\frac{\partial}{\partial x_j}\left\{v_{0k}(x)\frac{\partial}{\partial x_k}\left(\frac{\partial v_{0i}}{\partial x_j}+\frac{\partial v_{0j}}{\partial x_i}\right)\right\}\equiv F_i(x),\tag{2.5}$$

while by virtue of the condition on $\vec{v}_0(x)$ and $\vec{f}(x,0)$ $\vec{F}(x)\in L_2(\Omega)$. From the theorem on the orthogonal decomposition of the space $L_2(\Omega)$ there follows [7] that $\vec{v}_t(x,0)-æ\Delta\vec{v}_t(x,0)$ and $grad\ p$ are obtained as the projections of $\vec{F}(x)$ onto the subspaces $\overset{\circ}{J}(\Omega)$ and $G(\Omega)$. Then, solving the Dirichlet problem

$$æ\Delta\vec{v}_t(x,0)-\vec{v}_t(x,0)=-Pr_{\overset{\circ}{J}(\Omega)}\vec{F}(x),\ x\in\Omega;\quad \vec{v}_t\big|_{\partial\Omega}=0,\tag{2.6}$$

we find that [9] $\vec{v}_t(x,0)\in W_2^2(\Omega)\cap H(\Omega)$, and, by virtue of the first fundamental inequality for elliptic equations, we shall have for $\vec{v}_t(x,0)$ the estimate:

$$\int_\Omega\{\vec{v}_t^{\,2}(x,0)+æ\vec{v}_{xt}^{\,2}(x,0)\}dx\le C_4(\|\vec{f}(x,0)\|_{L_2(\Omega)},\|\vec{v}_0(x)\|_{W_2^3(\Omega)}).\tag{2.7}$$

In order to prove the solvability of problem (6) we make use of Galerkin's method. Let $\{\vec{\varphi}_k(x)\}$ , $k=1,2,\ldots$, be a complete system of functions in $W_2^3(\Omega)\cap H(\Omega)$ which can be assumed to be orthonormalized in $L_2(\Omega)$ . Therefore, for any function $\vec{v}_0(x)\in W_2^3(\Omega)\cap H(\Omega)$ there exists a sequence of functions $\{\vec{v}_{(n)}(x)\}$ :

$$\vec{v}_{(n)}(x)=\sum_{k=1}^{n}C_{kn}^{\circ}\vec{\varphi}_k(x),\quad n=1,2,\ldots,\tag{2.8}$$

which converges to $\vec{v}_0(x)$ in the norm of $W_2^3(\Omega)$ as $n\to\infty$.

We shall seek the approximate solutions $\vec{v}^n(x,t)$ of the problem (6) in the form

$$\vec{v}^n(x,t)=\sum_{k=1}^{n} C_{kn}(t)\vec{\varphi}_k(x)\ ,\quad n=1,2,\dots, \tag{2.9}$$

where the functions $C_{kn}(t)$ are obtained from the system of differential equations

$$\int_\Omega \{L_i(\vec{v}^n,\vec{v}_0(x);\rho)-f_i(x,t)\}\varphi_{il}(x)dx\equiv\int_\Omega \vec{v}^n_t\vec{\varphi}_l dx+æ\int_\Omega \vec{v}^n_{xt}\vec{\varphi}_{lx}dx+\nu\int_\Omega \vec{v}^n_x\vec{\varphi}_{lx}dx-$$

$$-\int v^n_k\vec{v}\vec{\varphi}_{lx_k}dx+æ\int_\Omega v_{0k}(x)\left(\frac{\partial v^n_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\varphi_{lix_j}dx-\int_\Omega \vec{f}\vec{\varphi}dx=0,\ l=1,\dots,n,\ 0<t\leq T \tag{2.10}$$

and the Cauchy initial conditions

$$C_{ln}\big|_{t=0}=C^0_{ln}\ ,\quad l=1,\dots,n. \tag{2.11}$$

We multiply the $l$-th equation of (2.10) by $C_{ln}(t)$ and we sum with respect to $l$ from 1 to $n$. Integrating by parts and making use of equalities (1.4) and (1.5) (with $\vec{\omega}\equiv\vec{v}^n$ and $\vec{V}\equiv\vec{v}_0(x)$ ), we obtain the equality:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\vec{v}^n)^2dx+\frac{æ}{2}\frac{d}{dt}\int_\Omega(\vec{v}^n_x)^2dx+\nu\int_\Omega(\vec{v}^n_x)^2dx=\int_\Omega\vec{f}\vec{v}^n dx-æ\int_\Omega\frac{\partial v_{0k}}{\partial x_i}\frac{\partial v^n_i}{\partial x_j}\frac{\partial v^n_j}{\partial x_i}dx, \tag{2.12}$$

from which, applying Hölder's inequality, making use of Gronwall's lemma (see [7, Chap. VI, Lemma 1]), and using the fact that

$$\|\vec{v}_n(x,0)\|_{W_2^1(\Omega)}\leq C\,\|\vec{v}_0(x,0)\|_{W_2^1(\Omega)}\leq C_4(\Omega)\max_{\bar\Omega}|\vec{v}_{0x}|\ , \tag{2.13}$$

we obtain the inequality

$$\max_{0\leq t\leq T}\int_\Omega\{(\vec{v}^n(x,t))^2+æ(\vec{v}^n_x)^2\}dx+\nu\iint_{Q_T}(\vec{v}^n_x)^2dQ\leq C_5\left(\|\vec{f}\|_{L_2(Q_T)},\max_{\Omega}|\vec{v}_{0x}|,T,\Omega\right),n=1,2,\dots, \tag{2.14}$$

where the constant $C_5$ does not depend on $æ$. From estimates (2.14) and the orthonormality of $\{\vec{\varphi}_k(x)\}$ in $L_2(\Omega)$, there follows an a priori estimate for the possible solutions of Cauchy problem (2.10), (2.11):

$$\max_{0\leq t\leq T}\sum_{l=1}^{n}C^2_{ln}(t)\equiv\max_{0\leq t\leq T}\|\vec{v}^n(x,t)\|^2_{L_2(\Omega)}\leq C_5,\quad n=1,2,\dots, \tag{2.15}$$

which ensures the unique solvability of Cauchy problem (2.10), (2.11) for any $n=1,2,\dots$ on any finite time interval $[0,T]$.

We multiply the $l$-th equation of (2.10) by $\frac{dC_{ln}}{dt}$, we sum with respect to $l$ from 1 to $n$, and we write the obtained equality for $t=0$. Integrating then by parts, we obtain the equality:

$$\int_\Omega \{|\vec{v}_t^n(x,0)|^2 + æ|\vec{v}_{xt}^n|^2\}dx = \nu\int_\Omega \vec{v}_t^n(x,0)\Delta\vec{v}_{(n)}dx - \int_\Omega \vec{v}_t^n(x,0)v_{(n)k}(x)\frac{\partial \vec{v}_{(n)}(x)}{\partial x_k}dx +$$

$$+æ\int_\Omega \vec{v}_t^n(x,0)\frac{\partial}{\partial x_j}\left\{v_{0k}(x)\frac{\partial}{\partial x_k}\left(\frac{\partial v_{(n)i}}{\partial x_j}+\frac{\partial v_{(n)j}}{\partial x_i}\right)\right\}dx + \int_\Omega \vec{f}(x,0)\vec{v}_t^n(x,0)dx,\ n=1,2,\ldots, \tag{2.16}$$

from which, applying the inequalities of Hölder and Cauchy, inequality (1.6), and making use of the fact that

$$\|\vec{v}_{(n)}\|_{W_2^3(\Omega)} \leq C\,\|\vec{v}_0\|_{W_2^3(\Omega)},\quad n=1,2,\ldots, \tag{2.17}$$

we obtain estimate [see (27)]:

$$\int_\Omega \{|\vec{v}_t^n(x,0)|^2 + æ|\vec{v}_{xt}^n(x,0)|^2\}dx \leq C_6\left(\|\vec{f}(x,0)\|_{L_2(\Omega)}, \|\vec{v}_0\|_{W_2^3(\Omega)}\right), \tag{2.18}$$

which is uniform with respect to $n=1,2,\ldots$ and $æ>0$.

We differentiate Eqs. (2.10) with respect to $t$, we multiply the $\ell$-th equation by $\frac{dC_{\ell n}}{dt}$, and we sum with respect to $\ell$ from 1 to $n$. Integrating by parts and applying once again equalities (1.4) and (1.5) (this time with $\vec{\omega} \equiv \vec{v}_t^n$ and $\vec{V}=\vec{v}_0(x)$), we obtain the equality:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \{|\vec{v}_t^n|^2 + æ|\vec{v}_{xt}^n|^2\}dx + \nu\int_\Omega |\vec{v}_{xt}^n|^2 dx = \int_\Omega \vec{f}_t\vec{v}_t^n dx + \int_\Omega v_{kt}^n\vec{v}_{x_k}^n\vec{v}_t^n dx - æ\int_\Omega \frac{\partial v_{0k}}{\partial x_i}\frac{\partial v_{it}^n}{\partial x_j}\frac{\partial v_{jt}^n}{\partial x_i}dx,\quad 0<t\leq T,\ n=1,2,\ldots \tag{2.19}$$

If $\Omega$ is a three-dimensional domain, then, estimating the second integral on the right-hand side with the aid of the Hölder inequality and of inequality (1.6) and applying estimate (2.14) we obtain:

$$\left|\int_\Omega v_{kt}^n\vec{v}_{x_k}^n\vec{v}_t^n dx\right| \leq \frac{C_5^{1/2}C_\Omega^2}{æ^{1/2}}\int_\Omega |\vec{v}_{xt}^n|^2 dx. \tag{2.20}$$

Then from (2.19) we obtain the inequality:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \{|\vec{v}_t^n(x,t)|^2 + æ|\vec{v}_{xt}^n|^2\}dx + \nu\int_\Omega |\vec{v}_{xt}^n|^2 dx \leq \left(\|\vec{f}_t\|_{L_2(\Omega)}^2 + \max_\Omega|\vec{v}_{0x}| + \frac{C_5^{1/2}C_\Omega^2}{æ^{3/2}}\right)\int_\Omega \{|\vec{v}_t^n|^2 + æ|\vec{v}_{xt}^n|^2\}dx,$$
$$0\leq t\leq T,\ n=1,2,\ldots, \tag{2.21}$$

from which, applying Gronwall's lemma and making use of estimate (2.18), we obtain the estimate:

$$\max_{0\leq t\leq T}\int_\Omega \{|\vec{v}_t^n(x,t)|^2 + æ|\vec{v}_{xt}^n|^2\}dx + \nu\iint_{Q_T}|\vec{v}_{xt}^n|^2 dQ \leq C_7\left(\|\vec{f},\vec{f}_t\|_{L_2(Q_T)}, \|\vec{v}_0\|_{W_2^3(\Omega)}; æ\right),\quad n=1,2,\ldots, \tag{2.22}$$

where $C_7\to\infty$ for $æ\to 0$.

If $\Omega$ is a two-dimensional domain, then, applying for the estimate of the second integral in the right-hand side of equality (2.19), instead of inequality (1.6) the inequality [7, Chap. I]:

$$\|\vec{\omega}\|^2_{L_4(\Omega)} \leq \sqrt{2}\,\|\vec{\omega}\|_{L_2(\Omega)} \cdot \|\vec{\omega}_x\|_{L_2(\Omega)}, \tag{2.23}$$

we obtain:

$$\left|\int_\Omega v^n_{kt}\vec{v}^n_{x_k}\vec{v}^n_t dx\right| \leq \frac{\nu}{2}\int_\Omega |\vec{v}^n_{xt}|^2 dx + \frac{1}{\nu}\int_\Omega |\vec{v}^n_x|^2 dx \int_\Omega |\vec{v}^n_t|^2 dx. \tag{2.24}$$

Then from (2.19) we obtain instead of (2.21) the following inequality:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \{|\vec{v}^n_t|^2 + æ|\vec{v}^n_{xt}|^2\} dx + \frac{\nu}{2}\int_\Omega |\vec{v}^n_{xt}|^2 dx \leq \left(\|\vec{f}_t\|^2_{L_2(\Omega)} + \max_{\Omega}|\vec{v}_{\circ x}| + \frac{1}{\nu}\int_\Omega |\vec{v}^n_x|^2 dx\right)\int_\Omega (|\vec{v}^n_t|^2 + æ|\vec{v}^n_{xt}|^2) dx, \tag{2.25}$$

$$0<t\leq T, \quad n=1,2,\dots,$$

from which, applying once again Gronwall's lemma and making use of estimate (2.14), by virtue of which

$$\iint_{Q_T} |\vec{v}^n_x|^2 dQ \leq \frac{C_5}{\nu}, \tag{2.26}$$

we obtain instead of inequalities (2.22) the estimate

$$\max_{0\leq t\leq T}\int_\Omega \left(|\vec{v}^n_t(x,t)|^2 + æ|\vec{v}^n_{xt}|^2\right) dx + \iint_{Q_T} |\vec{v}^n_{xt}|^2 dQ \leq C_8(\nu, \|\vec{f}, f_t\|_{L_2(Q_T)}, \|\vec{v}_\circ\|_{W^3_2(\Omega_T)}), \tag{2.27}$$

$$n=1,2,\dots,$$

where the constant $C_8$ remains bounded for $æ \to 0$.

We prove now that for a fixed $\ell$ and $n \geq \ell$, the functions $\psi_{n,\ell}(t) \equiv \int_\Omega [\vec{v}^n_t(x,t)\vec{\varphi}_\ell(x) + æ\vec{v}^n_{xt}\vec{\varphi}_{\ell x}] dx$ form a uniformly bounded and equicontinuous family of functions on $[0,T]$. The uniform boundedness of $\{\psi_{n,\ell}(t)\}$ follows from estimates (2.22) or (2.27):

$$\max_{0\leq t\leq T}|\psi_{n,\ell}(t)| \leq \max_{0\leq t\leq T}\|\vec{v}^n_t\|_{L_2(\Omega)} + æ^{1/2}\max_{0\leq t\leq T}\|\vec{v}^n_{xt}\|_{L_2(\Omega)} \cdot æ^{1/2}\|\vec{\varphi}_{\ell x}\|_{L_2(\Omega)} \leq C_9(\ell), \quad n=1,2,\dots, \tag{2.28}$$

where the constant $C_9$ depends on the constant $C_7$ in the three-dimensional case and on the constant $C_8$ (i.e., does not depend on $æ>0$) in the two-dimensional case. For the proof of the equicontinuity of $\{\psi_{n,\ell}(t)\}$, we differentiate the Eqs. (2.10) with respect to $t$ and we write the result in the form

$$\frac{d}{dt}\psi_{n,\ell}(t) = \nu\int_\Omega \vec{v}^n_t \Delta\vec{\varphi}_\ell dx + \int_\Omega (\vec{v}^n_{kt}\vec{v}^n + v^n_k\vec{v}^n_t)\vec{\varphi}_{\ell x_k} dx - æ\int_\Omega v_{\circ k}(x)\left(\frac{\partial v^n_{it}}{\partial x_j} + \frac{\partial v^n_{jt}}{\partial x_i}\right)\varphi_{\ell i x_j x_k} dx +$$

$$+\int_\Omega \vec{f}_t(x,t)\vec{\varphi}_\ell dx, \quad 0<t\leq T, \tag{2.29}$$

from this equality, integrating with respect to $t$ from $t$ to $t+\Delta t\leq T$ , estimating the integral in the right-hand side by Hölder's inequality and making use of estimates (2.14), (2.22), and (2.27), we obtain the inequality:

$$|\psi_{n,\ell}(t+\Delta t)-\psi_{n,\ell}(t)|\leq C_{10}(\ell)\Big(\sqrt{\Delta t}+\int_t^{t+\Delta t}\|\vec{f}_t\|_{L_2(\Omega)}dt\Big),\quad n=1,2,\dots, \tag{2.30}$$

where the constant $C_{10}$ depends on $C_7$ in the three-dimensional case and on $C_8$ in the two-dimensional case.

Estimates (2.14), (2.22), (2.28), (2.30) in the three-dimensional case and estimates (2.14), (2.27), (2.28), (2.30) (uniform relative to $\varkappa>0$ ) in the two-dimensional case allow us to conclude (see [7, Chap. VI]) that from the totality $\{\vec{v}^n(x,t)\}$ of Galerkin approximations for the solutions of the problem (6) one can extract at least one subsequence $\{\vec{v}^{n_i}(x,t)\}$ , for which for $n_i\to\infty$ $\vec{v}^{n_i}, \vec{v}^{n_i}_x, \vec{v}^{n_i}_t, \vec{v}^{n_i}_{xt}$ converge weakly in $L_2(\Omega)$, uniformly with respect to $t\in[0,T]$ , to the functions $\vec{v}$ , $\vec{v}_x$ , $\vec{v}_t$ , $\vec{v}_{xt}$ and these functions are continuous with respect to $t\in[0,T]$ in the weak topology of $L_2(\Omega)$ , $\vec{v}^{n_i}$ converge to $\vec{v}$ strongly in $L_2(Q_T)$ and for the limit function $\vec{v}(x,t)$ , by virtue of the properties of weak convergence, estimates (2.2), (2.3) or (2.2), (2.4) will hold. Then, with the aid of the known arguments [7, Chap. VI] one proves that the limit function $\vec{v}(x,t)$ will be one of the "strong" generalized solutions of the IBV problem (6). Theorem 2.1 is proved.

Let $\Omega$ be a two-dimensional domain. In this case for the "strong" generalized solutions $\vec{v}^{\varkappa}(x,t)$ of problem (6) one has estimates (2.2), (2.4), whose right-hand sides do not depend on $\varkappa>0$. Then from the totality $\{\vec{v}^{\varkappa}(x,t)\}$ one can extract at least one subsequence $\{\vec{v}^{\varkappa_m}(x,t)\}$ , for which for $\varkappa_m\to 0$ we have

$$\varkappa_m\iint_{Q_t}\Big\{\vec{v}^{\varkappa_m}_{xt}\vec{\Phi}_x-v_{ok}(x)\Big(\frac{\partial v_i^{\varkappa_m}}{\partial x_j}+\frac{\partial v_j^{\varkappa_m}}{\partial x_i}\Big)\frac{\partial^2\Phi}{\partial x_j\partial x_k}\Big\}dQ\to 0,\ \forall\vec{\Phi}\in\overset{\circ}{W}{}^{2,0}_2(Q_T)\cap\overset{\circ}{J}(Q_T), \tag{2.31}$$

$\vec{v}^{\varkappa_m}$ converges strongly in $L_2(Q_T)$ to $\vec{v}(x,t)$, $\vec{v}^{\varkappa_m}_x$ and $\vec{v}^{\varkappa_m}_t$ converge weakly in $L_2(Q_T)$ to $\vec{v}_x$ and $\vec{v}_t$ , the limit function $\vec{v}(x,t)$ satisfies trivially the inequality

$$\iint_{Q_t}\big(|\vec{v}_x|^2+|\vec{v}_t|^2+|\vec{v}_{xt}|^2\big)dQ\leq C_{11}(C_1,C_3,T), \tag{2.32}$$

it satisfies the integral identity

$$\iint_{Q_t}\Big\{\vec{v}_t\vec{\Phi}+\nu\vec{v}_x\vec{\Phi}_x+v_k\frac{\partial\vec{v}}{\partial x_k}\vec{\Phi}\Big\}dQ=\iint_{Q_t}\vec{f}\vec{\Phi}\,dQ,\quad 0<t\leq T, \tag{2.33}$$

for $\forall\vec{\Phi}(x,t)\in\overset{\circ}{W}{}^{2,0}_2(Q_T)\cap\overset{\circ}{J}(Q_T)$ and the initial condition $\vec{v}\big|_{t=0}=\vec{v}_0(x)$, i.e., it is a generalized solution in the sense of Ladyzhenskaya of the first IBV problem for the nonstationary system of Navier—Stokes equations [7, Chap. VI]. Since such a generalized solution is unique [7, Chap. VI], we have proved

THEOREM 2.2. Let $\Omega$ be a two-dimensional bounded domain and let $\vec{f}, \vec{f}_t \in L_2(Q_T)$, $\vec{v}_0(x) \in W_2^3(\Omega) \cap H(\Omega)$. Then for $\varkappa \to 0$ the totality $\{\vec{v}^{\varkappa}(x,t)\}$ of "strong" generalized solutions of the initial- and boundary-value problem (6) converges (in the above-described sense) to a unique generalized solution $\vec{v}(x,t)$, in the sense of Ladyzhenskaya, of the first IBV problem for the nonstationary system of Navier–Stokes equations.

The analysis of the proof of Theorem 2.1 shows that estimate (2.4), uniform relative to $\varkappa > 0$, as well as Theorem 2.2 which results from it, hold in the large not only for two-dimensional domains $\Omega$, but also for such three-dimensional domains for which the first IBV problem for the Navier–Stokes equations is solvable in the large in the class of the generalized solutions of Ladyzhenskaya, e.g., for three-dimensional domains obtained by the rotation about some axis of a plane domain not containing the axis of rotation, under the condition that the cylindrical components of the velocity vector and the free term do not depend on the polar angle $\theta$ [7, Chap. VI]. For arbitrary three-dimensional flows, the estimate (2.4) and the assertion which follows from it regarding the limiting process of the totality $\{\vec{v}^{\varkappa}(x,t)\}$ for $\varkappa \to 0$ to the solution of the Navier–Stokes equation take place in the small under the conditions, similar to those smallness conditions under which the first IBV problem for the three-dimensional nonstationary system of Navier–Stokes equations is solvable in the class of generalized solutions of Ladyzhenskaya [7, Chap. VI]. We give one of the variants of these conditions, introducing first necessary notations (see [7, Chap VI]; see also [5], where similar estimates have been obtained for the solutions of the system $\frac{\partial \vec{v}}{\partial t} - \nu\Delta\vec{v} + v_k \frac{\partial \vec{v}}{\partial x_k} - \varkappa \frac{\partial \Delta \vec{v}}{\partial t} + \operatorname{grad} p = \vec{f}, \operatorname{div} \vec{v} = 0$).

We set $\psi_{\varkappa}^2(0) = \|\vec{v}_t(x,0)\|^2_{L_2(\Omega)} + \varkappa \|\vec{v}_{xt}(x,0)\|^2_{L_2(\Omega)}$,*

$$C_{12} = \max_{\Omega} |\vec{v}_{0x}|, \quad C_{13} = \max_{0 \le t \le T} \|\vec{f}\|_{L_2(\Omega)}, \quad C_{14} = \psi_{\frac{\nu}{2C_{12}}}(0) + \|\vec{f}_t\|_{L_{2,1}(Q_T)}. \tag{2.34}$$

Then we have

THEOREM 2.3. Let $\Omega$ be a three-dimensional bounded domain, let $\vec{f}, \vec{f}_t \in L_2(Q_T)$, $\vec{v}_0(x) \in W_2^3(\Omega) \cap H(\Omega)$ and assume that the following "conditions of smallness of the data of problem (6)" are satisfied:

$$\varkappa C_{12} < \frac{\nu}{2}, \quad \frac{\nu}{2} - C_{\Omega}^2 \sqrt{\frac{2C_1^{1/2}}{\nu}(\sqrt{2}C_{14} + C_{13})} \equiv \gamma > 0, \tag{2.35}$$

where $C_{\Omega}$ and $C_1$ are the constants from inequalities (1.6) and (2.2). Then, for the "strong" generalized solution $\vec{v}^{\varkappa}(x,t)$ of the IBV problem (6), whose existence is guaranteed for any $\varkappa > 0$ by Theorem 2.1, we have uniformly relative to $\varkappa \in [0,1]$ estimate (2.2) and the estimates:

$$\max_{0 \le t \le T} \|\vec{v}_x^{\varkappa}(x,t)\|_{L_2(\Omega)} \le \left[\frac{2C_1^{1/2}}{\nu}(\sqrt{2}C_{14} + C_{13})\right]^{1/2}, \tag{2.36}$$

$$\max_{0 \le t \le T} \psi_{\varkappa}^2(t) + 2\gamma \iint_{Q_T} |\vec{v}_{xt}^{\varkappa}|^2 dQ \le \psi_1^2(0) + 2C_{14}^2. \tag{2.37}$$

*The estimate of $\psi_{\varkappa}(0)$ is given in terms of the data of problem (6) by inequality (2.7).

For $\varkappa \to 0$ the totality $\{\vec{v}^{\varkappa}(x,t)\}$ converges (in the sense described in Theorem 2.2) to a unique generalized solution, in the sense of Ladyzhenskaya, of the first IBV problem for the Navier–Stokes equations.

In conclusion, we note that the first IBV problem for the Navier–Stokes equations, whose given data satisfy the second of the "smallness conditions" (2.35), has trivially a generalized solution from the class of Ladyzhenskaya [7, Chap. VI].

## 3. Existence in the Large of a "Weak" Generalized Solution of the Initial- and Boundary-Value Problem (6)

By a "weak" generalized solution of the IBV problem (6), where the known vector $\vec{V} \equiv \vec{V}(x,t)$ has a finite $\max\limits_{Q_T}\{|\vec{V}(x,t)| + |\vec{V}_x(x,t)|\} \equiv A_{\vec{V}}$ and $\vec{V}|_{t=0} = \vec{v}_0(x)$, we mean a function $\vec{v}(x,t) \in H(\Omega)$, $0 \le t \le T$, having a finite $\max\limits_{0\le t\le T}\|\vec{v}\|_{H(\Omega)}$, weakly continuous with respect to $t \in [0,T]$ in $W_2^1(\Omega)$ and satisfying the integral identity

$$\iint\limits_{Q_T}\left\{\vec{v}\vec{\Phi}_t + \varkappa\vec{v}_x\vec{\Phi}_{xt} - \nu\vec{v}_x\vec{\Phi}_x + v_k\vec{v}\vec{\Phi}_{x_k} + \varkappa V_k(x,t)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\frac{\partial^2\Phi_i}{\partial x_j \partial x_k}\right\}dQ -$$

$$-\int\limits_{\Omega}\left\{\vec{v}_0(x)\vec{\Phi}(x,0) + \varkappa\vec{v}_{0x}\vec{\Phi}_x(x,0)\right\}dQ = -\iint\limits_{Q_T}\vec{f}\vec{\Phi}\,dQ \tag{3.1}$$

for $\forall\vec{\Phi}(x,t) \in \overset{\circ}{W}{}_2^{2,1}(Q_T) \cap \overset{\circ}{J}(Q_T), \vec{\Phi}(x,T) = 0$. Let us prove that we have

<u>THEOREM 3.1.</u> Let $\vec{f}(x,t) \in L_2(Q_T)$, $\vec{v}_0(x) \in H(\Omega)$. Then the IBV problem (6) has at least one "weak" generalized solution and for this solution one has the energy estimate:

$$\max_{0\le t\le T}\int\limits_{\Omega}\left\{|\vec{v}(x,t)|^2 + \varkappa\vec{v}_x^2\right\}dx + \nu\iint\limits_{Q_t}\vec{v}_x^2\,dQ \le C_1(\|\vec{f}\|_{L_2(Q_T)}, \|\vec{v}_0\|_{H(\Omega)}, T, A_{\vec{V}}). \tag{3.2}$$

As before, for the proof of the weak solvability of problem (6) we make use of the Galerkin method and we shall seek the approximate solutions $\vec{v}^n(x,t)$ in the form (2.9) from the system of differential equations (2.10) and Cauchy initial conditions (2.11), (2.8). For $\{\vec{v}^n(x,t)\}$ we have the equality, similar to (2.12):

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}\left[|\vec{v}^n|^2 + \varkappa|\vec{v}_x^n|^2\right]dx + \nu\int\limits_{\Omega}|\vec{v}_x^n|^2 dx = \int\limits_{\Omega}\vec{f}\vec{v}^n dx - \varkappa\int\limits_{\Omega}\frac{\partial V_k(x,t)}{\partial x_i}\frac{\partial v_i^k}{\partial x_j}\frac{\partial v_j^n}{\partial x_i}\,dt, \quad 0 < t \le T, \tag{3.3}$$

from which, applying the maximization operation, Hölder's inequality, making use of Gronwall's lemma [7, Chap. VI, Lemma 1] and using the fact that

$$\|\vec{v}^n(x,0)\|_{H(\Omega)} \le C\|\vec{v}_0\|_{H(\Omega)}, \quad n = 1, 2, \ldots, \tag{3.4}$$

we obtain the estimate:

$$\max_{0\le t\le T}\int\limits_{\Omega}\left\{|\vec{v}^n|^2 + \varkappa|\vec{v}_x^n|^2\right\}dx + \nu\iint\limits_{Q_T}|\vec{v}_x^n|^2 dQ \le C_2\left(\|\vec{f}\|_{L_2(Q_T)}, \|\vec{v}_0\|_{H(\Omega)}, T, A_{\vec{V}}\right), \; n = 1, 2, \ldots. \tag{3.5}$$

From estimate (3.5) and from the orthonormality of $\{\vec{\varphi}_\ell(x)\}$ in $L_2(\Omega)$ we obtain once again a priori estimate (2.15) for the possible solutions of Cauchy problem (2.10), (2.11), (2.8), guaranteeing its solvability for each $n = 1,2,\ldots$ on any finite interval $[0,T]$.

In addition, from estimate (3.5) and from the theorem of weak compactness of a bounded set in a Hilbert space it follows that from the totality $\{\vec{v}^n(x,t)\}$ of Galerkin approximations one can extract a subsequence $\{\vec{v}^{n_m}(x,t)\}$, which converges to some function $\vec{v}(x,t)$ weakly in $L_2(Q_T)$ together with the first derivatives with respect to $x$ and $\vec{v}, \vec{v}_x \in L_2(Q_T)$, $\operatorname{div} \vec{v} = 0$ in $Q_T$ and $\vec{v}|_{\partial Q_T} = 0$.

Let us show now that for fixed $\ell$ and $n \geq \ell$ the functions $\Phi_{n,\ell}(t) \equiv \int_\Omega \{\vec{v}^n(x,t)\vec{\varphi}_\ell(x) + æ\vec{v}^n_x \vec{\varphi}_{\ell x}\}dx$, $0 \leq t \leq T$, form a uniformly bounded and equicontinuous family of functions on $[0,T]$ (see [7, Chap. VI, Sec. 7]). The uniform boundedness of $\{\Phi_{n,\ell}(t)\}$ follows from estimate (3.5):

$$\max_{0\leq t\leq T} |\Phi_{n,\ell}(t)| \leq \max_{0\leq t\leq T} \|\vec{v}^n(x,t)\|_{L_2(\Omega)} + æ \max_{0\leq t\leq T} \|\vec{v}^n_x\|_{L_2(\Omega)} \|\vec{\varphi}_\ell\|_{H(\Omega)} \leq C_3(\ell, C_2),\ n = 1,2,\ldots \tag{3.6}$$

In order to prove the equicontinuity of $\{\Phi_{n,\ell}(t)\}$ we rewrite Eq. (2.10) in the form

$$\frac{d}{dt}\Phi_{n,\ell}(t) = -\nu\int_\Omega \vec{v}^n_x \vec{\varphi}_{\ell x} dx + \int_\Omega v^n_k \vec{v}^n \vec{\varphi}_{\ell x_k} dx + æ\int_\Omega V_k(x,t)\left(\frac{\partial v^n_i}{\partial x_j} + \frac{\partial v^n_j}{\partial x_i}\right)\varphi_{\ell i x_j x_k} dx + \int_\Omega \vec{f}\vec{\varphi}_\ell dx, \tag{3.7}$$

and from these equalities, integrating with respect to $t$, from $t$ to $t+\Delta t \leq T$, applying the maximization operation, estimating the integrals on the right-hand side by Hölder's inequality, and making use of inequality (1.6) and of estimate (3.5), we obtain the inequality:

$$|\Phi_{n,\ell}(t+\Delta t) - \Phi_{n,\ell}(t)| \leq C_4(\ell, C_2, A_{\vec{V}})\left(\sqrt{\Delta t} + \int_t^{t+\Delta t} \|\vec{f}\|_{L_2(\Omega)} dt\right),\ n \geq \ell. \tag{3.8}$$

From the properties of the function $\vec{f}(x,t)$ there follows that, for a fixed $\ell$ and $n \geq \ell$, the right-hand side of the last equality tends to zero, as $\Delta t \to 0$, uniformly with respect to $n \geq \ell$.

The functions $\Phi_{n,\ell}(t)$, for a fixed $æ > 0$, represent (except for normalization) the inner products of $\vec{v}^n(x,t)$ and $\vec{\varphi}_\ell(x)$ in $W_2^1(\Omega)$, $0 \leq t \leq T$. By the diagonalization process, one can extract from $\{\Phi_{n,\ell}(t)\}$ a subsequence $\{\Phi_{n_m,\ell}(t)\}$, for which the functions $\Phi_{n_m,\ell}(t)$, for any fixed $\ell$, converge for $m \to \infty$ uniformly on $[0,T]$ to the functions $\varphi_\ell(t) \in C[0,T]$, $\ell = 1,2,\ldots$. In terms of the functions $\varphi_\ell(t)$, the limit function $\vec{v}(x,t)$ of $\{\vec{v}^{n_m}(x,t)\}$ is determined for all $t \in [0,T]$ as: $\vec{v}(x,t) = \sum_{\ell=1}^{\infty} \varphi_\ell(t)\vec{\varphi}_\ell(x)$, and for $m \to \infty$ $\vec{v}^{n_m}(x,t)$ converges to $\vec{v}(x,t)$ weakly in $W_2^1(\Omega)$ uniformly relative to $t \in [0,T]$: $(\vec{v}^{n_m} - \vec{v}, \vec{\varphi})_{W_2^1(\Omega)} \to 0$, $0 \leq t \leq T$, $\forall \vec{\varphi}(x) \in W_2^1(\Omega)$, and $\vec{v}(x,t)$ is continuous with respect to $t \in [0,T]$ in the weak topology of $W_2^1(\Omega)$. Then, with the aid of Friedrich's lemma one can easily show (see [7, Chap. VI]) that the approximations $\vec{v}^{n_m}(x,t)$ converge to $\vec{v}(x,t)$ strongly in $L_2(Q_T)$. In addition, for the limit function $\vec{v}(x,t)$, by virtue of the properties of weak convergence, we shall have the energy estimate (3.2). Finally, with the aid of known arguments one verifies (see [7, Chap. VI]) that the limit function $\vec{v}(x,t)$ satisfies also the integral identity (3.1), i.e., it is one of the "weak" generalized solutions of the IBV problem (6). Theorem 3.1 is proved.

Energy estimate (3.2) and estimates (3.6), (3.8), as well as the similar estimates for the weak solutions of the Navier–Stokes equations [7, Chap. VI], form the theoretical basis for the construction of stable explicit finite-difference schemes for the IBV problem (6).

Quasilinear equations (1) are obtained from the equations of the motion of a continuous medium in the Cauchy form

$$\frac{d\vec{v}}{dt}=\operatorname{div} T+\vec{f}(x,t),\ \operatorname{div}\vec{v}=0 \tag{3.9}$$

for the following determining equations, connecting the stress tensor $T(\vec{v},p)$ and the strain tensor $D(\vec{v})=(D_{ij}(\vec{v}))$, $D_{ij}(\vec{v})=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)$ (see [1, 5]):

$$T=-pE+2\nu D+2\varkappa\frac{dD}{dt}, \tag{3.10}$$

where $E$ is the identity matrix. If the determining equation is taken in the form

$$T=-pE+2\nu D+2\varkappa\frac{dD'}{dt}, \tag{3.11}$$

where $D'(\vec{v})=(D'_{ij}(\vec{v}))$, $D'_{ij}(\vec{v})=\frac{1}{2}\frac{\partial v_i}{\partial x_j}$ ,* then from (3.9) we obtain the quasilinear equations

$$\frac{\partial\vec{v}}{\partial t}-\nu\Delta\vec{v}+v_k\frac{\partial\vec{v}}{\partial x_k}-\varkappa\left\{\frac{\partial\Delta\vec{v}}{\partial t}+\frac{\partial}{\partial x_j}\left[v_k\frac{\partial^2\vec{v}}{\partial x_j\partial x_k}\right]\right\}+\operatorname{grad} p=\vec{f},\ \operatorname{div}\vec{v}=0, \tag{3.12}$$

more general than the equations of problem (2), where, however, the nonlinear terms which are additional to the Navier–Stokes equations have a specific divergent-type structure.

We shall solve Eqs. (3.12) in the cylinder $Q_T$ under the IBV conditions

$$\vec{v}\big|_{t=0}=\vec{v}_0(x),\ x\in\Omega;\ \vec{v}\big|_{\partial Q_T}=0 \tag{3.13}$$

and by a "weak" generalized solution of IBV problem (3.12), (3.13) we mean a function $\vec{v}(x,t)\in H(\Omega)$, $0\leq t\leq T$ , having a finite $\max\limits_{0\leq t\leq T}\|\vec{v}\|_{H(\Omega)}$ , weakly continuous with respect to $t\in[0,T]$ in $W_2^1(\Omega)$ and satisfying the integral identity

$$\iint\limits_{Q_T}\Big\{\vec{v}\vec{\Phi}_t+\varkappa\vec{v}_x\vec{\Phi}_{xt}-\nu\vec{v}_x\vec{\Phi}_x-v_k\vec{v}_{x_k}\vec{\Phi}+\varkappa v_k\frac{\partial\vec{v}}{\partial x_j}\frac{\partial^2\vec{\Phi}}{\partial x_j\partial x_k}+\vec{f}\vec{\Phi}\Big\}dQ-\int\limits_{\Omega}\big[\vec{v}_0\vec{\Phi}(x,0)+\varkappa\vec{v}_{0x}\vec{\Phi}_x(x,0)\big]dx=0 \tag{3.14}$$

*We note that since $\operatorname{div}\vec{v}=0$, we have $\operatorname{div}D(\vec{v})\equiv\left(\frac{\partial}{\partial x_j}D_{ij}(\vec{v})\right)=\operatorname{div}D'(\vec{v})=\frac{\Delta\vec{v}}{2}$.

for $\forall \vec{\Phi}(x,t) \in \overset{0}{W}{}_2^{2,1}(Q_T) \cap \overset{\circ}{J}(Q_T), \vec{\Phi}(x,T)=0$. Then, similar to Theorem 3.1 (see also [7, Chap. VI, Theorem 20]) one proves

THEOREM 3.2. Let $\vec{f}(x,t) \in L_2(Q_T)$, $\vec{v}_0(x) \in H(\Omega)$. Then the IBV problem (3.12), (3.13) has at least one "weak" generalized solution which can be obtained by Galerkin's method and for which we have the energy estimate:

$$\max_{0\le t\le T} \int_\Omega [\vec{v}^2(x,t) + ae\vec{v}_x^2] dx + \nu \iint_{Q_T} \vec{v}_x^2 dQ \le C_5(\|\vec{f}\|_{L_2(Q_T)}, \|\vec{v}_0\|_{H(\Omega)}, T) \tag{3.15}$$

The proof of Theorem 3.2 can be carried out since for the Galerkin approximations $\{\vec{v}^n(x,t)\}$ for the solutions of problem (3.12), (3.13) instead of equality (3.3) one has the equality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(\vec{v}^n)^2 + ae(\vec{v}_x^n)^2] dx + \nu \int_\Omega (\vec{v}_x^n)^2 dx = \int_\Omega \vec{f}\vec{v} dx, \quad 0<t\le T, \tag{3.16}$$

from which one obtains estimate (3.5), uniform with respect to $n=1,2,\dots$, with a constant $C_2$, depending only on $\|\vec{f}\|_{L_2(Q_T)}$, $\|\vec{v}_0\|_{H(\Omega)}$ and $T$. With the aid of this estimate, for the functions $\varphi_{n,\ell}(t)$, introduced in the proof of Theorem 3.1, satisfying in this case the equations [see (3.7)]:

$$\frac{d}{dt}\varphi_{n,\ell}(t) = -\nu\int_\Omega \vec{v}_x^n \vec{\varphi}_{\ell x} dx - \int_\Omega v_k^n \vec{v}_{x_k}^n \vec{\varphi}_\ell dx + ae\int_\Omega v_k \frac{\partial \vec{v}^n}{\partial x_j} \vec{\varphi}_{\ell x_j x_k} dx - \int_\Omega \vec{f}\vec{\varphi}_\ell dx, \tag{3.17}$$

we obtain, assuming that the elements $\vec{\varphi}_\ell(x), \ell=1,2,\dots,$ forming a basis in $H(\Omega)$, are sufficiently smooth so that $\max_\Omega |\vec{\varphi}_\ell, \vec{\varphi}_{\ell xxx}| < \infty$, inequalities of the form (3.6) and (3.8) where the constants $C_3$ and $C_4$ depend only on $\|\vec{f}\|_{L_2(Q_T)}$, $\|\vec{v}_0\|_{H(\Omega)}$, $T$, and $\ell$.

The inequalities of type (3.5), (3.6), (3.8) and Freidrichs' lemma [7, Chap. VI] allow us to conclude (see the proof of Theorem 3.1; see also [7, Chap. VI]) that from the totality $\{\vec{v}^n\}$ of Galerkin approximations for the solutions of problem (3.12), (3.13) one can extract a subsequence $\{\vec{v}^{n_m}\}$, which converges strongly in $L_2(Q_T)$ to the limit function $\vec{v}(x,t)$ and for which $\vec{v}_x^{n_m}$ converges weakly in $L_2(Q_T)$ to the derivative $\vec{v}_x$. After this, with the aid of the same arguments as in the proof of Theorem 3.1, one verifies that the limit function $\vec{v}(x,t)$ is weakly continuous with respect to $t \in [0,T]$ in $W_2^1(\Omega)$, satisfies the integral identity (3.14), and estimate (3.15) holds for it.

For sufficiently smooth solutions of IBV problem (3.12), (3.13), the energy estimate (3.15) is a consequence of the fact that for such solutions we have the equality:

$$\int_\Omega \vec{v}\frac{\partial}{\partial x_j}\left[v_k \frac{\partial^2 \vec{v}}{\partial x_j \partial x_k}\right] dx = -\frac{1}{2}\int_\Omega v_k \frac{\partial}{\partial x_k}\sum_{i,j}\left(\frac{\partial v_i}{\partial x_j}\right)^2 dx = 0 . \tag{3.18}$$

In conclusion, we note that if the determining equation is taken in the form

$$T=-pE+2\nu D+2\varkappa\frac{dD''}{dt}, \tag{3.19}$$

where $D''(\vec{v})=(D''_{ij}(\vec{v}))$, $D''_{ij}(\vec{v})=\frac{1}{2}\frac{\partial v_j}{\partial x_i}$ , then we obtain from (3.9) the equations

$$\frac{\partial v_i}{\partial t}-\nu\Delta v_i+v_k\frac{\partial v_i}{\partial x_k}-\varkappa\frac{\partial v_k}{\partial x_j}\frac{\partial^2 v_i}{\partial x_i\partial x_k}+\frac{\partial p}{\partial x_i}=f_i\,,\ i=1,2,3,\ \operatorname{div}\vec{v}=0. \tag{3.20}$$

## 4. Solvability in the Large of the Modified Initial- and Boundary-Value Problems of Heat Convection

In the present section we investigate the solvability of the IBV problem (7), where the heat-transfer equation is taken in the Boussinesq approximation

$$\frac{\partial S}{\partial t}-\chi\Delta S+v_k\frac{\partial S}{\partial x_k}=0, \tag{4.1}$$

and of more general IBV problems of heat convection in a viscous incompressible fluid, subjected to Eqs. (6), where the heat transfer takes into account partially (although in linearized form) of the dispersion of the energy in the fluid in the form of heat as a consequence of the kinematic and relaxational viscosity of the fluid*:

$$\begin{gathered}L_i(\vec{v},\vec{V};p)=f_i(x,t)+gS\gamma_i\,,\ i=1,2,3;\ \operatorname{div}\vec{v}=0\,,\ (x,t)\in Q_T,\\ \frac{\partial S}{\partial t}-\chi\Delta S+v_k\frac{\partial S}{\partial x_k}=\frac{\partial V_i}{\partial x_j}\Big(\nu+\varkappa V_k\frac{\partial}{\partial x_k}\Big)\Big(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\Big),\ (x,t)\in Q_T,\\ \vec{v}\big|_{t=0}=\vec{v}_0(x),\quad S\big|_{t=0}=S_0(x)\,,\ x\in\Omega;\ \vec{v}\big|_{\partial Q_T}=0\,,\ S\big|_{\partial Q_T}=0,\end{gathered} \tag{4.2}$$

where $\vec{V}$ is a known vector.

The merit of system (6), generated by the presence of the "regulating" term $-\varkappa\frac{\partial\Delta\vec{v}}{\partial t}$ and consisting in the fact that for $\vec{V}\equiv\vec{V}(x)$ in the general three-dimensional case there exists such a good ("strong") generalized solution of the IBV problem (6) as Ladyzhenskaya's solution of the similar initial-value problem for the Navier—Stokes equations, whose existence has been proved only in the small, is preserved also in the investigation of the solvability of the IBV problems of thermal convection (7) and (4.2). In addition, if $\vec{V}\equiv\vec{V}(x,t)$ , then for problems (7) and (4.2) one can prove, similarly as for problem (6), the existence in the large of a "weak" generalized solution.

By a "strong" generalized solution of the IBV problem (7), where the known vector $\vec{V}\equiv\vec{v}_0(x)\in W_2^3(\Omega)\cap H(\Omega)$ , we mean a pair of functions $\{\vec{v}(x,t),S(x,t)\}:\vec{v}(x,t)\in H(\Omega),\ S(x,t)\in\mathring{W}_2^1(\Omega)$,

*In connection with this we note that the equation of heat transfer in a viscous incompressible fluid, whose motion is described by Eqs. (1), has the form (see [11, Chap. V], [5]):

$$\frac{\partial S}{\partial t}-\chi\Delta S+v_k\frac{\partial S}{\partial x_k}=\frac{\partial v_i}{\partial x_j}\Big(\nu+\varkappa\frac{\partial}{\partial t}+\varkappa v_k\frac{\partial}{\partial x_k}\Big)\Big(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\Big).$$

$0 \le t \le T$, having derivatives $\vec{v}_t$, $\vec{v}_{xt}$, $S_t \in L_2(\Omega)$, $0 \le t \le T$, $S_{xt} \in L_2(Q_T)$, having a finite $\max\limits_{0 \le t \le T} \int\limits_\Omega \{\vec{v}_x^2 + \vec{v}_t^2 + \vec{v}_{xt}^2 + S_x^2 + S_t^2\} dx$, satisfying the integral identities

$$\iint\limits_{Q_T} \{\vec{v}_t \vec{\Phi} + \varkappa \vec{v}_{xt} \vec{\Phi}_x + \nu \vec{v}_x \vec{\Phi}_x + v_k \frac{\partial \vec{v}}{\partial x_k} - \varkappa v_{0k}(x)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\frac{\partial^2 \Phi_i}{\partial x_j \partial x_k}\} dQ = g \iint\limits_{Q_t} S \varphi_3 \, dQ +$$

$$+ \iint\limits_{Q_t} \vec{f} \vec{\Phi} \, dQ, \quad 0 < t \le T, \quad \forall \vec{\Phi}(x,t) \in \overset{\circ}{W}{}_2^{2,0}(Q_T) \cap \overset{\circ}{J}(Q_T), \tag{4.3}$$

$$M_t(S;\psi) \equiv \iint\limits_{Q_t} \{S_t \psi + \chi S_x \psi_x + v_k \frac{\partial S}{\partial x_k} \psi\} dQ = 0, \ 0 < t \le T, \ \forall \psi(x,t) \in \overset{\circ}{W}{}_2^{1,0}(Q_T) \tag{4.4}$$

and subjected to the initial conditions

$$\vec{v}\big|_{t=0} = \vec{v}_0(x), \quad S\big|_{t=0} = S_0(x), \quad x \in \Omega, \tag{4.5}$$

We prove that we have

THEOREM 4.1. Let $\vec{f}, \vec{f}_t \in L_2(Q_T)$, $\vec{v}_0(x) \in W_2^3(\Omega) \cap H(\Omega)$, $S_0(x) \in W_{2,0}^2(\Omega)$. Then IBV problem (7) has at least one "strong" generalized solution and this solution can be obtained by the Galerkin method and for it we have the following estimates: the energy inequality

$$\max_{0 \le t \le T} \int\limits_\Omega \{|\vec{v}(x,t)|^2 + \varkappa \vec{v}_x^2 + S^2\} dx + \iint\limits_{Q_T} (\nu \vec{v}_x^2 + \chi S_x^2) dQ \le C_1(\|\vec{f}\|_{L_2(Q_T)}, \max_{\bar{\Omega}} |\vec{v}_{0x}|, \|S_0\|_{L_2(\Omega)}); \tag{4.6}$$

in the three-dimensional case the estimate

$$\max_{0 \le t \le T} \int\limits_\Omega (\vec{v}_t^2 + \vec{v}_{xt}^2 + S_t^2) dx + \chi \iint\limits_{Q_T} S_{xt}^2 \, dQ \le C_2(\|\vec{f}, \vec{f}_t\|_{L_2(Q_T)}, \|\vec{v}_0\|_{W_2^3(\Omega)}, \|S_0\|_{W_2^2(\Omega)}, \varkappa), \tag{4.7}$$

where $C_2 \to \infty$ when $\varkappa \to 0$; in the two-dimensional case, uniformly with respect to $\varkappa \in [0,1]$ the estimate*

$$\max_{0 \le t \le T} \int\limits_\Omega \{\vec{v}_t^2 + \varkappa \vec{v}_{xt}^2 + S_t^2\} dx + \iint\limits_{Q_T} (\vec{v}_{xt}^2 + S_{xt}^2) dQ \le C_3(\|\vec{f}, \vec{f}_t\|_{L_2(Q_T)}, \|\vec{v}_0\|_{W_2^3(\Omega)}, \|S_0\|_{W_2^2(\Omega)}, \nu, \chi). \tag{4.8}$$

In order to prove the solvability of problem (7) we make use once again of the Galerkin method. Let $\{\vec{\varphi}_k(x)\}$, $k = 1, 2, \dots$ be a complete system of functions in $W_2^3(\Omega) \cap H(\Omega)$ and let $\{\psi_k(x)\}$, $k = 1, 2, \dots$, be a system of functions, complete in $W_{2,0}^2(\Omega)$, which will be assumed to be orthonormal in $L_2(\Omega)$. We shall seek the approximate solution $\{\vec{v}^n, S^n\}$ in the form

$$\vec{v}^n(x,t) = \sum_{k=1}^n C_{kn}(t) \vec{\varphi}_k(x), \quad S^n(x,t) = \sum_{k=1}^n \mathcal{D}_{kn}(t) \psi_k(x), \quad n = 1, 2, \dots, \tag{4.9}$$

*See the remark to Theorem 2.2.

from the system of differential equations [see more explicitly in (2.10)]

$$\int_\Omega \{L_i(\vec{v}^n,\vec{v}_0;\rho)-f_i\}\varphi_{i\ell}(x)dx = g\int_\Omega S^n\varphi_{3\ell}(x)dx\,,\ \ell=1,\dots,n,\ 0<t\leq T, \tag{4.10}$$

$$\int_\Omega S^n_t\psi_m(x)dx+\chi\int_\Omega S^n_{xt}\psi_{mx}dx-\int_\Omega v^n_k S^n\psi_{mx_k}dx=0\,,\ m=1,\dots,n,\ 0<t\leq T, \tag{4.11}$$

and the Cauchy initial conditions

$$C_{\ell n}\big|_{t=0}=\overset{\circ}{C}_{\ell n}\,,\ \mathcal{D}_{\ell n}\big|_{t=0}=\overset{\circ}{\mathcal{D}}_{\ell n}\ ,\ \ \ell=1,\dots,n, \tag{4.12}$$

where $\overset{\circ}{C}_{\ell n}$ and $\overset{\circ}{\mathcal{D}}_{\ell n}$ are the coefficients from the sequences of functions

$$\vec{v}_{(n)}(x)=\sum_{k=1}^{n}\overset{\circ}{C}_{kn}\vec{\varphi}_k(x)\ ,\ \ S_{(n)}(x)=\sum_{k=1}^{n}\overset{\circ}{\mathcal{D}}_{kn}\psi_k(x), \tag{4.13}$$

converging for $n\to\infty$ to the initial functions $\vec{v}_0(x)$ and $S_0(x)$ in the norms of $W_2^3(\Omega)$ and $W_2^2(\Omega)$ , respectively.

We proceed to obtain the a priori estimates for the Galerkin approximations $\{\vec{v}^n, S^n\}$ . For their derivation we rely heavily on the similar estimates obtained in Secs. 2, 3 for the Galerkin approximations $\vec{v}^n$ in the problem (6) and we shall indicate in detail only those (few) changes which are caused by the transfer of heat in the fluid.

First of all for $\{\vec{v}^n, S^n\}$ we obtain the equalities [see (2.12)]

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\left[|\vec{v}^n|^2+\varkappa|\vec{v}^n_x|^2\right]dx+\nu\int_\Omega|\vec{v}^n_x|^2dx=\int_\Omega\vec{f}\vec{v}^n dx-\varkappa\int_\Omega\frac{\partial v_{0k}}{\partial x_i}\frac{\partial v^n_i}{\partial x_j}\frac{\partial v^n_i}{\partial x_i}+g\int_\Omega S^n v^n_3 dx, \tag{4.14}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|S^n|^2dx+\chi\int_\Omega|S^n_x|^2dx=0\,,\ \ \int_\Omega v^n_k S^n S^n_{x_k}dx=0,\ \ 0<t\leq T, \tag{4.15}$$

from which, similar to inequality (2.14), we obtain the energy estimate

$$\max_{0\leq t\leq T}\int_\Omega\left\{|\vec{v}^n|^2+|S^n|^2+\varkappa|\vec{v}^n_x|^2\right\}dx+\iint_{Q_T}\left(\nu|\vec{v}^n_x|^2+\chi|S^n_x|^2\right)dQ\leq C_4\left(\|\vec{f}\|_{L_2(Q)},\max_{\bar\Omega}|\vec{v}_{0x}|,\|S_0\|_{L_2(\Omega)}\right), \tag{4.16}$$
$$n=1,2,\dots\ .$$

From inequality (4.16) and the orthonormality of the systems $\{\vec{\varphi}_k(x)\}$ and $\{\psi_k(x)\}$ in $L_2(\Omega)$ it follows that the Galerkin approximations $\{\vec{v}^n, S^n\}$ can be constructed for each $n=1,2,\dots$ on any finite time interval $[0,T]$ .

Then, with the aid of Eqs. (4.10) and (4.11) and of estimate (4.16) one shows [see (3.6)-(3.8)] that the functions $\Phi_{n,\ell}(t)=\int_\Omega\{\vec{v}^n(x,t)\vec{\varphi}_\ell(x)+\varkappa\vec{v}^n_x\vec{\varphi}_{\ell x}\}\,dx$ and $\omega_{n,\ell}(t)\equiv\int_\Omega S^n(x,t)\psi_\ell(x)dx$, $0\leq t\leq T$ , for fixed $\ell$ and $n\geq\ell$ , form a uniformly bounded and equicontinuous family of functions on $[0,T]$ :

$$\max_{0\leq t\leq T}|\Phi_{n,\ell}(t)|\leq C_5(\ell,C_4)\,,\ \ \max_{0\leq t\leq T}|\omega_{n,\ell}(t)|\leq C_4,\ \ n\geq\ell, \tag{4.17}$$

$$|\Phi_{n,\ell}(t+\Delta t)-\Phi_{n,\ell}(t)| \leq C_6\left(\ell, C_4, \max_{\bar{\Omega}}|\vec{v}_0|\right)\left(\sqrt{\Delta t}+\int_t^{t+\Delta t}\|\vec{f}\|_{L_2(\Omega)}dt\right),$$
$$|\omega_{n,\ell}(t+\Delta t)-\omega_{n,\ell}(t)| \leq C_7(\ell, C_4)\sqrt{\Delta t}, \quad n \geq \ell, \quad 0 \leq t \leq T. \tag{4.18}$$

Similar to inequality (2.18) is the estimate, uniform with respect to $n=1,2,\ldots$ and $\varkappa>0$,

$$\int_\Omega\left\{|\vec{v}_t^n(x,0)|^2+\varkappa|\vec{v}_{xt}^n|^2+|S_t^n(x,0)|^2\right\}dx \leq C_8\left(\|\vec{f}(x,0)\|_{L_2(\Omega)}, \|\vec{v}_0\|_{W_2^3(\Omega)}, \|S_0\|_{W_2^2(\Omega)}\right). \tag{4.19}$$

Then, similar to equality (2.19), one obtains the equality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left\{|\vec{v}_t^n(x,t)|^2+\varkappa|\vec{v}_{xt}^n|^2+|S_t^n|^2\right\}dx+\int_\Omega\left(\nu|\vec{v}_{xt}^n|^2+\chi|S_{xt}^n|^2\right)dx = \\
&=\int_\Omega\vec{f}_t\vec{v}_t^n dx+\int_\Omega v_{kt}^n\vec{v}_{x_k}^n\vec{v}_t^n dx-\varkappa\int_\Omega\frac{\partial v_{0k}}{\partial x_i}\frac{\partial v_{it}^n}{\partial x_j}\frac{\partial v_{it}^n}{\partial x_i}dx+ \\
&\quad+g\int_\Omega S_t^n v_{3t}^n dx+\int_\Omega v_{kt}^n S_{x_k}^n S_t^n dx, \quad 0<t\leq T, \ n=1,2,\ldots
\end{aligned} \tag{4.20}$$

(we note that $\int_\Omega v_k^n S_t S_{tx_k} dx=0$ ). In the right-hand side of equality (4.20) there are two new terms in comparison with equality (2.19). The first one can be estimated by Hölder's inequality, while in order to estimate the second term it is necessary to distinguish between the three- and two-dimensional domains $\Omega$ [see (2.20) and (2.24), however, this is necessary only if we follow the dependence of the constant on $\varkappa>0$ in the final inequality].

If $\Omega$ is a three-dimensional domain, then, estimating the last integral in the right-hand side of (4.20) with the aid of Hölder's and Cauchy's inequalities and inequality (1.6) and making use of estimate (4.16), we obtain

$$\left|\int_\Omega v_{kt}^n S_{x_k}^n S_t^n dx\right| \leq \frac{\chi}{2}\int_\Omega|S_{xt}^n|^2dx+\frac{C_\Omega^4 C_4}{2\chi\varkappa}\int_\Omega|\vec{v}_{xt}^n|^2dx. \tag{4.21}$$

After this we obtain from equalities (4.20) the estimate:

$$\begin{aligned}
&\max_{0\leq t\leq T}\int_\Omega\left\{|\vec{v}_x^n|^2+\varkappa|\vec{v}_{xt}^n|+|S_t^n|^2\right\}dx+\iint_{Q_T}\left(\nu|\vec{v}_{xt}^n|^2+\chi|S_{xt}^n|^2\right)dQ \leq \\
&\leq C_9\left(\|\vec{f}, \vec{f}_t\|_{L_2(Q_T)}, \|\vec{v}_0\|_{W_2^3(\Omega)}, \|S_0\|_{W_2^2(\Omega)}, \varkappa\right), \quad n=1,2,\ldots,
\end{aligned} \tag{4.22}$$

where $C_9 \to \infty$ for $\varkappa \to 0$ [see (2.22)].

If $\Omega$ is a two-dimensional domain, then, estimating the last integral in the right-hand side of the equality (4.20) with the aid of Hölder's and Cauchy's inequalities and of inequality (2.23), we obtain:

$$\left|\int_\Omega v^n_{kt} S^n_{x_k} S^n_t\,dx\right| \le \frac{\nu}{2}\int_\Omega |\vec{v}^n_{xt}|^2 dx + \frac{\chi}{2}\int_\Omega |S^n_{xt}|^2 dx + C_{10}(\nu,\chi)\int_\Omega |S^n_x|^2 dx \cdot \int_\Omega \left(|\vec{v}^n_t|^2 + |S^n_t|^2\right) dx, \tag{4.23}$$

and, by virtue of (4.16), $\int_\Omega |S^n_x|^2 dx \in L_1(0,T)$ . Then, from equalities (4.20) one obtains the estimate [see (2.27)]:

$$\max_{0\le t\le T}\int_\Omega \left(|\vec{v}^n_t|^2 + æ|\vec{v}^n_{xt}|^2 + |S^n_t|^2\right) dx + \iint_{Q_T}\left(|\vec{v}^n_{xt}|^2 + |S^n_{xt}|^2\right) dQ \le$$
$$\le C_{11}\left(\|\vec{f},\vec{f}_t\|_{L_2(Q_T)}, \|\vec{v}_0\|_{W^3_2(\Omega)}, \|S_0\|_{W^2_2(\Omega)}, \nu, \chi\right), \quad n=1,2,\dots, \tag{4.24}$$

where the constant $C_{11}$ remains bounded as $æ\to 0$.

Finally, with the aid of Eqs. (4.10) and (4.11), differentiated with respect to $t$, and of the estimates (4.22) or (4.24) one shows that the functions

$$\Psi_{n,\ell}(t) \equiv \int_\Omega \left\{\vec{v}^n_t(x,t)\vec{\varphi}_\ell(x) + æ\vec{v}^n_{xt}\vec{\varphi}_{\ell x}\right\} dx \quad\text{and}\quad \pi_{n,\ell}(t) \equiv \int_\Omega S^n_t(x,t)\psi_\ell(x)\,dx,\ 0\le t\le T,$$

form, for a fixed $\ell = 1,2,\dots$ and $n\le\ell$ , a uniformly bounded and equicontinuous family of functions on $[0,T]$ :

$$\max_{0\le t\le T}|\Psi_{n,\ell}(t)|,\quad \max_{0\le t\le T}|\pi_{n,\ell}(t)| \le C_{12}(\ell),\quad n=1,2,\dots \tag{4.25}$$

$$|\Psi_{n,\ell}(t+\Delta t) - \Psi_{n,\ell}(t)| \le C_{13}(\ell)\left(\sqrt{\Delta t} + \int_t^{t+\Delta t}\|\vec{f}_t\|_{L_2(\Omega)} dt\right),$$
$$|\pi_{n,\ell}(t+\Delta t) - \pi_{n,\ell}(t)| \le C_{14}(\ell)\sqrt{\Delta t},\quad n\ge\ell,\ 0\le t\le T, \tag{4.26}$$

where the constants $C_{12}$–$C_{14}$ depend on the constant $C_9$ if $\Omega$ is a three-dimensional domain and on the constant $C_{11}$ if $\Omega$ is a two-dimensional domain.

Estimates (4.16), (4.22), (4.25), (4.26) in the case of a three-dimensional domain $\Omega$ and estimates (4.16), (4.24)-(4.26) in the case of a two-dimensional domain $\Omega$ are sufficient in order to conclude, with the aid of the known arguments ([7, Chap. VI]; see also the end of the proof of Theorem 2.1), the proof of Theorem 4.1.

Moreover, estimates (4.16)-(4.18) remain valid, obviously, also in the case when $\vec{V}\equiv \vec{V}(x,t)\in \mathring{C}^{1,0}(\bar{Q}_T)\cap \mathring{J}(Q_T)$, $\vec{V}|_{t=0} = \vec{v}_0(x)$ , and allow us to prove, as in the case of the IBV problem (6), the existence of at least one "weak" generalized solution of the IBV problem (7), which, by analogy with problem (6), we define as a pair of functions $\{\vec{v}(x,t), S(x,t)\}$: $\vec{v}(x,t)\in H(\Omega)$, $S(x,t)\in L_2(\Omega)$, $0\le t\le T$ , having a finite $\max_{0\le t\le T}\|\vec{v}\|_{H(\Omega)}$ and $\iint_{Q_T} S_x^2\,dQ$ , respectively, and satisfying the integral identities

$$\iint_{Q_T}\left\{\vec{v}\vec{\Phi}_t + æ\vec{v}_x\vec{\Phi}_{xt} - \nu\vec{v}_x\vec{\Phi}_x + v_k\vec{v}\vec{\Phi}_{x_k} + æV_k(x,t)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\frac{\partial^2 \Phi_i}{\partial x_j\partial x_k}\right\} dQ -$$

$$-\int_\Omega \{\vec{v}_0(x)\vec{\Phi}(x,0)+æ\vec{v}_{0x}\vec{\Phi}_x(x,0)\}dx=-\iint_{Q_T}\vec{f}\vec{\Phi}dQ-g\iint_{Q_T}S\Phi_3 dQ,$$

$$\forall\vec{\Phi}(x,t)\in \overset{\circ}{W}{}_2^{2,1}(Q_T)\cap \overset{\circ}{J}(Q_T)\ ,\quad \vec{\Phi}(x,T)=0, \tag{4.27}$$

$$\iint_{Q_T}\{S\psi_t-\chi S_x\psi_x+v_k S\psi_{x_k}\}dQ=\int_\Omega S_0(x)\psi(x,0)dx,\ \forall\psi(x,t)\in \overset{\circ}{W}{}_2^{2,1}(Q_T),\ \psi(x,T)=0 \tag{4.28}$$

As we have just mentioned, one has the

THEOREM 4.2. Let $\vec{f}(x,t)\in L_2(Q_T)$, $\vec{v}_0(x)\in H(\Omega)$, $S_0(x)\in L_2(\Omega)$. Then IBV problem (7) has at least one "weak" generalized solution, whose solution can be obtained by the Galerkin method and for it we have the energy inequality:

$$\max_{0\le t\le T}\int_\Omega\left(|\vec{v}(x,t)|^2+æ\vec{v}_x^{\,2}+S^2\right)dx+\iint_{Q_T}(\nu\vec{v}_x^{\,2}+\chi S_x^2)\,dQ\le$$

$$\le C_{15}\left(\|\vec{f}\|_{L_2(Q_T)},\|\vec{v}_0\|_{H(\Omega)},\|S_0\|_{L_2(\Omega)},\ \max_{\bar{Q}_T}|\vec{V}_x|\right). \tag{4.29}$$

By a "strong" generalized solution of IBV problem (4.2), where the known vector $\vec{V}\equiv \vec{v}_0(x)\in W_2^3(\Omega)\cap H(\Omega)$ and $\max_{\Omega}|\vec{v}_{0xxx}|<\infty$, we mean a pair of functions $\{\vec{v}(x,t),S(x,t)\}$, having the same differential properties as the "strong" generalized solution of problem (7), satisfying integral identity (4.3) and the integral identity

$$M_t(S,\psi)=\iint_{Q_t}\left\{\nu\frac{\partial v_{0i}}{\partial x_j}\psi-æv_{0k}\left(\frac{\partial^2 v_{0i}}{\partial x_j\partial x_k}\psi+\frac{\partial v_{0i}}{\partial x_j}\frac{\partial\psi}{\partial x_k}\right)\right\}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)dQ,\ 0<t\le T,\ \forall\psi(x,t)\in \overset{\circ}{W}{}_2^{1,0}(Q_T) \tag{4.30}$$

and subjected to initial conditions (4.5). Similarly to Theorem 4.1 one proves

THEOREM 4.3. Let $\vec{f},\vec{f}_t\in L_2(Q_T)$, $\vec{v}_0(x)\in W_2^3(\Omega)\cap H(\Omega)$ and $\max_{\Omega}|\vec{v}_{0xx}|<\infty$, $S_0(x)\in \overset{\circ}{W}{}_2^2(\Omega)$. Then, IBV problem (4.2) has at least one "strong" generalized solution $\{\vec{v},S\}$, this solution can be obtained by the Galerkin method, and for it we have estimates (4.6)-(4.8), where the constants $C_1-C_3$ depend also on $\max_{\bar{\Omega}}|\vec{v}_{0xx}|$.

In conclusion, we note that for IBV problem (7) [and also for problem (4.2)], where the known vector $\vec{V}(x,t)\in C^{1,0}(\bar{Q}_T)\cap \overset{\circ}{J}(Q_T)$, one has the uniqueness of a "good" generalized solution which is defined as a pair of functions $\{\vec{v}(x,t),S(x,t)\}$, subjected to the following conditions: $\vec{v}(x,t)\in H(\Omega), 0\le t\le T$, it has a finite $\max_{0\le t\le T}\|\vec{v}\|_{H(\Omega)}$ and has derivatives $\vec{v}_{xx}\in L_2(Q_T)$; $S(x,t)\in V_2(Q_T)\cap L_{q,\tau}(Q_T)$ with $q$ and $\tau$, satisfying one of the conditions (see [7, Chap. VI]):

$$\frac{1}{\tau}+\frac{n}{2q}=\frac{1}{2},\ \tau\in[2,\infty);\quad q\in(n,\infty]\quad \text{or}\quad q>n,\ \tau=\infty,\ n=2,3; \tag{4.31}$$

and satisfies the integral identities

$$\iint\limits_{Q_t}\{\vec{v}\vec{\Phi}_t+\varkappa\vec{v}_x\vec{\Phi}_{xt}-\nu\vec{v}_x\vec{\Phi}_x-v_k\vec{v}_{x_k}\vec{\Phi}-\varkappa V_k(x,t)\frac{\partial}{\partial x_k}\Big(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\Big)\Phi_{ix_j}+\vec{f}\vec{\Phi}+g\,S\Phi_3\}dQ- \tag{4.32}$$

$$-\iint\limits_{\Omega}\{\vec{v}(x,t)\vec{\Phi}(x,t)+\varkappa\vec{v}_x\vec{\Phi}_x\}dx+\int\limits_{\Omega}\{\vec{v}_0(x)\vec{\Phi}(x,0)+\varkappa\vec{v}_{0x}(x)\vec{\Phi}_x\}dx=0,\quad 0<t\leq T,$$

$$\iint\limits_{Q_t}\big(S\psi_t-\chi S_x\psi_x+v_k S\psi_{x_k}\big)dQ-\int\limits_{\Omega}S(x,t)\psi(x,t)dx+\int\limits_{\Omega}S_0(x)\psi(x,0)dx=0,\ 0<t\leq T, \tag{4.33}$$

for any $\vec{\Phi}(x,t)\in H(\Omega), 0\leq t\leq T$, and such that $\vec{\Phi}_t, \vec{\Phi}_{xt}\in L_2(\Omega), 0\leq t\leq T$, and for any $\psi(x,t)\in \mathring{W}_2^{2,1}(Q_T)$.

The proof of this statement is obtained by combining the proof of Theorem 1.1 with Ladyzhenskaya's method of the proof of the uniqueness of the generalized solution from $V_2(Q_T)\cap L_{q,\tau}(Q_T)$, with $q$ and $\tau$ subjected to one of conditions (4.31), for the IBV problem for the nonstationary Navier—Stokes equations (see [7, Chap. VI]).

For the modified equations of magnetohydrodynamics (8), the presence in the equations of the motion of the fluid of the "regulating" term $-\varkappa\frac{\partial\Delta\vec{v}}{\partial t}$ does not cause the effect which it gives in the investigation of the solvability of the IBV problems (6) and (7). The "strong" generalized solutions of IBV problem (8), i.e., the solution $(\vec{v},\vec{H})$, which has derivatives $\vec{v}_x, \vec{v}_{tx}, \vec{H}_t\in L_2(\Omega), 0\leq t\leq T$, and $\vec{H}_{xt}\in L_2(Q_T)$, in the three-dimensional case exists only in the small, while in the large there exists a "strong" generalized solution in the two-dimensional case (more precisely, the two-dimensionality, i.e., the vanishing of one of the components and the dependence of the remaining two components only on two space variables, is required only from the vector of magnetic intensity $\vec{H}$), and in the three-dimensional case there exists a "weak" generalized solution (solution of Hopf type), i.e., a solution $(\vec{v},\vec{H})$, possessing derivatives $\vec{v}_x\in L_2(\Omega)$, $0\leq t\leq T$, $\vec{H}_x\in L_2(Q_T)$ and such that $\vec{v}$ is weakly continuous with respect to $t\in[0,T]$ in $W_2^1(\Omega)$, while $\vec{H}(x,t)$ is weakly continuous in $L_2(\Omega)$. All these statements are proved basically in the same way as the corresponding results for the standard equations of magnetohydrodynamics [12]. Small necessary variations in the arguments occur in Secs. 2 and 4.

## 5. A Priori Estimates for the Solutions of Problems (9) and (10)

Let $\Omega$ be a two-dimensional domain with a smooth boundary $\partial\Omega$, situated in the half-plane $\{\tau,z:\tau>0\}$ at a distance $z=0$ from the axis $\delta>0$, $Q_T=\Omega\times(0,T)$, $0<T<\infty$. By the generalized solution of IBV problem (9) we mean a function $\psi(\tau,z,t)$, which together with all the derivatives occurring in Eq. (9), belongs to $L_2(Q_T)$ and satisfies Eq. (9) almost everywhere in $Q_T$. We show that we have

THEOREM 5.1. Let $F(\tau,z,t)\in L_2(Q_T)$, $\psi_0(\tau,z)\in W_2^4(\Omega)$, $\psi_0|_{\partial\Omega}=\frac{\partial\psi_0}{\partial n}|_{\partial\Omega}=0$. Then for any generalized solution of problem (9) we have the estimates:

$$\max_{0\le t\le T}\int_\Omega[|\nabla\psi|^2+\varkappa(\psi_{rr}^2+\psi_{rz}^2+\psi_{zz}^2)]d\Omega+\nu\iint_{Q_T}(\psi_{rr}^2+\psi_{rz}^2+\psi_{zz}^2)dQ\le C_1(\delta,\|F\|_{L_2(Q_T)},\|\psi_0\|_{W_2^2(\Omega)}), \tag{5.1}$$

$$\max_{0\le t\le T}\|\psi\|_{W_2^4(\Omega)}\le C_2(\delta,\nu,\varkappa,T,\|F\|_{L_2(Q_T)},\|\psi_0\|_{W_2^4(\Omega)}), \tag{5.2}$$

$$\iint_{Q_T}\{\psi_t^2+|\nabla\psi_t|^2+\psi_{trr}^2+\psi_{trz}^2+\psi_{tzz}^2\}dQ\le C_3(\delta,\nu,\varkappa,C_1,C_2), \tag{5.3}$$

where $C_2$ and $C_3\to\infty$ for $\varkappa\to0$ and $\delta\to0$.

In order to prove estimate (5.1) we multiply Eq. (9) by $\psi$ and we integrate over $Q_t$, $0<t\le T$. Then, making use of the fact that the integral containing the nonlinear terms is equal to zero:

$$\iint_{Q_t}\psi\left\{\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\frac{D\psi-\varkappa D^2\psi}{r}-\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\frac{D\psi-\varkappa D^2\psi}{r}\right\}dQ=0, \tag{5.4}$$

integrating with respect to time and integrating by parts with respect to the space variables, we obtain the equality (we note that $D\psi\equiv\Delta\psi-\frac{1}{r}\frac{\partial\psi}{\partial r}$ ):

$$\frac{1}{2}\int_\Omega\left(|\nabla\psi|^2+\frac{\psi}{r}\frac{\partial\psi}{\partial r}\right)d\Omega\Big|_{t=0}^{t=t}+\frac{\varkappa}{2}\int_\Omega D\psi\left(\Delta\psi-\frac{\partial}{\partial r}\frac{\psi}{r}\right)d\Omega\Big|_{t=0}^{t=t}+\nu\iint_{Q_t}D\psi\left(\Delta\psi-\frac{\partial}{\partial r}\frac{\psi}{r}\right)dQ=-\iint F\psi dQ \tag{5.5}$$

from which, applying Hölder's inequality, the second fundamental inequality for the Laplace operator [7], and Gronwall's lemma, we obtain estimate (5.1).

In order to prove estimate (5.2) we multiply Eq. (9) by $\frac{1}{r}(D\psi-\varkappa D^2\psi)$ and we integrate over $Q_t, 0<t\le T$. Making use of the fact that the integral containing the nonlinear terms is once again equal to zero:

$$\iint_{Q_t}\frac{D\psi-\varkappa D^2\psi}{r}\left\{\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\frac{D\psi-\varkappa D^2\psi}{r}-\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\frac{D\psi-\varkappa D^2\psi}{r}\right\}dQ=0 \tag{5.6}$$

and integrating with respect to time, we obtain the equality:

$$\frac{1}{2}\int_\Omega\frac{(D\psi-\varkappa D^2\psi)^2}{r}d\Omega\Big|_{t=0}^{t=t}+\nu\varkappa\iint_{Q_t}\frac{(D^2\psi)^2}{r}dQ=\nu\iint_{Q_t}\frac{D\psi D^2\psi}{r}dQ+\iint_{Q_t}\frac{F}{r}(D\psi-\varkappa D^2\psi)dQ \tag{5.7}$$

from which, applying Hölder's inequality, the second fundamental inequality for the biharmonic operator [7, Chap. I], and Gronwall's lemma, we obtain estimate (5.2).

In order to prove estimate (5.3) we multiply Eq. (9) by $\psi_t$ and we integrate over $Q_t$, $0<t\leq T$. Integrating by parts we obtain the equality:

$$\iint_{Q_t}\left(|\nabla\psi_t|^2+\frac{\psi_t}{\tau}\frac{\partial\psi_t}{\partial\tau}\right)dQ+\varkappa\iint_{Q_t}D\psi_t\left(\Delta\psi_t-\frac{\partial}{\partial\tau}\frac{\psi_t}{\tau}\right)dQ+\nu\iint_{Q_t}\psi_t D^2\psi\,dQ+$$
$$+\iint_{Q_t}\frac{D\psi-\varkappa D^2\psi}{\tau}\left(\frac{\partial\psi}{\partial\tau}\frac{\partial\psi_t}{\partial\tau}-\frac{\partial\psi}{\partial\tau}\frac{\partial\psi_t}{\partial z}\right)dQ=-\iint_{Q_t}F\psi_t\,dQ, \tag{5.8}$$

from which, applying Hölder's inequality, the second fundamental inequality for the Laplace operator, and estimate (5.2), we obtain estimate (5.3).

By the generalized solution of boundary-value problem (10) we mean a function $\psi(\tau,z)$, which belongs, together with all the derivatives occurring in Eq. (10), to $L_2(\Omega)$ and satisfies Eq. (10) almost everywhere in the domain $\Omega$. Similarly to Theorem 5.1, one proves

THEOREM 5.2. Let $F(\tau,z)\in L_2(\Omega)$. Then for any generalized solution of problem (10) we have the estimates:

$$\|\psi\|_{W_2^2(\Omega)}\leq C_4(\delta,\nu,\|F\|_{L_2(\Omega)}), \tag{5.9}$$

$$\|\psi\|_{W_2^4(\Omega)}\leq C_5(\delta,\nu,\varkappa,\|F\|_{L_2(\Omega)}), \tag{5.10}$$

where $C_5\to\infty$ for $\varkappa\to 0$ and $\delta\to 0$.

It is easy to verify (see also [8]) that the generalized solution of IBV problem (9) considered in Theorem 5.1 is unique (in the large), while the generalized solution of boundary-value problem (10) considered in Theorem 5.2 is unique in the small.

LITERATURE CITED

1. Ya. I. Voitkunskii, V. B. Amfilokhiev, and V. A. Pavlovskii, "The equations of motion of a fluid taking into account its relaxational properties," Tr. Leningr. Korablestr. Inst., No. 69, 16-24 (1970).
2. V. A. Pavlovskii, "On the problem of the theoretical description of weak aqueous solutions of polymers," Dokl. Akad. Nauk SSSR, 200, No. 4, 809-813 (1971).
3. A. P. Oskolkov, "Solvability in the large of the first boundary-value problem for a certain quasilinear third-order system that is encountered in the study of the motion of a viscous fluid," Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst., Akad. Nauk SSSR, 27, 145-160 (1972).
4. A. P. Oskolkov, "On the uniqueness and global solvability of the first boundary-value problem for systems of equations describing the motion of an aqueous solution of polymers," Tr. Leningr. Korablestr. Inst., No. 80, 44-48 (1972).
5. A. P. Oskolkov, "The uniqueness and solvability in the large of boundary-value problems for the equations of motion of aqueous solutions of polymers," Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst., Akad. Nauk SSSR, 38, 98-136 (1973).

6. A. P. Oskolkov, "On some model nonstationary systems in the theory of non-Newtonian fluids," Tr. Mat. Inst., Akad. Nauk SSSR, 127 (1975).
7. O. A. Ladyzhenskaya, Mathematical Theory of Viscous Incompressible Flow, Gordan and Breach, New York (1969).
8. A. P. Oskolkov, "On certain two-dimensional degenerate quasilinar equations that arise in the theory of non-Newtonian fluids," in: Probl. Mat. Anal., No. 5, Izd. Leningr. Univ. (1975).
9. A. P. Oskolkov, "A certain nonstationary quasilinear system with a small parameter that regularizes the system of Navier–Stokes equations," in: Probl. Mat. Anal., No. 4, Izd. Leningr. Univ. (1973), pp. 78-87.
10. O. A. Ladyzhenskaya and N. N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York (1968).
11. L. D. Landau and E. M. Lifshits, Mechanics of Continuous Media, Pergamon.
12. O. A. Ladyzhenskaya and V. A. Solonnikov, "Solution of some nonstationary problems of magnetohydrodynamics for a viscous incompressible fluid," Tr. Mat. Inst., Akad. Nauk SSSR, 59, 115-173 (1960).

# ADMISSIBLE GROUPS OF TRANSFORMATIONS FOR CERTAIN THIRD-ORDER QUASILINEAR EQUATIONS

A. P. Oskolkov

UDC 517.9

We construct the infinitesimal operators of Lie algebras for the broadest group of transformations leaving invariant some quasilinear third-order equations with two independent variables occurring in the mechanics of continuous media.

With the aid of the well-known procedure of continuation theory [1-3] we find the admissible groups of transformations for some third-order partial differential equations. In order to obtain these results it has been required to construct the third extension $\overset{\equiv}{X}$ of an operator.

1. The broadest group of transformations admissible for the equation

$$u_t - \nu u_{xx} - \varkappa u_{txx} = 0, \quad \nu, \varkappa \equiv \mathrm{const} > 0, \tag{1}$$

is infinite-dimensional. Its Lie algebra is generated by the infinitesimal operator

$$X = a_1 \frac{\partial}{\partial t} + a_2 \frac{\partial}{\partial x} + a_3 u \frac{\partial}{\partial u} + a_4(x,t) \frac{\partial}{\partial u}, \tag{2}$$

where $a_1, a_2, a_3$ are arbitrary constants while $a_4(x,t)$ is an arbitrary smooth solution of Eq. (1).

2. The broadest group of transformations admissible for the equations

$$u_t + u u_x - \nu u_{xx} - \varkappa (u_{txx} + u u_{xxx}) = 0, \tag{3}$$

$$u_t + u u_x - \nu u_{xx} - \varkappa (u_{txx} + u u_{xxx} + u_x u_{xx}) = 0, \tag{4}$$

$$u_t + u u_x + \alpha u_{xxx} = 0, \quad \alpha \equiv \mathrm{const} > 0, \tag{5}$$

has three parameters. Its Lie algebra is generated by the infinitesimal operator

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR, Vol. 52, pp. 158-159, 1975.

 0096-4104/78/0905-0790$07.50 