THE UNIQUENESS AND GLOBAL SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR THE EQUATIONS OF MOTION FOR AQUEOUS SOLUTIONS OF POLYMERS

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It has been experimentally established [1] that if we introduce in a viscous fluid, which moves around a body, a very small (up to a fraction of a hundredth percent) amount of special polymer substances, without changing the density and the viscosity of the fluid, then the motion of the fluid in a boundary layer will be significantly affected and the friction resistance of the moving body will decrease. The comparison of the physical characteristics of water and weak aqueous solutions of polymers [2-3] show that for practically identical values of the density and viscosity, these fluids differ sharply in their relaxational properties – the relaxational processes in the polymer solution being very slow in comparison with those in water.

It is known (see, e.g., [4]) that Newton's equation, connecting the stress tensor T, the strain tensor $D(\vec{v}) = (D_{ik}(\vec{v}))$, $D_{ik}(\vec{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$ and the pressure p for the motion of a viscous incompressible fluid has the form:

$$T = -pE + 2\nu D,$$

where ν is the kinematic viscosity coefficient. This relation has been obtained under the assumption that the fluid does not possess relaxation properties, i.e., it returns instantly to its initial state as soon as the exterior stresses applied to it have been removed. In [2-3] one suggests to take into account the relaxation properties of aqueous solutions of polymers with the aid of the following modification of the governing equation:

$$T = -pE + 2\nu D + 2\varkappa \frac{dD}{dt}, \tag{1}$$

where $\varkappa$ is the relaxational viscous coefficient and $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\,\mathrm{grad}$ is the Stokes derivative. Inserting the stress tensor (2) into the equation of the motion of a continuous medium, written in the Cauchy form

$$\frac{d\vec{v}}{dt} = \mathrm{div}\, T + \vec{f}, \tag{2}$$

we obtain the system of equations

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$$\frac{\partial v_i}{\partial t}-\nu\Delta v_i+v_k\frac{\partial v_i}{\partial x_k}-\varkappa\left(\frac{\partial \Delta v_i}{\partial t}+v_k\frac{\partial \Delta v_i}{\partial x_k}\right)-\varkappa\sum_{j,k=1}^{n}\frac{\partial v_j}{\partial x_k}\left(\frac{\partial^2 v_i}{\partial x_j \partial x_k}+\right.$$
$$\left.+\frac{\partial^2 v_k}{\partial x_i \partial x_j}\right)+\frac{\partial p}{\partial x_i}=f_i,\quad i=1,\ \ldots,n,\qquad \operatorname{div}\vec{v}=0, \tag{3}$$

and from this system, neglecting the terms containing the products of the first and second derivatives of $\vec{v}(x,t)$ with respect to the space derivatives since they are small compared to the velocities and deformations, we obtain the following system [2, 3]:

$$\frac{\partial \vec{v}}{\partial t}-\nu\Delta\vec{v}+v_k\frac{\partial \vec{v}}{\partial x_k}-\varkappa\left(\frac{\partial \Delta\vec{v}}{\partial t}+v_k\frac{\partial \Delta\vec{v}}{\partial x_k}\right)+\operatorname{grad}=\vec{f},\quad \operatorname{div}\vec{v}=0, \tag{4}$$

which will be the main object of investigation in the present paper. For $\varkappa=0$ systems (3) and (4) reduce to the system of the Navier–Stokes equations

$$\frac{\partial v}{\partial t}-\nu\Delta\vec{v}+v_k\frac{\partial \vec{v}}{\partial x_k}+\operatorname{grad} p=\vec{f},\quad \operatorname{div}\vec{v}=0. \tag{5}$$

If we introduce a new unknown function $\vec{u}=\varkappa\Delta\vec{v}-\vec{v}$, then system (4) can be rewritten in the following form:

$$\frac{\partial \vec{u}}{\partial t}+v_k\frac{\partial \vec{u}}{\partial x_k}+\frac{\nu}{\varkappa}\vec{u}+\frac{\nu}{\varkappa}\vec{v}-\operatorname{grad} p=-\vec{f},\quad \varkappa\Delta\vec{v}-\vec{v}=\vec{u},\quad \operatorname{div}\vec{v}=0. \tag{6}$$

2. Let Ω be a bounded domain in the two- or three-dimensional Euclidean space, let $\partial\Omega$ be the boundary of the domain Ω, $Q_T=\Omega\times[0,T]$, $0<T<\infty$, $\partial Q_T=\partial\Omega\times[0,T]$ the lateral surface of the cylinder Q_T, $\Omega_t\equiv\Omega$ the cross section of the cylinder Q_T by the plane $t=\mathrm{Const}$, $0\le t\le T$.

We shall consider in the cylinder Q_T the solution of system (4), satisfying the initial–boundary condition

$$\vec{v}\big|_{t=0}=\vec{v}_0(x),\ x\in\Omega;\quad \vec{v}\big|_{\partial Q_T}=0. \tag{7}$$

In Secs. 2-4 we shall prove uniqueness theorems of the solution of the initial–boundary-value problem (4), (7), which is the main object of investigation of the present paper, and also of the initial–boundary-value problem (3), (7) and of the boundary-value problems for the corresponding systems (3) and (4) of a stationary system.

By a generalized solution of the initial–boundary-value problem (4), (7) we mean a function $\vec{v}(x,t)\in\mathring{J}(Q_T)$ which has generalized derivatives $\vec{v}_x,\vec{v}_t,\vec{v}_{xx},\vec{v}_{xt},\vec{v}_{xxx}\in L_2(Q_T)$, satisfies the integral identity

$$\iint_{Q_t}[\vec{v}_t\vec{\Phi}+\nu v_x\vec{\Phi}_x+\varkappa\vec{v}_{xt}\vec{\Phi}_x+v_k\frac{\partial}{\partial x_k}(\vec{v}-\varkappa\Delta\vec{v})\vec{\Phi}]\,dQ=\iint_{Q_t}\vec{f}\vec{\Phi}\,dQ,\ 0<t\le T, \tag{8}$$

for any $\vec{\Phi}(x,t)\in\mathring{J}(Q_T)$, $\vec{\Phi}\big|_{\partial Q_T}=0$, having the same differential properties as $\vec{v}(x,t)$, and satisfies the initial–boundary conditions (7). It is easy to see that if $\vec{f}(x,t)\in L_{2,1}(Q_T)$, then all the integrals which occur in identity (8) make sense.

First of all we prove that we have the following theorem.

<u>THEOREM 1.</u> The generalized solution $\vec{v}(x,t)$ of the initial-boundary-value problem (4), (7), possessing in Q_T bounded derivatives $\vec{v}_x, \vec{v}_{xx}$, is unique.

Indeed, if problem (4), (7) has two solutions $\vec{v}_1$ and $\vec{v}_2$, possessing the properties indicated in Theorem 1, then their difference $\vec{\omega} \equiv \vec{v}_1 - \vec{v}_2$ satisfies the integral identity

$$\iint_{Q_t} \left[\vec{\omega}_t \vec{\Phi} + \nu \vec{\omega}_x \vec{\Phi}_x + \varkappa \vec{\omega}_{xt} \vec{\Phi}_x + v_{1k} \frac{\partial}{\partial x_k}\left(\vec{\omega} - \varkappa \Delta \vec{\omega}\right)\vec{\Phi} + \omega_k \frac{\partial}{\partial x_k}(\vec{v}_2 - \varkappa \Delta \vec{v}_2)\vec{\Phi}\right] dQ = 0, \tag{9}$$

and, in addition,

$$\vec{\omega}\big|_{t=0} = 0 \,, \quad \vec{\omega}\big|_{\partial Q_T} = 0. \tag{10}$$

Taking in (9) $\vec{\Phi} \equiv \vec{\omega}$, integrating by parts, and making use of (10), we obtain:

$$\frac{1}{2}\left(\|\vec{\omega}\|^2_{L_2(\Omega_t)} + \varkappa \|\vec{\omega}_x\|^2_{L_2(\Omega_t)}\right) + \nu \|\vec{\omega}_x\|^2_{L_2(Q_t)} - \iint_{Q_t} \left[\omega_k \frac{\partial \vec{\omega}}{\partial x_k}(\vec{v}_2 - \varkappa \Delta \vec{v}_2) + \varkappa \frac{\partial v_{1k}}{\partial x_i} \frac{\partial \omega_i}{\partial x_i} \frac{\partial \omega_i}{\partial x_k}\right] dQ = 0, \tag{11}$$

and from here, applying Hölder's inequality and making use of the boundedness of $\vec{v}_x$, $\vec{v}_{xx}$ in Q_T, we shall have:

$$\|\vec{\omega}\|^2_{L_2(\Omega_t)} + \varkappa \|\vec{\omega}_x\|^2_{L_2(\Omega_t)} \leq C_1(\varkappa, \vec{v}_1, \vec{v}_2)\left(\|\vec{\omega}\|^2_{L_2(Q_t)} + \varkappa \|\vec{\omega}_x\|^2_{L_2(Q_t)}\right), \quad 0 < t \leq T. \tag{12}$$

From the last inequality, making use of Gronwall's lemma [5, Chap. VI] and of (10), we obtain $\vec{\omega}(x,t) \equiv 0$, $(x,t) \in Q_T$.

The stationary system, corresponding to system (4), has the following form:

$$\nu \Delta \vec{v} - v_k \frac{\partial}{\partial x_k}(\vec{v} - \varkappa \Delta \vec{v}) + \operatorname{grad} p = \vec{f}(x), \quad \operatorname{div} \vec{v} = 0, \quad x \in \Omega. \tag{13}$$

We shall consider the solution of system (13) in the bounded domain Ω, satisfying the boundary condition

$$\vec{v}\big|_{\partial\Omega} = 0. \tag{14}$$

By a generalized solution of the boundary-value problem (13), (14) we mean a function $\vec{v}(x) \in H(\Omega)$ which has generalized derivatives $\vec{v}_{xx}$, $\vec{v}_{xxx} \in L_2(\Omega)$ and satisfies the integral identity

$$\int_\Omega \left[\nu \vec{v}_x \vec{\Phi}_x + v_k \frac{\partial}{\partial x_k}(\vec{v} - \varkappa \Delta \vec{v})\vec{\Phi}\right] dx = -\int_\Omega \vec{f} \vec{\Phi} \, dx \tag{15}$$

for any function $\vec{\Phi}(x)$ possessing the same properties as $\vec{v}(x)$.

We introduce the notation [5, Chap. VI]:

$$C^*_\Omega \equiv \max_{\vec{v} \in H(\Omega)} \left(\frac{\int_\Omega \vec{v}^2 dx}{\int_\Omega \vec{v}_x^2 \, dx}\right)^{1/2}$$

and we prove that, under well-defined conditions relative to the smallness of the data of the problem (13), (14), we have the following theorem.

THEOREM 2. The generalized solution $\vec{v}(x)$ of the boundary-value problem (13), (14), possessing in Ω bounded derivatives $\vec{v}_x$, $\vec{v}_{xx}$ and such that

$$\varkappa \max_{\Omega} |\vec{v}_x| + C^*_\Omega \max_{\Omega} |\vec{v} - \varkappa \Delta \vec{v}| < \nu, \tag{16}$$

is unique.

Indeed, for the difference $\vec{\omega}(x)$ of two possible solutions of problem (13), (14) we have the equality

$$\nu \|\vec{\omega}_x\|^2_{L_2(\Omega)} - \int_\Omega \omega_k \frac{\partial \vec{\omega}}{\partial x_k} (\vec{v}_2 - \varkappa \Delta \vec{v}_2) dx - \varkappa \int_\Omega \frac{\partial v_{1k}}{\partial x_i} \frac{\partial \omega_j}{\partial x_i} \frac{\partial \omega_j}{\partial x_k} dx = 0, \tag{17}$$

from which, applying Hölder's inequality for the estimate for the second integral and making use of the boundedness of $\vec{v}_x$ and $\vec{v}_{xx}$ in Ω , we obtain:

$$\nu \|\vec{\omega}_x\|^2_{L_2(\Omega)} \le \left(C^*_\Omega \max_\Omega |\vec{v}_2 - \varkappa \Delta \vec{v}_2| + \varkappa \max_\Omega |\vec{v}_{1x}|\right) \|\vec{\omega}_x\|^2_{L_2(\Omega)},\tag{18}$$

and from this inequality, making use of condition (16) and of the boundary condition $\vec{\omega}|_{\partial\Omega} = 0$, we obtain $\vec{\omega}(x) \equiv 0$, $x \in \Omega$.

3. Uniqueness theorems for the solutions of the initial-boundary-value problems similar to Theorems 1 and 2 for system (4) and (13) hold also for system (3) and its corresponding stationary system

$$\nu \Delta v_i - v_k \frac{\partial}{\partial x_k}(v_i - \varkappa \Delta v_i) + \varkappa \frac{\partial v_j}{\partial x_k}\left(\frac{\partial^2 v_k}{\partial x_i \partial x_j} + \frac{\partial^2 v_i}{\partial x_j \partial x_k}\right) + \frac{\partial p}{\partial x_i} = f_i, \; i = 1, \ldots, n, \; \operatorname{div} \vec{v} = 0. \tag{19}$$

By a generalized solution of the initial-boundary-value problem (3), (7) we mean a function $\vec{v}(x,t) \in \overset{\circ}{J}(Q_T)$ which has generalized derivatives $\vec{v}_x, \vec{v}_t, \vec{v}_{xx}, \vec{v}_{xt}, \vec{v}_{xxx} \in L_2(Q_T)$, satisfies the integral identity

$$\iint_{Q_t} \Big[\vec{v}_t \vec{\Phi} + \nu \vec{v}_x \vec{\Phi}_x + \varkappa \vec{v}_{xt} \vec{\Phi}_x + v_k \frac{\partial}{\partial x_k}(\vec{v} - \varkappa \Delta \vec{v}) \vec{\Phi} - \varkappa \frac{\partial v_j}{\partial x_k}\left(\frac{\partial^2 v_k}{\partial x_i \partial x_j} + \frac{\partial^2 v_i}{\partial x_j \partial x_k}\right) \Phi_i \Big] dQ = \iint_{Q_t} \vec{f} \vec{\Phi} dQ, \; 0 < t \le T, \tag{20}$$

for any $\vec{\Phi}(x,t) \in \overset{\circ}{J}(Q_T), \vec{\Phi}|_{\partial Q_T} = 0$ possessing the same smoothness properties as $\vec{v}(x,t)$, and satisfies the initial-boundary conditions (7). We show that we have

<u>THEOREM 3.</u> The generalized solution $\vec{v}(x,t)$ of the initial-boundary-value problem (3), (7), having in Q_T bounded derivatives $\vec{v}_x, \vec{v}_{xx}$, is unique.

Indeed, assume that problem (3), (7) has two solutions $\vec{v}_1$ and $\vec{v}_2$ and let $\vec{\omega} \equiv \vec{v}_1 - \vec{v}_2$. Then, reasoning in the same manner as in the proof of Theorem 1 and also integrating once by parts in the additional nonlinear terms, in which transformation for the vanishing of the boundary integrals the boundary condition $\vec{\omega}|_{\partial Q_T} = 0$ is sufficient, we obtain an equality which differs from equality (11) by the additional term

$$\varkappa\iint_{Q_t}\Big[-\Big(\frac{\partial^2 v_{2i}}{\partial x_j\partial x_k}+\frac{\partial^2 v_{2k}}{\partial x_i\partial x_j}\Big)\omega_i\frac{\partial\omega_i}{\partial x_k}+\frac{\partial v_{1j}}{\partial x_k}\frac{\partial\omega_i}{\partial x_j}\Big(\frac{\partial\omega_i}{\partial x_k}+\frac{\partial\omega_k}{\partial x_i}\Big)\Big]\,dQ. \tag{21}$$

From equalities (11)+(21) and also from equality (11), there follows inequality (12), which together with the condition $\vec\omega|_{t=0}=0$ gives $\vec\omega(x,t)\equiv 0$, $(x,t)\in Q_T$.

We now define the generalized solution of the boundary-value problem (19), (14) as a function $\vec v(x)\in H(\Omega)$ which has generalized derivatives $\vec v_{xx}$, $\vec v_{xxx}\in L_2(\Omega)$ and satisfies the integral identity

$$\int_\Omega\Big[\nu\vec v_x\vec\Phi_x-v_k\frac{\partial}{\partial x_k}(\vec v-\varkappa\Delta\vec v)\vec\Phi+\varkappa\frac{\partial v_i}{\partial x_k}\Big(\frac{\partial^2 v_k}{\partial x_i\partial x_j}+\frac{\partial^2 v_i}{\partial x_j\partial x_k}\Big)\Phi_i\Big]dx=\int_\Omega\vec f\vec\Phi\,dx \tag{22}$$

for any function $\vec\Phi(x)$ having the same properties as the solution $\vec v(x)$. Similarly to Theorem 2, one proves

THEOREM 4. The generalized solution of the boundary-value problem (19), (14), having in Ω bounded derivatives $\vec v_x$ and $\vec v_{xx}$ and such that

$$\varkappa\max_\Omega|\vec v_x|+C_\Omega^*\big(\varkappa\max_\Omega|\vec v_{xx}|+\max_\Omega|\vec v-\varkappa\Delta\vec v|\big)<\nu/2, \tag{23}$$

is unique.

4. In the two-dimensional case, the uniqueness theorems for the above-defined generalized solutions of the initial-boundary-value problems (3), (7); (4), (7) and of the boundary-value problems (19), (14); (13), (14) can be proved without assuming the derivative v_{xx} to be bounded in Q_T or Ω, respectively. At the foundation of these results is the known inequality [5, Chap. 1]:

$$\|u\|^4_{L_4(\Omega)}\leq 2\|u\|^2_{L_2(\Omega)}\|u_x\|^2_{L_2(\Omega)}, \tag{24}$$

valid for any function $u(x)\in\overset{\circ}{W}{}^1_2(\Omega)$ and any two-dimensional domain Ω.

THEOREM 5. Let Ω be a two-dimensional bounded domain. Then the generalized solution of the initial-boundary-value problem (3), (7), having in Q_T bounded derivative $\vec v_x$, is unique.

Indeed, if in equality (11) + (21) for the difference $\vec\omega(x,t)$ of two possible generalized solutions of problem (3), (7) one integrates by parts in the third and the fifth terms, then one obtains the equality:

$$\begin{aligned}\frac12\Big(\|\vec\omega\|^2_{L_2(\Omega_t)}+\varkappa\|\vec\omega_x\|^2_{L_2(\Omega_t)}\Big)+\nu\|\vec\omega_x\|^2_{L_2(Q_t)}+\iint_{Q_t}\omega_k\vec\omega\frac{\partial}{\partial x_k}(\vec v_2-\varkappa\Delta\vec v_2)\,dQ-\varkappa\iint_{Q_t}\frac{\partial v_{1k}}{\partial x_i}\frac{\partial\omega_j}{\partial x_k}\frac{\partial\omega_j}{\partial x}dQ+\\+\frac{\varkappa}{2}\iint_{Q_t}\omega_i\frac{\partial}{\partial x_k}\Big(\frac{\partial^2 v_{2i}}{\partial x_j\partial x_k}+\frac{\partial^2 v_{2k}}{\partial x_i\partial x_j}\Big)dQ+\varkappa\iint_{Q_t}\frac{\partial v_{1j}}{\partial x_k}\frac{\partial\omega_i}{\partial x_j}\Big(\frac{\partial\omega_i}{\partial x_k}+\frac{\partial\omega_k}{\partial x_i}\Big)dQ=0.\end{aligned} \tag{25}$$

For the estimate of the third and the fifth term in (25) we make use of the Hölder and Cauchy inequalities and of inequality (24), while for the estimate of the fourth and the sixth term we make use of the boundedness of $\vec v_x$ in Q_T. Then from (25) we obtain:

$$\frac12\big(\|\vec\omega\|^2_{L_2(\Omega_t)}+\varkappa\|\vec\omega_x\|^2_{L_2(\Omega_t)}\big)\leq C_2(\varkappa,\vec v_1)\iint_{Q_t}\vec\omega_x^2\,dQ+C_2'(\varkappa)\int_0^t\|\vec\omega\|^2_{L_4(\Omega_\tau)}\|\vec v_{2xxx}\|_{L_2(\Omega_\tau)}d\tau\leq$$

$$\leq C_2\|\vec{\omega}_x\|^2_{L_2(Q_t)}+C_2''\int_0^t\|\vec{\omega}\|_{L_2(\Omega_\tau)}\|\vec{\omega}_x\|_{L_2(\Omega_\tau)}\|\vec{v}_{2xxx}\|_{L_2(\Omega_\tau)}d\tau\leq \nu\|\vec{\omega}_x\|^2_{L_2(Q_t)}+ \\ +C_2'''\Big\{t+\int_0^t\|\vec{v}_{2xxx}\|^2_{L_2(\Omega_\tau)}d\tau\Big\}\max_{0\leq\tau\leq t}\Big[\|\vec{\omega}\|^2_{L_2(\Omega_\tau)}+\|\vec{\omega}_x\|_{L_2(\Omega_\tau)}\Big],\quad 0<t\leq T, \tag{26}$$

and from this inequality we obtain this estimate:

$$\max_{0\leq\tau\leq t}\Big(\|\vec{\omega}\|^2_{L_2(\Omega_\tau)}+\|\vec{\omega}_x\|^2_{L_2(\Omega_\tau)}\Big)\leq c\Big(t+\int_0^t\|\vec{v}_{2xxx}\|_{L_2(\Omega_\tau)}d\tau\Big)\max_{0\leq\tau\leq t}\Big(\|\vec{\omega}\|^2_{L_2(\Omega_\tau)}+\|\vec{\omega}_x\|^2_{L_2(\Omega_\tau)}\Big). \tag{27}$$

Selecting t_1, satisfying the condition

$$C\Big(t_1+\int_0^{t_1}\|\vec{v}_{2xxx}\|^2_{L_2(\Omega_\tau)}d\tau\Big)<1, \tag{28}$$

we obtain from (27) that $\vec{\omega}(x,t)\equiv 0$ in Q_t with $t<t_1$. After this, the fact that $\vec{\omega}(x,t)$ is equal to zero everywhere in Q_T is proved in steps with respect to t (see [5, Chap. VI, Sec. 2]).

In the stationary case we have the following theorem.

<u>THEOREM 6</u>. Let Ω be a two-dimensional bounded domain. Then the generalized solution of the boundary-value problem (19), (14), such that

$$\varkappa\max_{\Omega}|\vec{v}_x|+C_\Omega^*\big(\varkappa\|\vec{v}_{xxx}\|_{L_2(\Omega)}+\|(\vec{v}-\varkappa\Delta\vec{v})_x\|_{L_2(\Omega)}\big)<\nu/2\ , \tag{29}$$

is unique.

In the two-dimensional case, the finiteness of $\max_{\Omega}|\vec{v}_x|$, occurring in condition (29) of Theorem 6, follows for the generalized solutions $\vec{v}(x)\in\overset{\circ}{W}{}_2^3(\Omega)$ from S. L. Sobolev's imbedding theorem.

Similar uniqueness theorems hold also for the problems (3), (7); (13), (14). For example, in the stationary case we have

<u>THEOREM 6'</u>. Let Ω be a two-dimensional bounded domain. Then the generalized solution of the boundary-value problem (13), (14), satisfying the condition

$$\varkappa\max_{\Omega}|\vec{v}_x|+C_\Omega^*\|(\vec{v}-\varkappa\Delta\vec{v})_x\|_{L_2(\Omega)}<\nu. \tag{30}$$

is unique.

In the two-dimensional case, for system (3), (4) and its corresponding stationary systems (19) and (14) one can introduce the stream function ψ with the aid of the same relations as for the Navier–Stokes equations:

$$v_1=\frac{\partial\psi}{\partial x_2},\quad v_2=-\frac{\partial\psi}{\partial x_1}. \tag{31}$$

For system (4), the equation of the stream function $\psi(x_1,x_2,t)$ has the form:

$$\frac{\partial}{\partial t}(\Delta\psi-\varkappa\Delta^2\psi)-\nu\Delta^2\psi-\Big\{\frac{\partial\psi}{\partial x_1}\frac{\partial}{\partial x_2}(\Delta\psi-\varkappa\Delta^2\psi)-\frac{\partial\psi}{\partial x_2}\frac{\partial}{\partial x_1}(\Delta\psi-\varkappa\Delta^2\psi)\Big\}- \\ -\varkappa\Big\{\frac{\partial^2\psi}{\partial x_1\partial x_2}\Big(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\Big)\Delta\psi-\frac{\partial^2\Delta\psi}{\partial x_1\partial x_2}\Big(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\Big)\psi\Big\}=\frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}\equiv g(x_1,x_2,t). \tag{32}$$

This equation is solved in the cylinder Q_T under the boundary conditions

$$\psi\big|_{\partial Q_T}=0\ ,\ \frac{\partial\psi}{\partial n}\Big|_{\partial Q_T}=0 \tag{33}$$

and Cauchy initial conditions

$$\psi\big|_{t=0}=\psi_0(x_1,x_2)\,,\ \ \psi_{0x_1}=-v_{20}(x),\ \ \psi_{0x_2}=v_{10}(x). \tag{34}$$

The equation for the stream function $\psi(x_1,x_2)$, corresponding to stationary system (13), has, obviously, the form:

$$\nu\Delta^2\psi+\frac{\partial(\psi,\Delta\psi-\varkappa\Delta^2\psi)}{\partial(x_1,x_2)}-\varkappa\Big\{\frac{\partial^2\psi}{\partial x_1\partial x_2}\Big(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\Big)\Delta\psi-\frac{\partial^2\Delta\psi}{\partial x_1\partial x_2}\Big(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\Big)\psi\Big\}=g(x_1,x_2) \tag{35}$$

It is solved in the domain Ω under the boundary conditions $\psi\big|_{\partial\Omega}=\frac{\partial\psi}{\partial n}\big|_{\partial\Omega}=0$.

5. We consider now for the nonstationary system (4) the first initial-boundary-value problem, viz., we shall seek in the cylinder Q_T its solution $\vec{v}(x,t)$, $p(x,t)$, satisfying the following initial and boundary conditions:

$$\vec{v}\big|_{t=0}=\vec{v}_0(x),\quad \vec{v}\big|_{\partial Q_T}=0. \tag{36}$$

This problem was investigated for the first time in [6]. It has been proved there that if $v_0(x)\in \overset{\circ}{W}{}_2^2(\Omega)\cap\overset{\circ}{J}(\Omega)$ and $\vec{f}(x,t)\in L_{2,1}(Q_T)$,* then the problem (4), (36) has at least one weak solution (solution in the sense of E. Hopf), i.e., a solution $\vec{v}(x,t)\in\overset{\circ}{J}(Q_T)\cap\overset{\circ}{W}{}_2^{2,0}(Q_T)$ satisfying the integral identity

$$\iint\limits_{Q_t}\{-\vec{v}\vec{\Phi}_t+\nu\vec{v}_x\vec{\Phi}_x-v_k\vec{v}\vec{\Phi}_{x_k}-\varkappa\Delta\vec{v}\vec{\Phi}_t+\varkappa v_k\Delta\vec{v}\vec{\Phi}_{x_k}\}dQ+\int\limits_\Omega\vec{v}\vec{\Phi}\Big|_{t=t}dx-\varkappa\int\limits_\Omega\vec{v}_x\vec{\Phi}_x\Big|_{t=t}dx-\int\limits_\Omega(\vec{v}_0+\varkappa\Delta\vec{v}_0)\vec{\Phi}(x,0)dx=\iint\limits_{Q_t}\vec{f}\vec{\Phi}dQ\,,\ 0<t\leq T \tag{37}$$

for any $\vec{\Phi}(x,t)\in\overset{\circ}{W}{}_2^{2,1}(Q_T)\cap\overset{\circ}{J}(Q_T)$, and for any weak solution $\vec{v}(x,t)$ of problem (4), (36) we have the inequalities:

$$\max_{0\leq t\leq T}\int\limits_\Omega(\vec{v}^2(x,t)+2\varkappa\vec{v}_x^2+\varkappa^2|\Delta\vec{v}|^2)dx+\nu\iint\limits_{Q_T}\vec{v}_x^2dQ+\nu\varkappa\iint\limits_{Q_T}|\Delta\vec{v}|^2dQ\leq C_3\,, \tag{38}$$

$$\max_{Q_T}|\vec{v}(x,t)|\leq\frac{C_4(\Omega,C_3)}{\varkappa}, \tag{39}$$

where the constant C_3 depends only on $\|\vec{v}_0\|_{\overset{\circ}{W}{}_2^2(\Omega)}$ and $\|\vec{f}\|_{L_{2,1}(Q_T)}$ and it is independent of $\varkappa\geq 0$. In the present paper† we introduce the strong solution of problem (4), (36) (solution in the sense of Ladyzhenskaya [5]) which we define as a function $\vec{v}(x,t)\in\overset{\circ}{J}(Q_T)$, $\vec{v}\big|_{\partial Q_T}=0$, for which $\vec{v}_x$, $\vec{v}_t$, $\vec{v}_{xx}$, $\vec{v}_{xt}\in L_2(Q_T)$ and which satisfies the integral identity

*For the notations see [5].

†See also [8].

$$\iint\limits_{Q_T} (\vec{v}_t\vec{\Phi} + \nu\vec{v}_x\vec{\Phi}_x - v_k\vec{v}\vec{\Phi}_{x_k} + \varkappa\vec{v}_{xt}\vec{\Phi}_x + \varkappa v_k\Delta\vec{v}\vec{\Phi}_{x_k})dQ = \iint\limits_{Q_T} \vec{f}\vec{\Phi}dQ \tag{40}$$

for any $\vec{\Phi}(x,t) \in \overset{\circ}{W}{}_2^{2,0}(Q_T) \cap \overset{\circ}{J}(Q_T)$, and we show that the following existence theorem in the large for the strong solution of problem (4), (36) holds.

THEOREM 7. Let $\vec{v}_0(x) \in \overset{\circ}{W}{}_2^3(\Omega) \cap \overset{\circ}{J}(\Omega)$, $\vec{f}(x,t) \in L_2(Q_T)$. Then initial–boundary-value problem (4), (36) has at least one strong solution $\vec{v}(x,t)$ and for any such solution we have the inequality:

$$\max_{Q_T} |\vec{v}(x,t)| + \max_{[0,T]} \int\limits_{\Omega} (\vec{v}^2 + \vec{v}_x^2 + \vec{v}_{xx}^2)dx + \iint\limits_{Q_T} (\vec{v}_t^2 + \vec{v}_{xt}^2)dQ \leqslant C_5(\varkappa, \|\vec{v}_0\|_{W_2^3(\Omega)}, \|\vec{f}\|_{L_2(Q_T)}), \qquad C_5 \to \infty, \quad \varkappa \to 0 \tag{41}$$

For the proof of Theorem 7, just as for the proof of the existence theorem for the weak solution of the initial–boundary-value problem (4), (36) (see [6]), we make use of the method of introducing a vanishing viscosity. To this end, we consider in Q_T the fourth-order system with a small parameter ("vanishing viscosity")

$$\frac{\partial\vec{v}^\varepsilon}{\partial t} - \nu\Delta\vec{v}^\varepsilon + \varepsilon\Delta^2\vec{v}^\varepsilon + v_k^\varepsilon\frac{\partial\vec{v}^\varepsilon}{\partial x_k} - \varkappa\left(\frac{\partial\Delta\vec{v}^\varepsilon}{\partial t} + v_k\frac{\partial\Delta\vec{v}^\varepsilon}{\partial x_k}\right) + \operatorname{grad} p^\varepsilon = \vec{f}, \quad \operatorname{div}\vec{v}^\varepsilon = 0, \tag{42}$$

which for $\varepsilon = 0$ degenerates into system (4), and we shall solve it under the conditions

$$\vec{v}^\varepsilon\Big|_{t=0} = \vec{v}_0(x), \quad \vec{v}^\varepsilon\Big|_{\partial Q_T} = 0, \quad \Delta\vec{v}^\varepsilon\Big|_{\partial Q_T} = 0. \tag{43}$$

Following [6], by the strong solution of problem (42), (43) we mean a function $\vec{v}^\varepsilon \in \overset{\circ}{J}(Q_T)$, for which $\vec{v}_x^\varepsilon$, $\vec{v}_t^\varepsilon$, $\vec{v}_t^\varepsilon$, $\vec{v}_{xx}^\varepsilon \in L_2(Q_T)$ and which satisfies the integral identity

$$\iint\limits_{Q_T} (\vec{v}_t^\varepsilon\vec{\Phi} + \nu\vec{v}_x^\varepsilon\vec{\Phi}_x + \varepsilon\Delta\vec{v}^\varepsilon\Delta\vec{\Phi} - v_k^\varepsilon\vec{v}^\varepsilon\vec{\Phi}_{x_k} + \varkappa\vec{v}_{xt}^\varepsilon\vec{\Phi}_x + \varkappa v_k^\varepsilon\Delta\vec{v}^\varepsilon\vec{\Phi}_{x_k})dQ = \iint \vec{f}\vec{\Phi}dQ \tag{44}$$

for any $\vec{\Phi}(x,t) \in \overset{\circ}{W}{}_2^{2,0}(Q_T) \cap \overset{\circ}{J}(Q_T)$. It is proved in [6] that for a strong solution $\vec{v}^\varepsilon(x,t)$ of problem (42), (43) we have the inequality

$$\max_{Q_T} |\vec{v}^\varepsilon(x,t)| + \max_{[0,T]} \int\limits_{\Omega} (\vec{v}^{\varepsilon^2} + \vec{v}_x^{\varepsilon^2} + \vec{v}_{xx}^{\varepsilon^2})dx + \varepsilon\iint\limits_{Q_T} |\Delta\vec{v}_x^\varepsilon|^2 dQ \leqslant C_6(\varkappa, \|\vec{v}_0\|_{W_2^3(\Omega)}, \|\vec{f}\|_{L_2(Q_T)}) \tag{45}$$

We set now in (44) $\vec{\Phi} \equiv \vec{v}_t^\varepsilon$. Then, making use, for the estimate of the nonlinear terms, of Hölder's inequality, S. L. Sobolev's embedding theorem, and inequality (45),* we obtain for the strong solution $\vec{v}^\varepsilon$ of problem (42), (43) the following additional inequality:

$$\iint\limits_{Q_T} (\vec{v}_t^{\varepsilon^2} + \vec{v}_{xt}^{\varepsilon^2})dQ + \varepsilon\max_{[0,T]} \int\limits_{\Omega} |\Delta\vec{v}^\varepsilon|^2 dx \leqslant C_7(\varkappa, \|\vec{v}_0\|_{W_2^3(\Omega)}, \|\vec{f}\|_{L_2(Q_T)}), \tag{46}$$

where $C_6, C_7 \to \infty$, $\varkappa \to 0$.

In order to obtain the strong solution $\vec{v}^\varepsilon(x,t)$ of problem (42), (43) for every $\varepsilon > 0$, we make use, as in [6], of the modified Galerkin method. Let $\{\vec{\Phi}^l(x)\}$, $l = 1, 2,$ be a complete system of

*We emphasize that for the estimate of the integral $\varkappa\iint\limits_{Q_T} v_k^\varepsilon\Delta\vec{v}^\varepsilon\vec{v}_{tx_k}^\varepsilon dQ$ it is essential to make use of the boundedness of in Q_T, ensuring inequality (45).

functions in $\overset{\circ}{W}{}_2^2(\Omega)\cap \overset{\circ}{J}(\Omega)$, let $\vec{a}^{\ell}(x)$, $\ell=1,2,\ldots$ be the solutions of the boundary-value problems

$$\vec{a}^{\ell}-æ\Delta\vec{a}^{\ell}=\vec{\Phi}^{\ell}(x),\ x\in\Omega,\ \vec{a}^{\ell}\big|_{\partial\Omega}=0. \tag{47}$$

The system $\{\vec{a}^{\ell}(x)\}$ is also complete in $\overset{\circ}{W}{}_2^2(\Omega)\cap\overset{\circ}{J}(\Omega)$. We shall seek the approximate solution of problem (42), (43) in the form $\vec{v}^{\varepsilon,n}=\sum_{k=1}^{n}C_{kn}(t)\vec{a}^{k}(x)$, where $C_{kn}(t)$ are obtained from the equations

$$\int_{\Omega}(\vec{u}_t^{\varepsilon,n}\vec{\Phi}^{\ell}+\nu\vec{v}_x^{\varepsilon,n}\vec{\Phi}_x^{\ell}+\varepsilon\Delta\vec{v}^{\varepsilon,n}\Delta\vec{\Phi}^{\ell}-v_k^{\varepsilon,n}\vec{u}^{\varepsilon,n}\vec{\Phi}_{x_k}^{\ell})dx=\int_{\Omega}\vec{f}\vec{\Phi}^{\ell}dx,\ \ell=1,\ldots,n,\ t\geq 0, \tag{48}$$

where

$$\vec{u}^{\varepsilon,n}(x,t)\equiv\vec{v}^{\varepsilon,n}-æ\Delta\vec{v}^{\varepsilon,n}, \tag{49}$$

and the initial Cauchy conditions

$$C_{kn}(0)=(\vec{v}_0,\vec{a}^{k})_{W_2^2(\Omega)},\ k=1,\ldots,n,\quad n=1,2,\ldots \tag{50}$$

It is shown in [6] that the Galerkin approximations $\vec{v}^{\varepsilon,n}$ can be constructed for each $n=1,2,\ldots$ and for them we have the estimate

$$\max_{[0,T]}\|\vec{u}^{\varepsilon,n}\|^2_{L_2(\Omega)}+\varepsilon\iint_{Q_T}|\Delta\vec{v}_x^{\varepsilon,n}|^2dQ+\iint_{Q_T}(\varepsilon|\Delta\vec{v}^{\varepsilon,n}|^2+\nu\vec{v}_x^{\varepsilon,n\,2}+\nu æ|\Delta\vec{v}^{\varepsilon,n}|^2)dQ\leq C_8(\|\vec{v}_0\|_{W_2^2(\Omega)},\|\vec{f}\|_{L_2(Q_T)}), \tag{51}$$

uniform with respect to $\varepsilon\geq 0$ and $n=1,2,\ldots$. Then, multiplying (48) by $\frac{dC_{\ell n}}{dt}$ and summing with respect to ℓ from 1 to n, we obtain:

$$\|\vec{u}_t^{\varepsilon,n}\|^2_{L_2(\Omega)}+\nu(\vec{v}_x^{\varepsilon,n},\vec{u}_{xt}^{\varepsilon,n})_{L_2(\Omega)}+\varepsilon(\Delta\vec{v}^{\varepsilon,n},\Delta\vec{u}_t^{\varepsilon,n})_{L_2(\Omega)}+(v_k^{\varepsilon,n}\vec{u}^{\varepsilon,n},\vec{u}_{tx_k}^{\varepsilon,n})_{L_2(\Omega)}=(\vec{f},\vec{u}_t^{\varepsilon,n})_{L_2(\Omega)},\ t\geq 0, \tag{52}$$

and from here, integrating with respect to $t\in[0,T]$ and making use of Hölder's inequality and estimate (51), we obtain the estimate, uniform with respect to $æ\geq 0$,

$$\|\vec{u}_t^{\varepsilon,n}\|^2_{L_2(Q_T)}+\varepsilon\max_{[0,T]}\|\Delta\vec{v}_x^{\varepsilon,n}\|^2_{L_2(\Omega)}\leq\frac{1}{\varepsilon}C_9\left(\|\vec{v}_0\|_{W_2^3(\Omega)},\|\vec{f}\|_{L_2(Q_T)}\right),\ \varepsilon>0. \tag{53}$$

Then, since $\vec{v}^{\varepsilon,n}(x,t)$ is a "solution" of the boundary-value problem

$$\vec{v}^{\varepsilon,n}-æ\Delta\vec{v}^{\varepsilon,n}=\vec{u}^{\varepsilon,n},\ x\in\Omega,\ \vec{v}^{\varepsilon,n}\big|_{\partial\Omega}=0,\ t\geq 0, \tag{54}$$

for $\vec{v}^{\varepsilon,n}$ and $\vec{v}_t^{\varepsilon,n}$ the second fundamental inequality [7] holds (in order to obtain this inequality it is necessary to assume that the boundary $\partial\Omega$ is twice boundedly differentiable):

$$\|\vec{v}^{\varepsilon,n}\|_{W_2^2(\Omega)}\leq C_{10}(æ,\partial\Omega)\|\vec{u}^{\varepsilon,n}\|_{L_2(\Omega)},\ \|\vec{v}_t^{\varepsilon,n}\|_{W_2^2(\Omega)}\leq C_{10}\|\vec{u}_t^{\varepsilon,n}\|_{L_2(\Omega)},\ t\geq 0, \tag{55}$$

and then from inequalities (55) and inequalities (51) and (53) it follows that

$$\max_{[0,T]}\|\vec{v}^{\varepsilon,n}\|_{W_2^2(\Omega)}\leq C_{11}(C_8,C_{10}), \tag{56}$$

$$\int_0^T\|\vec{v}_t^{\varepsilon,n}\|^2_{W_2^2(\Omega)}dt\leq C_{12}(\tfrac{1}{\varepsilon}\cdot C_9,C_{10}). \tag{57}$$

Inequalities (51), (53), (56), (57) and the theorem on the weak compactness of bounded sets in a Hilbert space allow us to conclude (see [5, Chap. VI]) that from the sequence of the Galerkin approximations $\{\vec{v}^{\varepsilon,n}\}$ one can extract at least one subsequence $\{\vec{v}^{\varepsilon,n_k}\}$, which for $n_k \to \infty$ converges strongly in $L_2(Q_T)$ to the limit function $\vec{v}^{\varepsilon}(x,t)$ and for which $\vec{v}^{\varepsilon,n_k}$, $\vec{v}_t^{\varepsilon,n_k}$, $\vec{v}_{xt}^{\varepsilon,n_k}$, $\vec{v}_{xx}^{\varepsilon,n_k}$ converge weakly in $L_2(Q_T)$ to $\vec{v}_x^{\varepsilon}$, $\vec{v}_t^{\varepsilon}$, $\vec{v}_{xt}^{\varepsilon}$, $\vec{v}_{xx}^{\varepsilon}$, respectively. This limit function $\vec{v}^{\varepsilon}(x,t)$ will be the strong solution of the initial–boundary problem (42), (43). This statement is proved in the same way as the corresponding statement in the proof of the existence theorem of the strong solution of the initial–boundary-value problem for the system of Navier–Stokes equations [5, Chap. VI], but in this case the integral $\int_\Omega \vec{u}_t^{\varepsilon,n_k} \vec{\Phi}^{\ell} dx$ in Eqs. (48) has to be transformed by integration by parts to the form $\int_\Omega (\vec{v}_t^{\varepsilon,n_k} \vec{\Phi}^{\ell} + \varkappa \vec{v}_{tx_k}^{\varepsilon,n_k} \vec{\Phi}_{x_k}^{\ell}) dx$, $t \geqslant 0$.

Since for any strong solution $\vec{v}(x,t)$ of initial–boundary-value problem (42), (43) we have the estimates (45), (46), uniform with respect to $\varepsilon \geqslant 0$, from the totality $\{\vec{v}^{\varepsilon}(x,t)\}$ of the strong solutions of the problem (42), (43) one can select a subsequence $\{\vec{v}^{\varepsilon_i}\}$, which for $\varepsilon_i \to 0$ converges strongly in $L_2(Q_T)$ to the limit function $\vec{v}(x,t)$ and for which $\vec{v}_x^{\varepsilon_i}$, $\vec{v}_t^{\varepsilon_i}$, $\vec{v}_{xt}^{\varepsilon_i}$, $\vec{v}_{xx}^{\varepsilon_i}$ converge weakly in $L_2(Q_T)$ to $\vec{v}_x$, $\vec{v}_t$, $\vec{v}_{xt}$, $\vec{v}_{xx}$, respectively. In addition,

$$\varepsilon_i \iint_{Q_T} \Delta \vec{v}^{\varepsilon_i} \Delta \vec{\Phi} \, dQ \to 0, \ \varepsilon_i \to 0. \tag{58}$$

Making use of all these limiting relations and taking the limit in integral identity (44) with respect to the selected subsequence $\varepsilon_i \to 0$, we obtain that the limit function $\vec{v}(x,t)$ will satisfy the integral identity (40), i.e., it will be the strong solution of problem (4), (36). Inequality (41) is obtained by a limiting process for $\varepsilon_i \to 0$ from estimates (45), (46).

6. We return once again to the problem of the uniqueness of the initial–boundary-value problems considered in the present paper. In the theorems on the uniqueness of the generalized solutions, proved in Secs. 2-4, both in the formulation of the theorems and in the process of their proofs, we have assumed that these solutions possess the derivatives $\vec{v}_{xxx}(x,t) \in L_2(Q_T)$ (or $\vec{v}_{xxx}(x) \in L_2(\Omega)$). In the present section, assuming that Ω is a two-dimentional bounded domain, we prove uniqueness theorems for a class of generalized solutions, wider than in the Secs. 2-4, for which the existence of the derivatives $\vec{v}_{xxx}$ is not assumed but on the other hand $\vec{v}_{xx}(x,t) \in L_{4,1}(Q_T)$ (or $\vec{v}_{xx}(x) \in L_4(\Omega)$).

We define the strong generalized solution of initial–boundary-value problem (3), (7) as a function $\vec{v}(x,t) \in \mathring{J}(Q_T)$ which admits in Q_T bounded derivatives $\vec{v}_x$, possesses derivatives $\vec{v}_t$, $\vec{v}_{xt} \in L_2(Q_T)$ and derivatives $\vec{v}_{xx} \in L_{4,1}(Q_T)$, satisfies the integral identity

$$\begin{gathered}\iint_{Q_t} [\vec{v}_t \vec{\Phi} + \nu \vec{v}_x \vec{\Phi}_x + \varkappa \vec{v}_{xt} \vec{\Phi}_x - v_k \vec{v} \vec{\Phi}_{x_k} + \\ + \varkappa \frac{\partial^2 v_j}{\partial x_i \partial x_k} \Big(v_k \frac{\partial \Phi_j}{\partial x_i} + \frac{\partial v_k}{\partial x_i} \Phi_j \Big) - \varkappa \frac{\partial v_i}{\partial x_k} \Big(\frac{\partial^2 v_k}{\partial x_i \partial x_j} + \frac{\partial^2 v_i}{\partial x_j \partial x_k} \Big) \Phi_i] dQ = \iint_{Q_t} \vec{f} \vec{\Phi} \, dQ, \ 0 < t \leqslant T,\end{gathered} \tag{59}$$

for any $\vec{\Phi}(x,t) \in \mathring{J}(Q_T)$, $\vec{\Phi}|_{\partial Q_T} = 0$, has the same differential properties as $\vec{v}(x,t)$, and satisfies the initial–boundary conditions (7); let us prove that we have

<u>THEOREM 8.</u> Let Ω be a two-dimensional bounded domain. Then, the strong generalized solution of initial–boundary-value problem (3), (7) is unique.

Indeed, forming the integral identity for the difference $\vec{\omega}(x,t)=\vec{v}_1-\vec{v}_2$ of two possible strong generalized solutions of problem (3), (7), substituting in it $\vec{\Phi}=\vec{\omega}$, and integrating by parts, we obtain the equality

$$\begin{gathered}\frac{1}{2}\left(\|\vec{\omega}\|^2_{L_2(\Omega_t)}+\varkappa\|\vec{\omega}_x\|^2_{L_2(\Omega_t)}\right)+\nu\|\vec{\omega}_x\|^2_{L_2(Q_t)}-\iint\limits_{Q_t}\frac{\partial\vec{v}_2}{\partial x_k}\omega_k\vec{\omega}\,dQ+\varkappa\iint\limits_{Q_t}\frac{\partial^2 v_{1j}}{\partial x_i\partial x_k}\Big(\omega_k\frac{\partial\omega_j}{\partial x_i}+\\ +\omega_j\frac{\partial\omega_k}{\partial x_i}\Big)dQ-\varkappa\iint\limits_{Q_t}\frac{\partial v_{2k}}{\partial x_i}\frac{\partial\omega_j}{\partial x_i}\frac{\partial\omega_j}{\partial x_k}dQ+\varkappa\iint\limits_{Q_t}\Big[-\Big(\frac{\partial^2 v_{2i}}{\partial x_j\partial x_k}+\frac{\partial^2 v_{2k}}{\partial x_i\partial x_j}\Big)\omega_i\times\frac{\partial\omega_j}{\partial x_k}+\\ +\frac{\partial v_{1i}}{\partial x_k}\frac{\partial\omega_i}{\partial x_j}\Big(\frac{\partial\omega_i}{\partial x_k}+\frac{\partial\omega_k}{\partial x_i}\Big)\Big]dQ=0\ ,\quad 0<t\leq T,\end{gathered}\tag{60}$$

and from this equality, making use of the boundedness of $\vec{v}_x$ in Q_T and applying Hölder's inequality, we shall have:

$$\begin{gathered}\frac{1}{2}\left(\|\vec{\omega}\|^2_{L_2(\Omega_t)}+\varkappa\|\vec{\omega}_x\|^2_{L_2(\Omega_t)}\right)\leq C(\varkappa,\vec{v}_1,\vec{v}_2)\iint\limits_{Q_t}(\vec{\omega}^2+\vec{\omega}_x^2)dQ+\varkappa\int_0^t(\|\vec{v}_{1xx}\|_{L_4(\Omega_\tau)}+\\ +\|\vec{v}_{2xx}\|_{L_4(\Omega_\tau)})\times\|\vec{\omega}\|_{L_4(\Omega_\tau)}\|\vec{\omega}_x\|_{L_2(\Omega_\tau)}d\tau,\quad 0<t\leq T.\end{gathered}\tag{61}$$

From inequality (61), applying inequality (24), Friedrich's inequality, and the operation of maximization, we obtain the estimate:

$$\max_{0\leq\tau\leq t}\left(\|\vec{\omega}\|^2_{L_2(\Omega_\tau)}+\|\vec{\omega}_x\|^2_{L_2(\Omega_\tau)}\right)\leq C'\Big(t+\int_0^t(\|\vec{v}_{1xx}\|_{L_4(\Omega_\tau)}+\|\vec{v}_{2xx}\|_{L_4(\Omega_\tau)})d\tau\Big)\max_{0\leq\tau\leq t}\left(\|\vec{\omega}\|^2_{L_2(\Omega_\tau)}+\|\vec{\omega}_x\|^2_{L_2(\Omega_\tau)}\right),\quad 0<t\leq T,\tag{62}$$

and from this estimate, selecting t_1 satisfying the condition

$$C'\Big[t+\int_0^t(\|\vec{v}_{1xx}\|_{L_4(\Omega_\tau)}+\|\vec{v}_{2xx}\|_{L_4(\Omega_\tau)})d\tau\Big]<1,\tag{63}$$

we obtain that $\vec{\omega}(x,t)\equiv 0$ in Q_t with $t<t_1$. After this, the vanishing of $\vec{\omega}(x,t)$ in the entire cylinder Q_T is proved in steps with respect to t (see Sec. 4).

A similar uniqueness theorem holds also for the strong generalized solutions of initial-boundary-value problem (4), (7), for whose definition one has to omit in the left-hand side of the integral identity (59) the terms $-\varkappa\frac{\partial v_j}{\partial x_k}(\ldots)\Phi_i$.

We turn now to stationary problems. By the strong generalized solution of the boundary-value problem (19), (14) we mean a function $\vec{v}(x)\in H(\Omega)$, which possesses the derivatives $\vec{v}_{xx}\in L_4(\Omega)$ and satisfies the integral identity

$$\int_\Omega\Big[\nu\vec{v}_x\vec{\Phi}_x-v_k\vec{v}\vec{\Phi}_{x_k}+\varkappa\frac{\partial^2 v_j}{\partial x_i\partial x_k}\Big(v_k\frac{\partial\Phi_j}{\partial x_i}+\frac{\partial v_k}{\partial x_i}\Phi_j\Big)-\varkappa\frac{\partial v_j}{\partial x_k}\Big(\frac{\partial^2 v_k}{\partial x_i\partial x_j}+\frac{\partial^2 v_i}{\partial x_j\partial x_k}\Big)\Phi_i\Big]dx=\int_\Omega\vec{f}\vec{\Phi}\,dx\tag{64}$$

for any $\vec{\Phi}(x)\in H(\Omega)$ having the same smoothness as $\vec{v}(x)$. From Sobolev's embedding theorem it follows that the strong generalized solution $\vec{v}(x)$ of problem (19), (14) has in Ω bounded derivatives $\vec{v}_x$.

With the same arguments as those used for Theorem 8, one proves the following theorem.

THEOREM 8'. Let Ω be a two-dimensional bounded domain. Then the strong generalized solution of boundary-value problem (19), (14), satisfying

$$(\text{æ} + C_\Omega^*) \max_\Omega |\vec{v}_x| + \text{æ}\, C_\Omega^{*1/2} \|\vec{v}_{xx}\|_{L_4(\Omega)} < 1/2, \tag{65}$$

is unique.

A similar uniqueness theorem in the small takes place also for the strong generalized solution of boundary-value problem (13), (14), which is defined as a function $\vec{v}(x) \in H(\Omega)$, possessing derivatives $\vec{v}_{xx} \in L_4(\Omega)$ and satisfying the identity

$$\int_\Omega [\nu \vec{v}_x \vec{\Phi}_x - v_k \vec{v} \vec{\Phi}_{x_k} + \text{æ} \frac{\partial^2 v_j}{\partial x_i \partial x_k} (v_k \frac{\partial \varphi_j}{\partial x_i} + \frac{\partial v_k}{\partial x_i} \varphi_j)] dx = \int_\Omega \vec{f} \vec{\Phi} dx$$

for any $\vec{\Phi} \in H(\Omega)$ having derivatives $\vec{\Phi}_{xx} \in L_4(\Omega)$.

THEOREM 8". Let Ω be a two-dimensional bounded domain. Then the strong generalized solution of boundary-value problem (13), (14), satisfying the condition

$$(\frac{\text{æ}}{2} + C_\Omega^*) \max_\Omega |\vec{v}_x| + \frac{\text{æ}}{2} C_\Omega^{*1/2} \|\vec{v}_{xx}\|_{L_4(\Omega)} < \nu, \tag{65'}$$

is unique.

7. Now we turn to the problem of the relationship between the solutions of initial-boundary-value problem (4), (36) for $\text{æ} \to 0$ and the solutions of problem (5), (36) for the system of Navier-Stokes equations into which system (4) degenerates for $\text{æ} \to 0$. For system (4), until now nobody has succeeded in answering this question; it has not been successfully proved that for the weak (and, moreover, the strong) solution of problem (4), (36) converges strongly in $L_2(Q_T)$ at least to a weak solution of problem (5), (36). Therefore, instead of system (4) we consider the following simplified system

$$\frac{\partial \vec{v}}{\partial t} - \nu \Delta \vec{v} + a_k(x,t) \frac{\partial \vec{v}}{\partial x_k} - \text{æ} \frac{\partial \Delta \vec{v}}{\partial t} - \text{æ} v_k \frac{\partial \Delta \vec{v}}{\partial x_k} + \operatorname{grad} p = \vec{f}(x,t), \quad \operatorname{div} \vec{v} = 0, \tag{66}$$

which for $\text{æ} = 0$ becomes Ozin's nonstationary system

$$\frac{\partial \vec{v}}{\partial t} - \nu \Delta \vec{v} + a_k(x,t) \frac{\partial \vec{v}}{\partial x_k} + \operatorname{grad} p = \vec{f}, \quad \operatorname{div} v = 0, \quad \vec{a}(x,t) \in \mathring{J}(Q_T) \cap \mathring{C}(\bar{Q}_T), \tag{67}$$

but for every $\text{æ} > 0$ retains the terms which describe the relaxational properties of the fluid. As before, we shall solve systems (66) and (67) in Q_T under initial-boundary conditions (36).

For initial-boundary-value problem (66), (36) one can, in analogy with original problem (4), (36), determine the weak and strong generalized solutions, which, as before, are determined by integral identities (37) and (40), respectively, where $v_k \vec{v} \vec{\Phi}_{x_k} \equiv a_k \vec{v} \vec{\Phi}_{x_k}$. After this the situation is exactly the same as in the case of problem (4), (36), i.e., by the introduction of a vanishing viscosity one proves the following existence theorems for the weak and strong solutions of problem (66), (36).

THEOREM 9. Let $\vec{v}_0(x) \in \mathring{W}_2^2(\Omega) \cap \mathring{J}(\Omega)$, $\vec{f}(x,t) \in L_{2,1}(Q_T)$. Then problem (66), (36) has at least one weak solution $\vec{v}(x,t)$ and for any such solution we have the estimate:

$$\text{æ}^{1/2} \max_{Q_T} |\vec{v}(x,t)| + \max_{[0,T]} \int_\Omega (\vec{v}_x^2 + \text{æ} |\Delta \vec{v}|^2) dx + \nu \iint_{Q_T} |\Delta \vec{v}|^2 dQ \le C_{13}, \tag{68}$$

where the constant C_{13} is determined only by the norms $\|\vec{v}_0\|_{W_2^3(\Omega)}$ and $\|\vec{f}\|_{L_{2,1}(Q_T)}$, $\max_{Q_T}|\vec{a}(x,t)|$.

<u>THEOREM 10.</u> Let $\vec{v}_0(x)\in \overset{\circ}{W}{}_2^3(\Omega)\cap \overset{\circ}{J}(\Omega)$, $\vec{f}(x,t)\in L_2(Q_T)$. Then problem (66), (36) has at least one strong solution $\vec{v}(x,t)$ and for any such solution, in addition to estimate (68), we have the inequality

$$\iint\limits_{Q_T}(\vec{v}_t^2+\varkappa\vec{v}_{xt}^2)\,dQ\leq C_{14}, \tag{69}$$

where the constant C_{14} depends only on $\|\vec{v}_0\|_{W_2^3(\Omega)}$, $\|\vec{f}\|_{L_2(Q_T)}$, $\max_{Q_T}|\vec{a}|$, C_{13} and does not depend on $\varkappa\geq 0$.

We emphasize that the occurrence of $\varkappa$ in inequalities (68), (69) is completely different than in inequalities (38), (39), (41) for the solutions of problem (4), (36) and this allows us to prove the following theorem.

<u>THEOREM 11.</u> Let $\vec{v}_0(x)\in \overset{\circ}{W}{}_2^3(\Omega)\cap \overset{\circ}{J}(\Omega)$, $\vec{f}(x,t)\in L_2(Q_T)$. Then for $\sim$ any strong solution $\vec{v}^{\varkappa}(x,t)$ of problem (66), (36) tends to the unique strong solution (Ladyzhenskaya's solution) of initial-boundary-value problem (67), (36).

We recall [5, Chap. IV] that the strong solution of problem (67), (36) is defined as a function $\vec{v}(x,t)\in\overset{\circ}{J}(Q_T)$ for which $\vec{v}_x$, $\vec{v}_t$, $\vec{v}_{xt}\in L_2(Q_T)$ and which satisfies the integral identity

$$\iint\limits_{Q_T}(\vec{v}_t\vec{\Phi}+\nu\vec{v}_x\vec{\Phi}_x-a_k\vec{v}\,\vec{\Phi}_{x_k})\,dQ=\iint\limits_{Q_T}\vec{f}\,\vec{\Phi}\,dQ \tag{70}$$

for any $\vec{\Phi}(x,t)\in\overset{\circ}{W}{}_2^{1,0}(Q_T)\cap\overset{\circ}{J}(Q_T)$. For the proof of Theorem 11 it is sufficient to note that, by virtue of inequalities (68), (69), from the totality $\{\vec{v}^{\varkappa}(x,t)\}$ of strong solutions of problem (66), (36) one can extract a subsequence $\{\vec{v}^{\varkappa_i}\}$, which for $\varkappa_i\to 0$ converges strongly in $L_2(Q_T)$ to the limit function $\vec{v}(x,t)$ and for which $\vec{v}_x^{\varkappa_i}$, $\vec{v}_t^{\varkappa_i}$ converge weakly in $L_2(Q_T)$ to $\vec{v}_x$, $\vec{v}_t$, respectively. In addition, by virtue of inequalities (68) and (69) we have

$$\varkappa_i\iint\limits_{Q_T}\vec{v}_{xt}^{\varkappa_i}\vec{\Phi}_x\,dQ\to 0,\quad \varkappa_i\iint\limits_{Q_T}v_k^{\varkappa_i}\Delta\vec{v}^{\varkappa_i}\vec{\Phi}_{x_k}\,dQ\to 0,\ \ \varkappa_i\to 0. \tag{71}$$

Now, if in the integral identity (40), with $v_k\vec{v}\vec{\Phi}_{x_k}\equiv a_k\vec{v}\vec{\Phi}_{x_k}$, we take the limit as $\varkappa_i\to 0$, we obtain that the limit function $\vec{v}(x,t)$ satisfies identity (70), i.e., it is a strong solution of the linearized problem (67), (36), which, as is known, is unique.

8. We consider one more particular case of the system (4), the system

$$\frac{\partial\vec{v}}{\partial t}-\nu\Delta\vec{v}+v_k\frac{\partial\vec{v}}{\partial x_k}-\varkappa\frac{\partial\Delta\vec{v}}{\partial t}+\operatorname{grad}p=\vec{f},\quad \operatorname{div}\vec{v}=0, \tag{72}$$

which we shall solve in the cylinder Q_T under the initial-boundary conditions (36). This system also has a real physical sense; it describes the flow of a viscous incompressible Newtonian fluid which requires $\frac{\varkappa}{\nu}$ units of time in order to be set in motion under the action of a suddenly applied force. A model for such a fluid has been suggested by Voigt and is characterized by the following defining equation:

$$T = -pE + 2\nu\left(1 + \frac{\varkappa}{\nu}\frac{\partial}{\partial t}\right)D. \tag{73}$$

Inserting this stress tensor into equations of motion (2), we obtain system (72).

For the initial-boundary-value problem (72), (36) we introduce two generalized solutions: a strong and a weak one. The strong solution of problem (72), (36) is defined as a function $\vec{v}(x,t) \in \overset{\circ}{J}(Q_T)$, for which $\vec{v}_x, \vec{v}_t, \vec{v}_{xt} \in L_2(Q_T)$ and which satisfies the integral identity

$$\iint_{Q_T} (\vec{v}_t\vec{\Phi} + \nu\vec{v}_x\vec{\Phi}_x + \varkappa\vec{v}_{xt}\vec{\Phi}_x - v_k\vec{v}\vec{\Phi}_{x_k})dQ = \iint_{Q_T} \vec{f}\vec{\Phi}dQ \tag{74}$$

for any $\vec{\Phi}(x,t) \in \overset{\circ}{W}{}_2^1(Q) \cap J(Q_T)$ while the weak solution of problem (72), (36) is defined as a function $\vec{v}(x,t) \in J(Q_T)$ for which $\vec{v}^2$, $v_x \in L_2(Q_T)$ and which satisfies the integral identity

$$\iint_{Q_t} (-\vec{v}\vec{\Phi}_t + \nu\vec{v}_x\vec{\Phi}_x - v_k\vec{v}\vec{\Phi}_{x_k} - \varkappa\vec{v}_x\vec{\Phi}_{xt})dQ + \int_\Omega \vec{v}\vec{\Phi}\Big|_{t=t}dx + \varkappa\int_\Omega \vec{v}_x\vec{\Phi}_x\Big|_{t=t}dx -$$
$$- \int_\Omega [\vec{v}_0(x)\vec{\Phi}(x,0) + \varkappa\vec{v}_{0x}(x)\vec{\Phi}_x(x,0)]dx = \iint_{Q_T} \vec{f}\vec{\Phi}dQ \quad 0 < t \leq T, \tag{75}$$

for any $\vec{\Phi}(x,t) \in \overset{\circ}{W}{}_2^1(Q_T) \cap J(Q_T)$ and such that $\vec{\Phi}_{xt} \in L_2(Q_T)$. As it will be shown below, for any weak, and moreover also for a strong solution of problem (72), (36), we have the energy inequality

$$\max_{[0,T]} \int_\Omega (\vec{v}^2(x,t) + \varkappa\vec{v}_x^2)dx + \iint_{Q_T} \vec{v}_x^2 dQ \leq C_{15}. \tag{76}$$

where the constant C_{15} depends only on $\|\vec{f}\|_{L_2(Q_T)}$, $\|\vec{v}_0\|_{W_2^1(\Omega)}$, and T.

First of all we prove the following theorem.

<u>THEOREM 12.</u> The weak (and therefore, also the strong) solution of problem (72), (36) is uniquely defined.

Indeed, assume that problem (72), (36) has two weak solutions $\vec{v}_1$ and $\vec{v}_2$. Then their difference satisfies the integral identity

$$\iint_{Q_t} (-\vec{w}\vec{\Phi}_t + \nu\vec{w}_x\vec{\Phi}_x - \varkappa\vec{w}_x\vec{\Phi}_{xt} - w_k\vec{v}_1\vec{\Phi}_{x_k} - v_{2k}\vec{w}\vec{\Phi}_{x_k})dQ + \int_\Omega (\vec{w}\vec{\Phi} + \varkappa\vec{w}_x\vec{\Phi}_x)\Big|_{t=t}dx = 0 \tag{77}$$

$\forall\vec{\Phi} \in \overset{\circ}{W}{}_2^1(Q_T) \cap \overset{\circ}{J}(Q_T)$, $\vec{\Phi}_{xt} \in L_2(Q_T)$. Taking in identity (77)

$$\vec{\Phi}(x,t) = \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \vec{w}(x,\tau)d\tau, \quad 0 < t < T, \quad \varepsilon > 0 \tag{78}$$

(here we assume without loss of generality that $\vec{w}(x,t) \equiv 0$ for $t < 0$), taking then the limit for $\varepsilon \to 0$ and making use of estimate (76), we obtain for $\vec{w}(x,t)$ the equality (see [5, 1st ed., pp. 174-175]):

$$\frac{1}{2}\int_\Omega (\vec{w}^2(x,t) + \varkappa\vec{w}_x^2)dx + \nu\iint_{Q_t} \vec{w}_x^2 dQ = \iint_{Q_t} \vec{v}_1 w_k \vec{w}_{x_k} dQ, \quad 0 < t \leq T \tag{79}$$

Now we make use of the following well-known inequality [5, Chap. I]:

$$\|u\|_{L_4(\Omega)} \leq C_\Omega \|u_x\|_{L_2(\Omega)}, \tag{80}$$

which holds $\forall u(x) \in \overset{\circ}{W}{}_2^{1}(\Omega)$ and for any bounded domain Ω. Then, estimating the right-hand side of equality (79) which the aid of Holder's inequality and inequality (80) and making use of the a priori estimate (76), we obtain:

$$\Big|\iint_{Q_t} \vec{v}_1 \omega_k \vec{\omega}_{x_k} dQ\Big| \leq C_{16}(C_{15}, C_\Omega, \tfrac{1}{\varkappa}, T) \iint_{Q_t} \vec{\omega}_x^2 dQ. \tag{81}$$

Setting $y(t) \equiv \iint_{Q_t} \vec{\omega}_x^2 dQ$, from (79), (81) we obtain the differential inequality

$$\frac{\varkappa}{2} y'(t) \leq C_{16} y(t), \quad 0 < t \leq T, \quad y(0) = 0, \tag{82}$$

from which it follows that $y(t) \equiv 0$, $0 \leq t \leq T$, and therefore, also $\vec{\omega}(x,t) \equiv 0$, $(x,t) \in Q$.

Proceeding to the proof of the existence theorem for the solutions of the initial-boundary-value problem (72), (36), we show that system (72) regularizes the Navier–Stokes system of equations (5) in the following sense: 1) if $\vec{f}(x,t)$, $\vec{f}_t \in L_2(Q_T)$, $\vec{v}_0(x) \in W_2^2(\Omega) \cap \overset{\circ}{J}{}_2^1(\Omega)$, then problem (72), (36) has for each $\varkappa > 0$ a strong solution $\vec{v}^\varkappa(x,t)$ in the large; 2) under well-defined conditions of smallness on the given data ν^{-1}, $\vec{f}(x,t)$, $\vec{v}_0(x)$ of the problem and on the dimensions of the cylinder Q_T, for which the initial-boundary-value problem (5), (36) for the system of Navier–Stokes equations has a strong solution $\vec{v}(x,t)$ (Ladyzhenskaya's solution), the strong solution $\vec{v}^\varkappa(x,t)$ of problem (72), (36) goes, for $\varkappa \to 0$, into the strong solution $\vec{v}(x,t)$ of problem (5), (36) for the system of Navier–Stokes equations; 3) if $\vec{f}(x,t) \in L_2(Q_T)$, $\vec{v}_0(x) \in \overset{\circ}{J}{}_2^1(\Omega)$, then problem (72), (36) has for each $\varkappa > 0$ a weak solution $\vec{v}^\varkappa(x,t)$ in the large and for $\varkappa \to 0$ this solution goes into the weak solution (Hopf's solution) of problem (5), (36) for the system of Navier–Stokes equations.

At the foundation of all these results are the a priori estimates for the solutions of initial-boundary-value problem (72), (36). These estimates are derived from the equalities

$$\frac{1}{2}\frac{d}{dt}\Big(\|\vec{v}\|^2_{L_2(\Omega)} + \varkappa\|\vec{v}_x\|^2_{L_2(\Omega)}\Big) + \nu\|\vec{v}_x\|^2_{L_2(\Omega)} = (\vec{f}, \vec{v})_{L_2(\Omega)}, \quad 0 < t \leq T, \tag{83}$$

$$\frac{1}{2}\frac{d}{dt}\Big(\|\vec{v}_t\|^2_{L_2(\Omega)} + \varkappa\|\vec{v}_{xt}\|^2_{L_2(\Omega)}\Big) + \nu\|\vec{v}_{xt}\|^2_{L_2(\Omega)} + \int_\Omega v_{kt}\vec{v}_{x_k}\vec{v}_t dx = (\vec{f}_t, \vec{v}_t)_{L_2(\Omega)}, \quad 0 < t \leq T, \tag{84}$$

$$\|\vec{v}_t\|^2_{L_2(\Omega)} + \varkappa\|\vec{v}_{xt}\|^2_{L_2(\Omega)} + \frac{\nu}{2}\frac{d}{dt}\|\vec{v}_x\|^2_{L_2(\Omega)} + \int_\Omega v_k\vec{v}_{x_k}\vec{v}_t dx = (\vec{f}, \vec{v}_t)_{L_2(\Omega)}, \quad 0 < t \leq T, \tag{85}$$

which, in turn, are derived with the aid of integration by parts from the equalities

$$\left.\begin{aligned} &\int_\Omega L_\varkappa\vec{v}\,\vec{v}\,dx = \int_\Omega \vec{f}\vec{v}\,dx, \quad \int_\Omega (L_\varkappa\vec{v})_t\vec{v}_t dx = \int_\Omega \vec{f}_t\vec{v}_t dx, \\ &\int_\Omega L_\varkappa\vec{v}\cdot\vec{v}_t dx = \int_\Omega \vec{f}\vec{v}_t dx, \quad L_\varkappa\vec{v} \equiv \vec{v}_t - \nu\Delta\vec{v} + v_k\vec{v}_{x_k} - \varkappa\frac{\partial\Delta\vec{v}}{\partial t} + \operatorname{grad} p, \; 0 < t \leq T, \end{aligned}\right\} \tag{86}$$

In order to obtain these estimates we shall make use, in a substantial manner, of inequality (80) and of Gronwall's lemma.

LEMMA 1. If $\vec{v}(x,t)$ satisfies for $t\in[0,T]$ relation (83), $\vec{f}\in L_2(Q_T)$ and $\vec{v}(x,0)=\vec{v}_0(x)\in \mathring{J}_2^1(\Omega)$, then for it we have the inequalities:

$$\|\vec{v}(x,t)\|^2_{L_2(\Omega)}+æ\|\vec{v}_x\|^2_{L_2(\Omega)}\leq e^T\left[\|\vec{v}_0\|^2_{L_2(\Omega)}+æ\|\vec{v}_{0x}\|^2_{L_2(\Omega)}+\|\vec{f}\|^2_{L_2(Q_T)}\right]\equiv C_{17}, \tag{87}$$

$$\frac{1}{2}\left(\|\vec{v}\|^2_{L_2(\Omega)}+æ\|\vec{v}_x\|^2_{L_2(\Omega)}\right)+\nu\|\vec{v}_x\|^2_{L_2(Q_t)}\leq C_{17}^{1/2}\|\vec{f}\|^2_{L_2(Q_T)}+\frac{1}{2}\left(\|\vec{v}_0\|^2_{L_2(\Omega)}+æ\|\vec{v}_{0x}\|^2_{L_2(\Omega)}\right)\equiv C_{18}, \quad 0<t\leq T. \tag{88}$$

Estimates (87), (88) are obtained from equality (83) with the aid of Hölder's inequality and Gronwall's lemma.

We introduce the following notations: $\varphi(t)=\|\vec{v}_x(x,t)\|_{L_2(\Omega)}$, $\psi^2_æ(t)=\|\vec{v}_t\|^2_{L_2(\Omega)}+æ\|\vec{v}_{xt}\|^2_{L_2(\Omega)}$, $F(t)=\|\vec{v}_{xt}\|_{L_2(\Omega)}$, $0\leq t\leq T$.

LEMMA 2. If $\vec{v}(x,t)$ satisfies for $t\in[0,T]$ relations (83), (84) and if $\vec{f},\vec{f}_t\in L_2(Q_T)$, $\vec{v}_0(x)\in W_2^2(\Omega)\cap \mathring{J}_2^1(\Omega)$, then for any $t\in[0,T]$ and any $æ>0$ we have the following inequalities:

$$\psi^2_æ(t)\leq e^{C_{19}(æ)T}\left\{\psi^2_æ(0)+\|\vec{f}\|^2_{L_2(Q_t)}\right\}\equiv C_{20}(t), \tag{89}$$

$$\psi^2_æ(t)+2\nu\int_0^t F^2(\tau)d\tau\leq C_{19}(æ)\int_0^t C_{20}(\tau)d\tau+\psi^2_æ(0)+\|\vec{f}\|^2_{L_2(Q_t)}, \tag{90}$$

where $C_{19}(æ)=2\max\{1;\frac{C_{17}^{1/2}C_\Omega^2}{æ^2}\}$.

Indeed, from equality (84), applying Hölder's and Cauchy's inequalities and making use of inequalities (87), (88), we obtain the inequality:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\psi^2_æ(t)&\leq\frac{1}{2}\frac{d}{dt}\left(\|\vec{v}_t\|^2_{L_2(\Omega)}+æ\|\vec{v}_{xt}\|^2_{L_2(\Omega)}\right)+\nu\|\vec{v}_{xt}\|^2_{L_2(\Omega)}=(\vec{f}_t,\vec{v}_t)_{L_2(\Omega)}-\int_\Omega v_{kt}\vec{v}_{x_k}\vec{v}_t\,dx\leq\\
&\leq\|\vec{f}_t\|_{L_2(\Omega)}\|\vec{v}_t\|_{L_2(\Omega)}+\|\vec{v}_t\|_{L_4(\Omega)}\|\vec{v}_x\|_{L_2(\Omega)}\leq\frac{1}{2}\left(\|\vec{f}_t\|^2_{L_2(\Omega)}+\|\vec{v}_t\|^2_{L_2(\Omega)}\right)+\\
&+C_\Omega^2\|\vec{v}_x\|_{L_2(\Omega)}\|\vec{v}_{xt}\|^2_{L_2(\Omega)}\leq\frac{1}{2}\|\vec{f}_t\|^2_{L_2(\Omega)}+\frac{1}{2}\left(\|\vec{v}_t\|^2_{L_2(\Omega)}+2æ\frac{C_{17}^{1/2}}{æ^2}\|\vec{v}_{xt}\|^2_{L_2(\Omega)}\right)\leq\frac{1}{2}\|\vec{f}_t\|^2_{L_2(\Omega)}+\frac{1}{2}C_{19}(æ)\psi^2_æ(t), \quad 0<t\leq T,
\end{aligned}\tag{91}$$

and from this inequality, applying Gronwall's lemma, we obtain estimate (89).

Then, inequality (91) can be rewritten in the following manner:

$$\frac{d}{dt}\psi^2_æ(t)+2\nu F^2(t)\leq\|\vec{f}_t\|^2_{L_2(\Omega)}+C_{19}(æ)\psi^2_æ(t),\ 0<t\leq T, \tag{92}$$

from where, integrating with respect to t from 0 to $t\leq T$ and making use of estimate (89), we obtain estimate (90).

LEMMA 3. If $\vec{v}(x,t)$ satisfies for $t\in[0,T]$ relations (83) and (85) and if $\vec{f}\in L_2(Q_T)$, $\vec{v}_0(x)\in \mathring{J}_2^1(\Omega)$, then for any $t\in[0,T]$ and for any $æ>0$ we have the inequality

$$\nu\varphi^2(t) + \int_0^t \psi_{\varkappa}^2(\tau)d\tau \leqslant C_{21}(\varkappa), \tag{93}$$

where the constant C_{21} depends only on $\|\vec{f}\|_{L_2(Q_T)}$, $\|\vec{v}_0\|_{W_2^1(\Omega)}$, C_Ω, and $\varkappa$ and $C_{21}(\varkappa) \to \infty$ for $\varkappa \to 0$.

Indeed, integrating equality (85) with respect to from 0 to $t \leqslant T$ and making use of the Hölder and Cauchy inequalities, we obtain:

$$\int_0^t \psi_{\varkappa}^2(\tau)d\tau + \nu\varphi^2(t) \leqslant \frac{\nu}{2}\int_\Omega \vec{v}_{0x}^2 dx + \frac{1}{2}\iint_{Q_t} \vec{f}^2 dQ + \frac{1}{2}\iint_{Q_t} \vec{v}_t^2 dQ + \frac{\varkappa}{2}\iint_{Q_t} \vec{v}_{xt}^2 dQ + \frac{3}{2\varkappa}\iint_{Q_t} \vec{v}^4 dQ. \tag{94}$$

From here, estimating the last integral in the right-hand side with the aid of inequality (80) and making use of the already proved estimate (87), we obtain estimate (93).

Now we prove some estimates which generalize the estimates obtained by Ladyzhenskaya for the strong solutions of the Navier–Stokes equations [5, Chap. VI]. Unlike estimates (89), (90), and (93), they hold uniformly with respect to $\varkappa \in [0,1]$ and are proved basically in the same way as the analogous estimates for the solutions of the Navier–Stokes equations.

<u>LEMMA 4.</u> Assume that the function $\vec{v}(x,t)$ satisfies for $t \in [0,T]$ relations (83), (84), $\vec{f} \in L_2(Q_T)$, $\vec{f}_t \in L_{2,1}(Q_T)$, $\vec{v}_0(x) \in W_2^2(\Omega) \cap \overset{\circ}{J}{}_2^1(\Omega)$, and assume that

$$\nu - C_\Omega^2 \sqrt{\frac{C_{17}^{1/2}}{\nu}(\sqrt{2}\,C_{22} + C_{23})} \equiv \gamma > 0, \tag{95}$$

where C_{17} is the constant from estimate (87) and $C_{22} = \psi_1(0) + \|\vec{f}_t\|_{L_{2,1}(Q_T)}$, $C_{23} = \max_{[0,T]} \|\vec{f}(x,t)\|_{L_2(\Omega)}$. Then $\forall t \in [0,T]$ we have, uniformly with respect to $\varkappa \in [0,1]$, the inequalities

$$\varphi(t) \leqslant \left[\frac{C_{17}^{1/2}}{\nu}(\sqrt{2}\,C_{22} + C_{23})\right]^{1/2}, \tag{96}$$

$$\psi_{\varkappa}(t) \leqslant C_{22}, \tag{97}$$

$$\psi_{\varkappa}^2(t) + 2\gamma\int_0^t F^2(\tau)d\tau \leqslant 2C_{22}^2 + \psi_1^2(0) \tag{98}$$

Indeed, from equality (84), with the aid of inequality (91), we obtain

$$\frac{1}{2}\frac{d}{dt}\psi_{\varkappa}^2(t) + [\nu - C_\Omega^2\varphi^2(t)]F^2(t) \leqslant \psi_{\varkappa}(t)\|\vec{f}_t\|_{L_2(\Omega)}. \tag{99}$$

On the other hand, from equality (83) and estimate (87) we have

$$\begin{aligned} \nu\varphi^2(t) &\leqslant \|\vec{v}\|_{L_2(\Omega)}\left(\|\vec{f}\|_{L_2(\Omega)} + \|\vec{v}_t\|_{L_2(\Omega)}\right) + \varkappa\|\vec{v}_x\|_{L_2(\Omega)}\|\vec{v}_{xt}\|_{L_2(\Omega)} \leqslant \\ &\leqslant C_{17}^{1/2}(\sqrt{2}\,\psi_{\varkappa}(t) + \|\vec{f}\|_{L_2(\Omega)}), \end{aligned} \tag{100}$$

and therefore

$$\varphi(t) \leqslant \left[\frac{C_{17}^{1/2}}{\nu}(\sqrt{2}\,\psi_{\varkappa}(t) + C_{23})\right]^{1/2}. \tag{101}$$

At the initial moment $t = 0$, by virtue of (95) and (101) we have

$$\nu - C_\Omega^2 \varphi^2(0) \geq \gamma > 0. \tag{102}$$

Then, as in the case of the Navier–Stokes equations, it is easy to show that also $\forall t \in [0,T]$

$$\nu - C_\Omega^2 \varphi^2(t) > 0, \tag{103}$$

and then from inequality (99) it follows at once that

$$\psi_\varkappa(t) \leq \psi_1(0) + \int_0^t \|\vec{f}\|_{L_2(\Omega)} dt \leq C_{22}. \tag{104}$$

After this, from inequality (101) we obtain estimate (96) and the inequality

$$\nu - C_\Omega^2 \varphi^2(t) \geq \gamma, \quad 0 \leq t \leq T. \tag{105}$$

Finally, from inequality (99), making use of estimate (104) and inequality (105), we obtain estimate (98).

Remark. From (72) it follows that

$$\vec{v}_t(x,0) - \Delta\vec{v}_t(x,0) + \operatorname{grad} p(x,0) = \vec{f}(x,0) + \nu\Delta\vec{v}_0(x) - v_{0k}(x)\vec{v}_{0x_k}(x) \equiv \vec{F}(x), \ x \in \Omega. \tag{106}$$

If $\vec{f}(x,0) \in L_2(\Omega)$, $\vec{v}_0(x) \in W_2^2(\Omega) \cap \overset{\circ}{J}{}_2^1(\Omega)$, then $\vec{F}(x) \in L_2(\Omega)$. Then $\vec{v}_t(x,0) - \Delta\vec{v}_t(x,0)$ and $\operatorname{grad} p(x,0)$ are obtained as the projections of $\vec{F}(x)$ onto the subspaces $\overset{\circ}{J}(\Omega)$ and $G(\Omega)$, respectively. Finally, solving the elliptic boundary-value problem

$$\Delta\vec{v}_t(x,0) - \vec{v}_t(x,0) = -Pr_{\overset{\circ}{J}(\Omega)} \vec{F}(x), \ x \in \Omega, \ \vec{v}_t(x,0)\big|_{\partial\Omega} = 0, \tag{107}$$

we find $\vec{v}_t(x,0)$, $\vec{v}_{tx}(x,0)$, and therefore also $\psi_1(0)$, which occur in estimates (96)-(98).

We mention one more variant of the conditions under which for the solutions of the initial – boundary-value problem (72), (36) one can obtain estimates of the type (96)-(98), uniform with respect to $\varkappa \in [0,1]$ (see [5, Chap. VI]).

LEMMA 5. Assume that the function $\vec{v}(x,t)$ satisfies the same conditions as in Lemma 4 and let β and k be positive numbers such that

$$\gamma_1 \equiv \nu - [\beta(\|\vec{v}_{0x}\|_{L_2(\Omega)}^2 + k)^2]^{1/3} > 0. \tag{108}$$

Assume further that

$$C_{24}(\beta) = \frac{1}{4\beta} + \frac{C_{17}}{8}\left(\frac{4}{3}\right)^6\left(\frac{7}{8}\right)^7 \beta^{-7/3}, \tag{109}$$

$$C_{25} = \psi_1^2(0) + 2C_{18}C_{24}C_{22}^2 \exp\frac{2C_{18}C_{24}}{\nu} + 2C_{22}^2 \exp\frac{C_{18}C_{24}}{\nu}, \tag{110}$$

where C_{17}, C_{18}, C_{22} are the constants from Lemmas 1 and 4. Then for all $t \in [0,T]$, where

$$T_1 \geq \frac{2\gamma_1 k^2}{C_{25}}, \tag{111}$$

we have, uniformly with respect to $\varkappa \in [0,1]$, the inequalities:

$$\varphi(t) \leq \left[\frac{C_{17}}{\nu}^{1/2}\left(\sqrt{2}\, C_{22} \exp \frac{C_{17}^{1/2} C_{24}}{\nu} + C_{25}\right)\right]^{1/2}, \tag{112}$$

$$\psi_\varkappa(t) \leq C_{22} \exp \frac{C_{17}^{1/2} C_{24}}{\nu}, \tag{113}$$

$$\psi_\varkappa^2(t) + 2\gamma_1 \int_0^t F^2(\tau)\, d\tau \leq C_{25}. \tag{114}$$

Finally, we show that if all the given data of the initial-boundary-value problem (72), (36) and its solution $\vec{v}(x,t)$ do not depend on one of the spatial variables (e.g., on x_3), then, as in the case of the Navier–Stokes equations, the functions $\varphi(t)$, $\psi_\varkappa(t)$, $\int_0^t F^2(\tau)\, d\tau$ can be estimated uniformly with respect to $\varkappa \in [0,1]$ without any restrictions on the smallness of the given data. Namely, we have the following lemma.

LEMMA 6. Assume that $\vec{v}(x,t)$ satisfies for $t \in [0,T]$ relations (83), (84), assume that $\vec{v}(x,t)$, $\vec{f}(x,t)$, and $\vec{v}_0(x)$ do not depend on x_3, and let $\vec{f} \in L_2(Q_T)$, $\vec{f}_t \in L_{2,1}(Q_T)$, $\vec{v}_0(x) \in W_2^2(\Omega) \cap \overset{\circ}{J}{}_2^1(\Omega)$. Then $\forall t \in [0,T]$ we have, uniformly with respect to $\varkappa \in [0,1]$, the inequalities:

$$\psi_\varkappa(t) \leq C_{22} \exp\left\{\frac{C_{18}}{\nu^2}\right\}, \tag{115}$$

$$\int_0^t F^2(\tau)\, d\tau \leq \frac{C_{22}}{\nu}\left\{1 + 2\exp\frac{C_{18}}{\nu^2} + 2\frac{C_{18}}{\nu^2}\exp\frac{2C_{18}}{\nu^2}\right\} \equiv C_{26}. \tag{116}$$

Indeed, making use of the fact that $\vec{v}(x,t)$ depends only on two spatial variables, we estimate the integral in equality (84) with the aid of the following inequality [5, Chap. 1]:

$$\|u\|_{L_4(\Omega)}^4 \leq 2\|u\|_{L_2(\Omega)}^2 \cdot \|u_x\|_{L_2(\Omega)}^2, \tag{117}$$

valid for any function $u(x) \in \overset{\circ}{W}{}_2^1(\Omega)$ and any two-dimensional domain:

$$\left|\int_\Omega v_{kt} \vec{v}_{x_k} \vec{v}_t\, dx\right| \leq \|\vec{v}_x\|_{L_2(\Omega)} \|\vec{v}_t\|_{L_4(\Omega)}^2 \leq \sqrt{2}\, \|\vec{v}_x\|_{L_2(\Omega)} \|\vec{v}_t\|_{L_2(\Omega)} \|\vec{v}_{xt}\|_{L_2(\Omega)} \leq \frac{\nu}{2} F^2(t) + \frac{2}{\nu}\|\vec{v}_x\|_{L_2(\Omega)}^2 \psi_\varkappa^2(t). \tag{118}$$

Then from equality (84) we obtain the inequality

$$\frac{d}{dt}\psi_\varkappa^2(t) + \nu F^2(t) \leq \frac{2}{\nu}\varphi^2(t)\psi_\varkappa^2(t) + \|\vec{f}_t\|_{L_2(\Omega)} \psi_\varkappa(t), \quad 0 < t \leq T, \tag{119}$$

from which, making use of Gronwall's lemma and estimate (88), we obtain estimates (115) and (116).

With the aid of the a priori estimates, obtained in Lemmas 1-6, one proves by the Galerkin method the following existence theorems for the strong solution of problem (72), (36).

THEOREM 13. Let $\vec{f},\ \vec{f}_t \in L_2(Q_T)$, $\vec{v}_0(x) \in W_2^2(\Omega) \cap \overset{\circ}{J}{}_2^1(\Omega)$. Then the initial-boundary-value problem (72), (36) has for every $\varkappa > 0$ a unique strong solution and for this solution the estimates of Lemmas 1 and 2 hold.

THEOREM 14. Let $\vec{f}(x,t) \in L_2(Q_T)$, $\vec{f}_t \in L_{2,1}(Q_T)$, $\vec{v}_0(x) \in W_2^2(\Omega) \cap \overset{\circ}{J}{}_2^1(\Omega)$ and assume that the initial data of problem (72), (36) satisfy one of the smallness conditions formulated in Lemmas 4, 5. Then problem (72), (36) has for each $\varkappa > 0$ a unique strong solution $\vec{v}^{\varkappa}(x,t)$, the estimates of Lemmas 1, 4 or 1, 5, respectively, hold for it, and for $\varkappa \to 0$ the solution $\vec{v}^{\varkappa}(x,t)$ converges to the unique strong solution $\vec{v}(x,t)$ of problem (5), (36) for the Navier–Stokes equations.

For the proof of the last assertion of Theorem 14, we note that, as it follows from the a priori estimates (96)-(98) or (112)-(114) for the solutions $\vec{v}^{\varkappa}(x,t)$ of problem (72), (36), from $\{\vec{v}^{\varkappa}\}$ one can extract a subsequence $\{\vec{v}^{\varkappa_i}\}$ which for $\varkappa_i \to 0$ converges strongly in $L_4(Q_T)$ to the limit function $\vec{v}(x,t)$ and for which $\vec{v}_x^{\varkappa_i}$, $\vec{v}_t^{\varkappa_i}$, $\vec{v}_{xt}^{\varkappa_i}$ converge weakly in $L_2(Q_T)$ to $\vec{v}_x$, $\vec{v}_t$, $\vec{v}_{xt}$, respectively. Finally, from the a priori estimate (97) or (113) it follows that we have, uniformly with respect to $\varkappa \in [0,1]$ and $t \in [0,T]$,

$$\varkappa \|\vec{v}_{xt}^{\varkappa}\|^2_{L_2(\Omega)} \leq C, \tag{120}$$

and from here it follows at once that $\varkappa_i \iint\limits_{Q_t} \vec{v}_{xt}^{\varkappa_i} \vec{\Phi}_x \, dQ \to 0, \varkappa_i \to 0 \ \forall \vec{\Phi} \in \overset{\circ}{J}{}_2^1(Q_T)$. Taking the limit in integral identity (74) for $\varkappa_i \to 0$, we obtain that the limit function $\vec{v}(x,t)$ will satisfy the integral identity

$$\iint\limits_{Q_T} (\vec{v}_t \vec{\Phi} + \nu \vec{v}_x \vec{\Phi}_x - v_k \vec{v} \vec{\Phi}_{x_k}) dQ = \iint\limits_{Q_T} \vec{f} \vec{\Phi} \, dQ, \ \forall \vec{\Phi} \in \overset{\circ}{J}{}_2^1(Q_T), \tag{121}$$

i.e., it will be a strong solution of problem (5), (36) for the Navier–Stokes equations [5, Chap. VI]. Since the strong solution of problem (5), (36) is unique, it follows that also the entire totality $\{\vec{v}^{\varkappa}\}$ of solutions of problem (72), (36) converges for $\varkappa \to 0$ to $\vec{v}(x,t)$.

THEOREM 15. Assume that none of the given data of problem (72), (36) depends on the Cartesian coordinate x_3, assume that Ω is a two-dimensional bounded domain and assume that $\vec{f}(x,t) \in L_2(Q_T)$, $\vec{f}_t \in L_{2,1}(Q_T)$, $\vec{v}_0(x) \in W_2^2(\Omega) \cap \overset{\circ}{J}{}_2^1(\Omega)$. Then problem (72), (36) has for each $\varkappa > 0$ a unique strong solution $\vec{v}^{\varkappa}(x,t)$, the estimates of Lemmas 1 and 6 hold for this solution and for $\varkappa \to 0$ these solutions $\vec{v}^{\varkappa}$ converge to the unique strong solution $\vec{v}(x,t)$ of problem (5), (36) for the system of Navier–Stokes equations, and also for the difference $\vec{v}^{\varkappa} - \vec{v} \equiv \vec{\omega}$ one has the estimate:

$$|\vec{\omega}|^2_{Q_T} \equiv \max_{[0,T]} \int\limits_{\Omega} \vec{\omega}^2(x,t)\,dx + \iint\limits_{Q_T} \vec{\omega}_x^2 \, dQ = O(\varkappa^2), \quad \varkappa \to 0. \tag{122}$$

In order to prove estimate (122), first of all we note that for $\vec{\omega}(x,t)$ we have the equality

$$\frac{1}{2}\int\limits_{\Omega} \vec{\omega}^2 dx + \nu \iint\limits_{Q_t} \vec{\omega}_x^2 \, dQ = \varkappa \iint\limits_{Q_t} \vec{\omega}_x \vec{v}_{xt}^{\varkappa} \, dQ - \iint\limits_{Q_t} \omega_k \vec{\omega} \vec{v}_{x_k}^{\varkappa} \, dQ, \quad 0 < t \leq T. \tag{123}$$

Applying, for the estimates of the integrals in the right-hand side of (123), the Hölder and Cauchy inequalities and also inequality (117), we obtain from (123) the inequality

$$\frac{1}{2}\int_\Omega \vec{\omega}^2 dx + \nu \iint_{Q_t} \vec{\omega}_x^2 dQ \leq \frac{\nu}{2}\iint_{Q_t} \vec{\omega}_x^2 dQ + \frac{æ^2}{2\nu}\iint_{Q_t} (\vec{v}^{æ}_{xt})^2 dQ + \sqrt{2}\Big(\iint_{Q_t} (\vec{v}^{æ}_x)^2 dQ\Big)^{1/2} |\vec{\omega}|^2_{Q_t}, \tag{124}$$

and from this inequality, making use of the estimate (116) of Lemma 6, we obtain the inequality

$$|\vec{\omega}|^2_{Q_t} \leq C_{27}(\nu, C_{26}) æ^2 + C_{28}(\nu)\Big(\iint_{Q_t} (\vec{v}^{æ}_x)^2 dQ\Big)^{1/2} |\vec{\omega}|^2_{Q_t}, \quad 0 < t \leq T. \tag{125}$$

By virtue of estimate (88) of Lemma 1, for sufficiently small $t < t_1 : C_{28}\Big(\iint_{Q_t} (\vec{v}^{æ}_x)^2 dQ\Big) \leq \frac{1}{2}$, therefore for $0 < t < t_1$ we have

$$|\vec{\omega}|^2_{Q_t} \leq 2C_{27} æ^2. \tag{126}$$

After this estimate one can prove (122) on the entire interval [0, T] by successive steps with respect to t (see [5, Chap. VI, Sec. 2]; see also [9]).

The generalized solutions of the initial–boundary-value problem (72), (36), whose existence has been proved in Theorems 13-15, possess better properties than are required by definition from the strong solution of problem (72), (36), since they have for each $t \in [0,T]$ derivatives $\vec{v}_t$, $\vec{v}_{xt} \in L_2(\Omega)$. Correspondingly, the conditions on $\vec{f}(x,t)$ are overstated (one needs the existence of the derivative $\vec{f}_t$ from $L_2(Q_T)$ or $L_{2,1}(Q_T)$). Therefore, the following theorem presents interest, being proved, as before, by the Galerkin method with the use of the a priori estimates given by Lemmas 1 and 3.

<u>THEOREM 16.</u> Let $\vec{f}(x,t) \in L_2(Q_T)$, $\vec{v}_0(x) \in W_2^2(\Omega) \cap \overset{\circ}{J}{}_2^1(\Omega)$. Then problem (72), (36) has for each $æ > 0$ a unique strong solution and for this solution the estimates of Lemmas 1 and 3 hold.

Now we proceed to the investigation of the weak solutions of problem (72), (36). Here the following theorem is fundamental.

<u>THEOREM 17.</u> Let $\vec{f}(x,t) \in L_2(Q_T)$, $\vec{v}_0(x) \in \overset{\circ}{J}{}_2^1(\Omega)$. Then problem (72), (36) has for each $æ > 0$ a unique weak solution $\vec{v}^{æ}(x,t)$ and for this solution we have the following estimates:

$$\max_{[0,T]} \int_\Omega [(\vec{v}^{æ})^2 + æ(\vec{v}^{æ}_x)^2] dx + \nu \iint_{Q_T} (\vec{v}^{æ}_x)^2 dQ \leq C_{28}, \tag{127}$$

and for sufficiently small $\tau > 0$, under the condition that $\partial\Omega \in C^{(2)}$

$$\int_0^T \Big\{ \|\Delta_\tau \vec{v}^{æ}\|^2_{L_2(\Omega)} + æ\|\Delta_\tau \vec{v}^{æ}_x\|^2_{L_2(\Omega)} \Big\} dt \leq C_{29} \cdot \tau^{2\gamma}, \tag{128}$$

where $\Delta_\tau \vec{v}^{æ} \equiv \vec{v}^{æ}(x, t+\tau) - \vec{v}^{æ}(x,t)$ while the constants C_{28}, C_{29} and the exponent $\gamma \leq 2/7$ depend only on $\|\vec{f}\|_{L_2(Q_T)}$, $\|\vec{v}_0\|_{W_2^1(\Omega)}$, T, $\partial\Omega \in C^{(2)}$ and do not depend on $æ \geq 0$.

The existence of the weak solution of problem (72), (36) is proved by the Galerkin method. The a priori estimate (127) [or, which is the same, (76)] is proved by the limiting process for $n \to \infty$ from the similar estimates for the Galerkin approximations $\vec{v}^{æ,n}(x,t)$ for the solutions of problem (72), (36), which, in turn, is proved in the same way as the estimates of Lemma 1. Finally, a priori estimate (128) is proved with the aid of estimate (127), basically in the same way as the similar estimate for the weak solutions $\vec{v}(x,t)$ of problem (5), (36) for the Navier–Stokes equations [10].

The a priori estimates (127) and (128) are uniform with respect to $\varkappa \geqslant 0$ and the well-known compactness criteria in $L_2(Q_T)$ allow us to prove:

<u>THEOREM 18.</u> Let $\vec{f} \in L_2(Q_T)$, $\vec{v}_0(x) \in \overset{\circ}{J}{}_2^1(\Omega)$, $\partial\Omega \in C^{(2)}$. Then from the totality $\{\vec{v}^{\varkappa}\}$ of the weak solutions of problem (72), (36) one can extract a subsequence $\{\vec{v}^{\varkappa_i}\}$ which converges for $\varkappa_i \to 0$ to the weak solution $\vec{v}(x,t)$ of problem (5), (36) for the Navier–Stokes equations.

In the two-dimensional case every totality $\{\vec{v}^{\varkappa}\}$ of weak solutions of problem (72), (36) converges to the unique weak solution $\vec{v}$ of problem (5), (36).

In [11], the existence theorem for the weak solution of problem (72), (36), under the same conditions on $\vec{f}$ and $\vec{v}_0$ as in Theorem 17, is proved by the method of finite differences. Also there it is proved that from the totality $\{\vec{v}_h^{\varkappa}\}$ of the multilinear solutions of the finite-difference problems, approximating problem (72), (36), one can extract a subsequence which for Δt, $\Delta x = h$, $\varkappa \to 0$ converges to the weak solution $\vec{v}(x,t)$ of problem (5), (36) for the system of Navier–Stokes equations.

9. We investigate the problem of the limiting process for $\varkappa \to 0$ in the case of the stationary problems. The stationary system, corresponding to (4), is

$$-\nu\Delta\vec{v} + v_k\frac{\partial\vec{v}}{\partial x_k} - \varkappa v_k\frac{\partial\Delta\vec{v}}{\partial x_k} + \operatorname{grad} p = \vec{f}(x), \quad \operatorname{div}\vec{v} = 0. \tag{129}$$

We shall solve system (129) in the bounded domain under the boundary condition Ω

$$\vec{v}\big|_{\partial\Omega} = 0. \tag{130}$$

In [6] one has introduced for problem (129), (130) the generalized solution $\vec{v}(x)$ from the space $\overset{\circ}{W}{}_2^2(\Omega) \cap \overset{\circ}{J}(\Omega)$, which is defined as that function from the indicated space which satisfies the integral identity

$$\nu\int_\Omega \vec{v}_x\vec{\Phi}_x\,dx - \int_\Omega v_k\vec{v}\vec{\Phi}_{x_k} + \varkappa\int_\Omega v_k\Delta\vec{v}\vec{\Phi}_{x_k}\,dx = \int_\Omega \vec{f}\vec{\Phi}\,dx \tag{131}$$

for any $\vec{\Phi}(x) \in H(\Omega)$, and one has proved that if $\vec{f}(x) \in L_2(\Omega)$, then problem (129), (130) has at least one generalized solution $\vec{v}(x) \in \overset{\circ}{W}{}_2^2(\Omega) \cap \overset{\circ}{J}(\Omega)$ and for any such solution the following energy inequality holds:

$$\nu\int_\Omega \vec{v}_x^2\,dx + \nu\varkappa\int_\Omega |\Delta\vec{v}|^2\,dx \leq C_{30}(\nu, \Omega, \|\vec{f}\|_{L_2(\Omega)}). \tag{132}$$

Inequality (132), whose right-hand side does not depend on $\varkappa \geqslant 0$, allows us to prove the following theorem.

<u>THEOREM 19.</u> Let $\vec{f}(x) \in L_2(\Omega)$. Then for $\varkappa \to 0$ the generalized solution $\vec{v}^{\varkappa}(x)$ of problem (129), (130) from $\overset{\circ}{W}{}_2^2(\Omega) \cap \overset{\circ}{J}(\Omega)$ tends to the generalized solution from $H(\Omega)$ (Ladyzhenskaya's solution) of the first boundary-value problem for the stationary Navier–Stokes system

$$-\nu\Delta\vec{v} + v_k\frac{\partial\vec{v}}{\partial x_k} + \operatorname{grad} p = \vec{f}(x); \quad \operatorname{div}\vec{v} = 0, \quad x \in \Omega, \quad \vec{v}\big|_{\partial\Omega} = 0. \tag{133}$$

We recall [5, Chap. V] that the generalized solution from $H(\Omega)$ of the boundary-value problem (133) is defined as a function $\vec{v}(x)\in H(\Omega)$ which satisfies the integral identity

$$\nu\int_\Omega \vec{v}_x\vec{\Phi}_x\,dx-\int_\Omega v_k\vec{v}\vec{\Phi}_{x_k}\,dx=\int_\Omega \vec{f}\vec{\Phi}\,dx,\quad \forall\vec{\Phi}\in H(\Omega). \tag{134}$$

It is proved in [5] that the generalized solution from $H(\Omega)$ of problem (133) exists for any $\vec{f}(x)\in L_2(\Omega)$.

Theorem 19 is an immediate consequence of inequality (132). Indeed, by virtue of (132), from totality $\{\vec{v}^{æ}\}$ of the generalized solutions of problem (129), (130) one can select a subsequence $\{\vec{v}^{æ_i}\}$, which for $æ_i\to 0$ converges weakly in $H(\Omega)$ and strongly in $L_4(\Omega)$ to the limiting function $\vec{v}(x)\in H(\Omega)$. From here it follows that $\int_\Omega v_k^{æ_i}\vec{v}^{æ_i}\vec{\Phi}_{x_k}\,dx\to\int_\Omega v_k\vec{v}\vec{\Phi}_{x_k}\,dx$, $\forall\vec{\Phi}(x)\in H(\Omega)$. Then, by virtue of (132),

$$æ_i\int_\Omega v_k^{æ_i}\Delta\vec{v}^{æ_i}\vec{\Phi}_{x_k}\,dx\to 0,\quad æ_i\to 0,\quad \forall\vec{\Phi}\in H(\Omega). \tag{135}$$

Taking in the integral identity (131) the limit when $æ_i\to 0$ we obtain that the limiting function $\vec{v}(x)\in H(\Omega)$ satisfies the integral identity (134), i.e., it is a generalized solution from $H(\Omega)$ of problem (133).

10. Let Ω be a three-dimensional unbounded domain, situated outside a smooth surface $\partial\Omega$. We consider in Ω the stationary system (129) and we shall solve for it the flow problem with zero boundary conditions at infinity:

$$\vec{v}\big|_{\partial\Omega}=0;\quad \vec{v}(x)\to 0,\quad |x|\to\infty. \tag{136}$$

We define the generalized solution of the problem (129), (136) from the class $\overset{\circ}{H}{}^{(2)}(\Omega)\equiv\overset{\circ}{W}{}_2^2(\Omega)\cap\overset{\circ}{J}(\Omega)$* as a function from this space which satisfies the integral identity (131) for any $\vec{\Phi}(x)\in\overset{\circ}{J}(\Omega)$ (see [5, Chap. V]) and we prove that we have the following theorem.

<u>THEOREM 20.</u> Let $\vec{f}(x)\in L_{6/5}(\Omega)\cap L_2(\Omega)$. Then flow problem (129), (136) has at least one generalized solution $\vec{v}(x)\in\overset{\circ}{H}{}^{(2)}(\Omega)$ and for any such solution we have the estimate:

$$\int_\Omega \vec{v}_x^2\,dx+æ\int_\Omega|\Delta\vec{v}|^2\,dx\le C_{31}, \tag{137}$$

where the constant C_{31} depends only on the norms $\|\vec{f}\|_{L_{6/5}(\Omega)}$ and $\|\vec{f}\|_{L_2(\Omega)}$ and on ν.

For $æ\to 0$ the generalized solution $\vec{v}^{æ}(x)$ of problem (129), (136) from the class $\overset{\circ}{H}{}^{(2)}(\Omega)$ tends to the generalized solution from $H(\Omega)$ (Ladyzhenskaya's solution) of flow problem (133), (136) for the stationary system of Navier–Stokes equations.

In order to prove the theorem we consider the sequence of extending domains Ω_n with exterior boundaries $\partial\Omega_n$, $n=1,2,\ldots$, exhausting in the limit the entire domain Ω and in each of them we approximate the system (129) by the following system with a small parameter $\varepsilon>0$ ([6]):

*Obviously, $\overset{\circ}{H}{}^{(2)}(\Omega)$ is the closure of the set $\overset{\circ}{J}(\Omega)$ of the smooth, finite, solenoidal vectors in Ω, in the norm induced by the inner product

$$[\vec{v},\vec{w}]_{H^{(2)}(\Omega)}\equiv\int_\Omega(\vec{v}_x\vec{w}_x+\vec{v}_{xx}\vec{w}_{xx})\,dx.$$

$$\varepsilon\Delta^2\vec{v}-\nu\Delta\vec{v}+v_k\frac{\partial}{\partial x_k}(\vec{v}-\varkappa\Delta\vec{v})+\mathrm{grad}\,p=\vec{f}(x),\quad \mathrm{div}\,\vec{v}=0, \tag{138}$$

which will be solved in the domain Ω_n under the following boundary conditions:

$$\vec{v}\big|_{\partial\Omega}=\Delta\vec{v}\big|_{\partial\Omega}=0,\quad \vec{v}\big|_{\partial\Omega_n}=\Delta\vec{v}\big|_{\partial\Omega_n}=0. \tag{139}$$

For each of problems (138), (139), $n=1,2,\dots$, we define, following [6], a generalized solution $\vec{v}^{\varepsilon,n}(x)$ from $\overset{\circ}{W}{}_2^3(\Omega)\cap\overset{\circ}{J}(\Omega)$ as a function of this space which satisfies the integral identity

$$-\varepsilon\int_{\Omega_n}\Delta\vec{v}_x^{\varepsilon,n}\vec{\Phi}_x\,dx+\nu\int_{\Omega_n}\vec{v}_x^{\varepsilon,n}\vec{\Phi}_x\,dx+\int_{\Omega_n}v_k^{\varepsilon,n}(\varkappa\Delta\vec{v}^{\varepsilon,n}-\vec{v}^{\varepsilon,n})\vec{\Phi}_{x_k}\,dx=\int_{\Omega_n}\vec{f}\vec{\Phi}\,dx \tag{140}$$

for any $\vec{\Phi}(x)\in\overset{\circ}{J}(\Omega_n)$ (or, which for the bounded domain Ω_n is the same, for any $\Phi(x)\in\overset{\circ}{W}{}_2^3(\Omega_n)\cap J(\Omega)$).

For the generalized solutions of problem (138), (139) from $\overset{\circ}{W}{}_2^3(\Omega)\cap\overset{\circ}{J}(\Omega)$ we have the inequality

$$\nu\int_{\Omega_n}\vec{v}_x^{\varepsilon,n\,2}dx+\nu\varkappa\int_{\Omega_n}|\Delta\vec{v}^{\varepsilon,n}|^2dx-\varepsilon\int_{\Omega_n}|\Delta\vec{v}^{\varepsilon,n}|^2dx-\varepsilon\varkappa\int_{\Omega_n}|\Delta\vec{v}_x^{\varepsilon,n}|^2\,dx\leqslant C_{32}, \tag{141}$$

in which the constant C_{32} depends only on ν and on the norms $\|\vec{f}\|_{L_{6/5}(\Omega)}$, $\|\vec{f}\|_{L_2(\Omega)}$ and depends neither on the dimensions of the domain Ω_n, nor on $\varepsilon,\varkappa\geqslant 0$. In order to prove inequality (141) it is sufficient to take in the integral identity (140) $\vec{\Phi}(x)=\varkappa\Delta\vec{v}^{\varepsilon,n}-\vec{v}^{\varepsilon,n}$ and then to estimate the right-hand side with the aid of the Hölder and Cauchy inequalities and of the well-known inequality [5, Chap. I]

$$\|\vec{v}^{\varepsilon,n}\|_{L_6(\Omega_n)}\leqslant(48)^{1/6}\|\vec{v}_x^{\varepsilon,n}\|_{L_2(\Omega_n)}, \tag{142}$$

valid in any three-dimensional domain, including the unbounded ones too.

On the basis of a priori estimate (141) one proves (see [6]) that problem (138), (139) has for each $n=1,2,\dots$ and $\varepsilon>0$ at least one generalized solution $\vec{v}^{\varepsilon,n}(x)$ from $\overset{\circ}{W}{}_2^3(\Omega)\cap\overset{\circ}{J}(\Omega)$ and that for each fixed $n=1,2,$ one can extract from the totality $\{\vec{v}^{\varepsilon,n}(x)\}$ of such solutions a sequence $\{\vec{v}^{\varepsilon_i,n}(x)\}$ which for $\varepsilon_i\to 0$ converges weakly in $\overset{\circ}{H}{}^{(2)}(\Omega_n)$ and strongly in $L_2(\Omega)$ and in $L_4(\Omega_n)$ to the limiting function $\vec{v}^n(x)\in\overset{\circ}{H}{}^{(2)}(\Omega)$. In addition,

$$\varepsilon_i\int_{\Omega_n}\Delta\vec{v}_x^{\varepsilon_i,n}\vec{\Phi}_x\,dx\to 0,\quad \varepsilon_i\to 0,\quad \forall\vec{\Phi}(x)\in\overset{\circ}{J}(\Omega_n). \tag{143}$$

Taking now the limit in integral identity (140) as $\varepsilon_i\to 0$, we obtain that the function $\vec{v}^n(x)$ satisfies the integral identity

$$\nu\int_{\Omega_n}\vec{v}_x^n\vec{\Phi}_x\,dx+\int_{\Omega_n}v_k^n(\varkappa\Delta\vec{v}^n-\vec{v}^n)\vec{\Phi}_x\,dx=\int_{\Omega_n}\vec{f}\vec{\Phi}\,dx \tag{144}$$

for any $\vec{\Phi}(x)\in\overset{\circ}{J}(\Omega_n)\subset\overset{\circ}{J}(\Omega)$. In addition, for any $\vec{v}^n(x)$ we have the inequality

$$\nu\int_{\Omega_n}\vec{v}_x^{n^2}\,dx+\nu\varkappa\int_{\Omega_n}|\Delta\vec{v}^n|^2\,dx\le C_{32},\tag{145}$$

where the constant C_{32} does not depend on $n=1,2,\ldots$ and on the dimensions of the domain Ω_n.

We extend each of the functions $\vec{v}_n(x)$ outside the domain Ω_n by assigning the value zero there and preserving the same notation for the extended function. It is easy to see that $\vec{v}^n(x)\in\overset{\circ}{H}{}^{(2)}(\Omega)$, $n=1,2,\ldots$, and $|\vec{v}^n|_{\overset{\circ}{H}{}^{(2)}(\Omega_n)}\equiv|\vec{v}^n|_{H^{(2)}(\Omega)}$. Then from inequality (145) there follows the inequality

$$\nu\int_{\Omega}\vec{v}_x^{n^2}\,dx+\nu\varkappa\int_{\Omega}|\Delta\vec{v}^n|^2\,dx\le C_{32},\quad n=1,2,\ldots.\tag{146}$$

From inequality (146) it follows that from the totality of extended functions $\{\vec{v}^n(x)\}$ one can extract a subsequence $\{\vec{v}^{n_i}(x)\}$, which converges weakly in $\overset{\circ}{H}{}^{(2)}(\Omega)$ and strongly in $L_2(|x|\le Const)$ and in $L_4(|x|\le Const)$ to the limiting function $\vec{v}(x)\in\dot{H}^{(2)}(\Omega)$. Then, assuming that in integral identity (144) we have $\vec{\Phi}(x)\in\dot{J}(\Omega)$ and taking the limit as $n_i\to\infty$, we obtain that the limiting function $\vec{v}(x)$ satisfies the integral identity

$$\nu\int_{\Omega}\vec{v}_x\vec{\Phi}_x\,dx+\int_{\Omega}v_k(\varkappa\Delta\vec{v}-\vec{v})\vec{\Phi}_{x_k}\,dx=\int_{\Omega}\vec{f}\vec{\Phi}\,dx\tag{147}$$

for any $\vec{\Phi}(x)\in\dot{J}(\Omega)$, i.e., it is the desired generalized solution of problem (129), (130) from $\overset{\circ}{H}{}^{(2)}(\Omega)$. Inequality (137) is obtained by the limiting process as $n_i\to\infty$ in inequality (146).

The second part of Theorem 20 is proved in the same way as Theorem 19 in the case of a bounded domain and it is a consequence of the fact that the constant C_{31} in inequality (137) does not depend on $\varkappa\ge0$. We mention only that a generalized solution of the flow problem (133), (136) for the Navier–Stokes system from $H(\Omega)$ is defined as a function $\vec{v}(x)\in H(\Omega)$ which satisfies the integral identity

$$\nu\int_{\Omega}\vec{v}_x\vec{\Phi}_x\,dx-\int_{\Omega}v_k\vec{v}\vec{\Phi}_{x_k}\,dx=\int_{\Omega}\vec{f}\vec{\Phi}\,dx\tag{148}$$

for any $\vec{\Phi}(x)\in\dot{J}(\Omega)$, and that, as proved in [5, Chaps. II and V], such a solution exists for any $\vec{f}(x)\in L_{6/5}(\Omega)$. From Sobolev's embedding theorem with a limiting exponent, from Leray's inequality [5, Chap. I]

$$\int_{\Omega}\frac{u^2(x)}{|x-y|^2}\,dx\le4\int_{\Omega}u_x^2\,dx,\qquad\forall y\in\Omega,\tag{149}$$

valid for any domain $\Omega\subseteq E_3$ and any smooth, finite function $u(x)$ in Ω, and from inequality (137) there follows that for a generalized solution $\vec{v}(x)$ of flow problem (129), (136) from the class $\dot{H}^{(2)}(\Omega)$ we also have the inequality:

$$\int_{\Omega}(\vec{v}^6+\vec{v}_x^6)\,dx+\int_{\Omega}\frac{\vec{v}^2+\vec{v}_x^2}{|x-y|^2}\,dx\le C_{33}(C_{31},\varkappa),\quad\forall y\in\Omega.\tag{150}$$

Inequalities (137) and (150) show in what sense the solution $\vec{v}(x)$ of flow problem (129), (136) from the class $\overset{\circ}{H}{}^{(2)}(\Omega)$ and also its derivative $\vec{v}_x$ tend to zero as $|x|\to\infty$.

11. Some simple cases of initial-boundary-value problems for Eqs. (3) and (4), describing the motion of aqueous solutions of polymers, admit, just as the case of the Navier-Stokes equations, considerable simplifications and can be solved in a closed form. One of these problems is the problem of the nonstationary motion of an aqueous solution of a polymer in the half-space $-\infty < y, z < \infty$, $x > 0$, caused by the harmonic oscillations of its bounding plane. For the Navier-Stokes equations this problem has been solved, e.g., in [12, Sec. 24].

Namely, we assume that an aqueous solution of a polymer, whose motion is described by Eqs. (3), is in contact with the plane $-\infty < y, z < \infty$ which performs a simple harmonic oscillation with frequency ω in the direction of the y axis. Then the problem consists in solving in the half-space $-\infty < y, z < \infty, x > 0$ system (3) with $\vec{f} \equiv 0$ and under the boundary conditions

$$v_x = 0, \quad v_z = 0, \; v_y = u_0 e^{i\omega t} \text{ for } x = 0. \tag{151}$$

The symmetry conditions of problem (3), (151) induce the same simplifications as in the case of the similar problem of Navier-Stokes equations [12] and we arrive to the solving of the following simple problem for the unique nonzero component $v_y \equiv v(x,t)$ of the velocity:

$$\frac{\partial v}{\partial t} - \varkappa \frac{\partial^3 v}{\partial t \partial x^2} = \nu \frac{\partial^2 v}{\partial x^2}, \quad x > 0; \quad v\Big|_{x=0} = u_0 e^{-i\omega t}, \quad -\infty < t < \infty. \tag{152}$$

We shall seek a solution of problem (152), periodic with respect to x and t, of the form $v(x,t) = u_0 e^{i(kx - \omega t)}$. Inserting this solution into (152), we obtain $k^2 = \frac{i\omega}{\nu - i\varkappa\omega}$, whence $k(\varkappa) = \pm[a(\varkappa) + ib(\varkappa)]$, where

$$a(\varkappa) = \sqrt{\frac{\omega}{2\nu}} \frac{\sqrt{-\frac{\varkappa\omega}{\nu} + \sqrt{1 + 2\left(\frac{\varkappa\omega}{\nu}\right)^2}}}{\sqrt{1 + \left(\frac{\varkappa\omega}{\nu}\right)^2}}, \quad b(\varkappa) = \sqrt{\frac{\omega}{2\nu}} \frac{1}{\sqrt{1 + \left(\frac{\omega\varkappa}{\nu}\right)^2} \sqrt{-\frac{\varkappa\omega}{\nu} + \sqrt{1 + 2\left(\frac{\varkappa\omega}{\nu}\right)^2}}}. \tag{153}$$

Then

$$v(x,t) = u_0 e^{-b(\varkappa)x} e^{i(a(\varkappa)x - \omega t)}. \tag{154}$$

As in the case of the Navier-Stokes equations, solution (154) represents a transverse wave, whose wave vector is perpendicular to the direction of motion of the bounding plane, and whose amplitude fades away exponentially with increasing $x > 0$.

The friction force acting on a unit area performing a harmonic oscillation along the y axis, is directed along the y axis and is computed according to (2). The unique nonzero component of the stress tensor turns out to be

$$t_{xy}(\varkappa) = \left(\nu \frac{\partial v}{\partial x} + \varkappa \frac{\partial^2 v}{\partial x \partial t}\right)\Big|_{x=0}. \tag{155}$$

Inserting (154) into (155), assuming u_0 real and separating the real part, we obtain

$$t_{xy}(\varkappa) = \nu u_0 A(\varkappa) \left\{ \frac{\varkappa\omega}{\nu} \cos(\omega t - \varphi(\varkappa)) - \cos(\omega t - \varphi(x) + \frac{\pi}{2}) \right\}, \tag{156}$$

where

$$A(\varkappa)=\sqrt{\frac{\omega}{2\nu}}\cdot\frac{\sqrt{1+\left(-\frac{\varkappa\omega}{\nu}+\sqrt{1+2\left(\frac{\varkappa\omega}{\nu}\right)^2}\right)^2}}{\sqrt{1+\left(\frac{\varkappa\omega}{\nu}\right)^2}\sqrt{-\frac{\varkappa\omega}{\nu}+\sqrt{1+2\left(\frac{\varkappa\omega}{\nu}\right)^2}}},\quad \operatorname{Ctg}\varphi(\varkappa)=-\frac{\varkappa\omega}{\nu}+\sqrt{1+2\left(\frac{\varkappa\omega}{\nu}\right)^2}. \tag{157}$$

For $\varkappa=0$ the obtained solution coincides with the solution of the similar problem for the Navier–Stokes equations [12], for which

$$t_{xy}(0)=-\sqrt{\omega\nu}\,u_0\operatorname{Cos}(\omega t+\tfrac{\pi}{4}). \tag{158}$$

If $\frac{\varkappa\omega}{\nu}$ is sufficiently small, then we have the asymptotic equality

$$\nu A(\varkappa)\left(1\pm\frac{\varkappa\omega}{\nu}\right)\simeq\sqrt{\nu\omega}\left(1\pm\frac{\varkappa\omega}{\nu}\right). \tag{159}$$

Since the cosines in (156) are in opposite phase, it follows from (156) and (159) that the friction resistance of a weakly concentrated solution of polymer can be in absolute value either smaller or greater than the friction resistance computed from (158) for the "usual" viscous fluid.

12. Investigated system (4) and its corresponding stationary system (129) represent a special case of system (3), which from the point of view of the quasilinear equations, because of the presence of the convective terms of the second order $-\varkappa v_{jx_k}(v_{ix_jx_k}+v_{kx_ix_j})$, is a system with a strong nonlinearity and difficult to investigate. Desiring nevertheless to take into account the effect of these terms, we linearize them, replacing v_{jx_k} by α_{jx_k}, where $\vec{\alpha}(x,t)$ is a given solenoidal vector, equal to zero on ∂Q_T and having bounded first and second derivatives with respect to the x 's and we also set

$$\max_{Q_T}|\vec{\alpha}_x(x,t)|\equiv C_{34}. \tag{160}$$

We shall solve the partially linearized system

$$\begin{gathered}v_{it}-\nu\Delta v_i+v_kv_{ix_k}-\varkappa(\Delta v_{it}+v_k\Delta v_{ix_k})-\varkappa\alpha_{jx_k}(v_{ix_jx_k}+v_{kx_ix_j})=f_i,\\ i=1,\dots,n,\qquad \operatorname{div}\vec{v}=0,\end{gathered} \tag{161}$$

in Q_T under initial–boundary conditions (36). For problem (161), (36), as well as for problem (4), (36), one can introduce weak and strong generalized solution, whose definitions differ from the definitions of the corresponding generalized solutions of problem (4), (36) only in the fact that in the left-hand side of integral identities (37) and (41) one adds the integral

$$\varkappa\iint_{Q_T}\alpha_{jx_k}(v_{ix_k}+v_{kx_i})\Phi_{ix_j}\,dQ, \tag{162}$$

linear with respect to $\vec{v}$. Therefore, in analogy with Theorem 7 and Theorem 1 of [6], one proves:

<u>THEOREM 21.</u> Let $\vec{v}_0(x)\in\overset{\circ}{W}{}_2^2(\Omega)\cap\overset{\circ}{J}(\Omega)$, and $\vec{f}(x,t)\in L_{2,1}(Q_T)$. Then problem (161), (36) has at least one weak solution $\vec{v}(x,t)$ and for any such solution estimates (38), (39) hold, where C_3 and C_4 depend on C_{34} from (160).

Let $\vec{v}_0(x)\in\overset{\circ}{W}{}_2^3(\Omega)\cap\overset{\circ}{J}(\Omega)$, $\vec{f}(x,t)\in L_2(Q_T)$. Then problem (161), (36) admits at least one strong solution $\vec{v}(x,t)$ and for any such solution estimate (41) holds, where C_5 also dpends on C_{34}.

Moreover, for systems obtained from systems (66) and (72) by the addition of the linear terms $-\varkappa\alpha_{jx_k}(v_{ix_jx_k}+v_{kx_ix_j})$, where $\vec{\alpha}(x,t)$ possesses the above-enumerated properties, all the results which have been proved for systems (66) and (72) are preserved.

The first boundary-value problem for the stationary system corresponding to (161) has at least one generalized solution $\vec{v}(x)\in \overset{\circ}{W}{}_2^2(\Omega)\cap \overset{\circ}{J}(\Omega)$ satisfying for any $\vec{\Phi}(x)\in H(\Omega)$ the integral identity

$$\nu\int_\Omega \vec{v}_x\vec{\Phi}_x\,dx-\int_\Omega v_k\vec{v}\vec{\Phi}_{x_k}dx+\varkappa\int_\Omega v_k\Delta\vec{v}\vec{\Phi}_{x_k}dx+\varkappa\int_\Omega \alpha_{jx_k}(v_{ix_k}+v_{kx_i})\Phi_{ix_j}dx=\int_\Omega \vec{f}\vec{\Phi}dx \tag{163}$$

if $\vec{f}(x)\in L_2(\Omega)$ and we have the condition

$$C_{35}\equiv \nu-\varkappa\max_\Omega|\vec{\alpha}_x|>0, \tag{164}$$

which arises at the estimation of the integral

$$\varkappa\int_\Omega \alpha_{jx_k}(v_{ix_k}+v_{kx_i})(v_i-\varkappa\Delta v_i)_{x_j}dx \tag{165}$$

and ensures for the solutions of the stationary problem under consideration an estimate of the type (132) in which the constant C_{30} depends also on the constant C_{35}.

13. A series of non-Newtonian fluids (see, e.g., [13]) are described by the following defining equation, more general than (1),

$$T=-pE+2\nu D(\vec{v})+\sum_{\ell=1}^{N}\varkappa_\ell\frac{d^\ell D(\vec{v})}{dt^\ell}, \tag{166}$$

where $\varkappa_\ell, \ell=1,\dots,N$ are given nonnegative constants (the relaxation viscosity coefficients of different orders). Inserting the stress tensor (166) into the equation of motion (2) and neglecting in the first approximation, as well as in the derivation of system (4), the terms containing the products of the derivatives of $\vec{v}$ with respect to the x 's and t, we obtain the system of equations

$$\frac{\partial\vec{v}}{\partial t}-\nu\Delta\vec{v}+v_k\frac{\partial\vec{v}}{\partial x_k}-\sum_{\ell=1}^{N}\left(\frac{\partial^\ell\Delta\vec{v}}{\partial t^\ell}+\ell v_k\frac{\partial^\ell\Delta\vec{v}}{\partial t^{\ell-1}\partial x_k}\right)+\operatorname{grad} p=\vec{f},\quad \operatorname{div}\vec{v}=0. \tag{167}$$

We shall solve system (167) in the cylinder Q_T under the following initial-boundary conditions

$$\left.\frac{\partial^m\vec{v}}{\partial t^m}\right|_{t=0}=v_m(x),\ x\in\Omega,\ m=0,1,\dots,N-1;\ \vec{v}\big|_{\partial Q_T}=0. \tag{168}$$

The initial-boundary problem (167) admits a unique classical solution. In order to prove this uniqueness theorem it is sufficient to multiply the equation for the difference $\vec{\omega}(x,t)$ of two possible solutions of problem (167), (168) by $\frac{\partial^{N-1}\vec{\omega}}{\partial t^{N-1}}$, to integrate the obtained equality with respect to Q_t, $0<t\leqslant T$, to perform the integration by parts in the same manner as we have done it in the proof of Theorem 1, and to make use of the Hölder inequality and of Gronwall's lemma.

The solvability of problem (167), (168) can be investigated by introducing the vanishing viscosity $\varepsilon\Delta^2\vec{v}$, in a similar way as the solvability of problem (4), (36) has been studied; however, problem (167), (168) has for $N\geqslant 2$ one significant distinction: because of the presence of the nonlinear terms

$v_k \frac{\partial^{\ell} \Delta \vec{v}}{\partial x_k \partial t^{\ell-1}}$, $\ell = 2, 3, \ldots, \mathcal{N}$, even its "weak" solution must possess derivatives with respect to t up to the order $\mathcal{N}-1-\left[\frac{\mathcal{N}-1}{2}\right]$. A detailed investigation of the solvability of problem (167), (168) will be given in a subsequent paper.

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