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THE CATEGORY OF FINITE SETS AND CARTESIAN CLOSED CATEGORIES

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Applying methods of the proof theory, it is shown that two canonical morphisms are equal in all Cartesian closed categories if and only if some of their realizations in the category of finite sets are equal. All realizations of formal combinations of objects using the functors x and hom are isomorphic in all Cartesian closed categories if and only if some of their realizations in the category of finite sets are isomorphic. On the base of these results, a purely syntactic decision algorithm for (extensional) isomorphism of formal combinations of objects and a new decision algorithm for equality of canonical morphisms are obtained.

Introduction

The present work studies some "universal" properties of the category of finite sets and maps with respect to Cartesian closed categories. To this end, we use the proof theory technique that has been applied by many authors to solve different problems of category theory (e.g., [1-8]).

Two questions are considered – on isomorphism of objects and on equality of canonical morphisms in Cartesian closed categories. Also, algorithmic problems associated with these questions are studied.

Explicitly, only two Cartesian closed categories are studied: the category of finite sets M and the HCC system (the Hilbert system for Cartesian closed categories). In HCC, formulas serve as objects and equivalence classes of proofs by some (decidable) equivalence relation $\equiv$ play the role of morphisms.

It turns out that $f \equiv g$ if and only if $I(f) = I(g)$ for some functor I: HCC $\to$ M (only cardinalities of sets assigned by the functor I to variables of HCC depend on f and g). On the base of this fact one can obtain a new decision algorithm for $\equiv$.

A similar result is obtained for isomorphism of formulas of HCC, but in this case it is possible to find a purely syntactic decision algorithm; to prove its correctness, the category of finite sets is used.

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In a certain sense, the category HCC is free in the class of Cartesian closed categories (cf. [3]), which allows us to derive conclusions related to other Cartesian closed categories (see Lemma 1, Sec. 1).

These results can be extended (with some exceptions) to other categories which are close to Cartesian closed in some respect, for instance, the category of indexed sets ([9]).

Information necessary for understanding this article is given in Sec. 1, questions related to isomorphism are considered in Sec. 2, and systems associated with equality of canonical morphisms are considered in Sec. 3.

1. We introduce necessary concepts. In this section, except for the part related to the connection of HCC to the category of finite sets, we follow [6]. A similar-in-form defition of a Cartesian closed category and the deductive system used for study of Cartesian closed categories is given in [3].

A Cartesian closed category

$$\mathcal{D} = (\mathcal{D}_0, T, \&, \supset, 0, \langle\ \rangle, l, r, \varepsilon, {}^+)$$

is defined by the following data: (DI) a category $\mathcal{D}_0$; (D2) an object $T \in \mathcal{O}b\,\mathcal{D}_0$; (D3) binary operations & and $\supset$ on $\mathcal{O}b\,\mathcal{D}_0$ in $\mathcal{O}b\,\mathcal{D}_0$; (D4) families of morphisms

$$0 = 0_A : A \to T\ ;\ l = l_{A,B} : A \& B \to A;\ r = r_{A,B} : A \& B \to B\ ;$$
$$\varepsilon = \varepsilon_{A,B} : (A \supset B) \& A \to B$$

(D5) a binary operation < > assigning to each pair of morphisms $A \xrightarrow{f} B$, $A \xrightarrow{g} C$ a morphism $\langle f, g\rangle : A \to B \& C$; (D6) a unary operation + assigning to each morphism $A \& B \xrightarrow{f} C$ a morphism $f^+ : A \to B \supset C$.

These data must satisfy the following conditions:

A1. $f1_A = 1_B f$ for $f:A \to B$ (the identity law).

A2. $f(gh) = (fg)h$ for $A \xrightarrow{h} B \xrightarrow{g} C \xrightarrow{f} D$ (associativity of composition).

A3. $0_A = f$ for $f:A \to T$ (T is terminal).

A4. $l_{B,C}\langle f, g\rangle = f$; $r_{B,C}\langle f, g\rangle = g$ for $A \xrightarrow{f} B$, $A \xrightarrow{g} C$.

A5. $\langle l_f, r_f\rangle = f$ for $f:A \xrightarrow{f} B \& C$.

A6. $\varepsilon\langle g^+ l_{AB}, r_{AB}\rangle = g$ for $A \& B \xrightarrow{g} C$.

A7. $(\varepsilon\langle h\, l_{AB}, r_{AB}\rangle)^+ = h$ for $h:A \to B \supset C$.

Fulfillment of the first two conditions means that $\mathcal{D}_0$ is a category.

This definition of a Cartesian closed category is slightly unusual but it is equivalent to the standard one (cf. [3]).

The HCC system (the Hilbert system for Cartesian closed categories):

The atomic formulas are the constant T and propositional variables $a_1, \ldots, a_n, \ldots$. Formulas are constructed from the variables and T using binary connectives & and $\supset$. The sequents of the HCC system are expressions of the form $A \to B$ where A and B are formulas. Proofs and deductible sequents are defined inductively. We will write $A \xrightarrow{f} B$ or $f:A \to B$ instead of "f is a proof of the segment $A \to B$." We will list axioms and rules of inference (proofs are written in the treelike form).

$$\text{H1}\quad 1_A : A \to A\ ; \qquad \frac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{gf} C}$$

$$\text{H2}\quad 0_A : A \to T.$$

H3 $l_{A,B}: A\&B \to A$; $r_{A,B}: A\&B \to B$; $\dfrac{C \xrightarrow{f} A \quad C \xrightarrow{g} B}{C \xrightarrow{\langle f,g\rangle} A\&B}$.

H4 $\varepsilon_{A,B}: (A \supset B)\&A \to B$; $\dfrac{A\&B \xrightarrow{h} C}{A \xrightarrow{h^{+}} B \supset C}$.

An equivalence relation between proofs is defined by the relations obtained from A1-A7 when = is replaced by $\equiv$. Taking the quotient of the set of proofs of HCC by this equivalence relation makes HCC into a Cartesian closed category.

Objects A and B of a category $\mathcal{D}$ are said to be isomorphic (written $A \simeq B$) if there exist morphisms $f: A \to B$ and $g: B \to A$ such that $gf = 1_B$ and $gf = 1_A$. We prefer the term "isomorphism" to the term "equivalence" adopted in category theory in order to avoid confusion with the equivalence of proofs in HCC.

Canonical morphisms in Cartesian closed categories are those that can be obtained from 1, O, l, r, ε using < >, +, and the composition. Two canonical morphisms (viewed as formal combinations of the symbols 1, O, l, etc.) are considered equal if their realizations in each Cartesian closed category are equal.

The following lemma holds (cf. Lemma 2.1 in [6]):

LEMMA 1. (a) Canonical morphisms f and g are equal if and only if $f \equiv g$. If A and B are formulas of HCC, then $A\xi \simeq B\xi$ for any permutation ξ of objects of an arbitrary Cartesian closed category as variables if and only if $A \simeq B$ in HCC.

For study of proofs in HCC, deductive terms are used. To each term, some formula is assigned as a type. The notation $t \in A$ means that t is a term of type A. Sometimes, instead of writing $t \in A$ we will write the superscript A at t. We give the definition of deductive terms.

(1) For each formula A there is a list of variables of type A, denoted x^A, y^A,... (perhaps with subscripts). There is also a constant $\mathbb{0}$ of type T. The variables and the constant $\mathbb{0}$ are terms.

(2) If $t \in (A \supset B)$, $s \in A$, then $(t, s) \in B$ [actually, $(t^{A \supset B}, s^A) \in B$].

(3) $\lambda x^A t^B \in (A \supset B)$.

(4) $l t^{A\&B} \in A$, $r t^{A\&B} \in B$.

(5) $\langle t_1^A, t_2^B \rangle \in A\&B$.

Variables in terms are divided in a usual way into tied (by the symbol λ) and free variables. Below, $\doteq$ denotes graphic equality of syntactic expressions and $t_{x^A}[s^A]$ denotes the result of substituting s^A for all free occurrences of the variable x^A in t with simultaneous *renaming* of tied occurrences of variables to avoid collisions.

To each proof f in HCC, a deductive term, denoted by $\tau(f)$, is assigned in the following way:

H1′ $\tau(1_A) \doteq x^A$; $\tau(gf) \doteq \tau(g)_{x^B}[\tau(f)]$ where, $f: A \to B$, $g: B \to C$.

H2′ $\tau(O_A) \doteq \mathbb{0}$.

H3′ $\tau(\langle f,g\rangle) \doteq \langle \tau(f), \tau(g)\rangle$; $\tau(l_{AB}) \doteq l x^{A\&B}$; $\tau(r_{AB}) \doteq r x^{A\&B}$

H4′ $\tau(\varepsilon_{AB}) \doteq (l x^{(A\supset B)\&A}, r x^{(A\supset B)\&A})$;

$\tau(h^{+}) \doteq \lambda x^B(\tau(h)_{x^{A\&B}}[\langle x^A, x^B\rangle])$, where $h: A\&B \to C$.

Here, x^A is some previously fixed variable of type A.

In $\tau(f)$, where $f: A \to B$ ($B \neq T$), only the variable x^A occurs freely. The deductive term with free variables $x_1^{A_1}, \ldots, x_n^{A_n}$ can be considered as a code of some multimorphism $A_1, \ldots, A_n \to B$. (The notion of a multicategory is considered in [2, 5, and 8].)

Equivalence of terms (which will also be denoted by $\equiv$) is defined by the following relations:

$t \equiv \lambda y^A(t, y^A)$, where $t \in A \supset B$ and y^A is a new variable

$$t \equiv \langle \ell t, r t \rangle \text{ , where } t \in A \& B;$$
$$(\lambda x^A t, s^A) \equiv t_{x^A}[s^A];$$
$$\ell \langle t_1, t_2 \rangle \equiv t_1,\ r\langle t_1, t_2 \rangle \equiv t_2;$$

$t_1 \equiv t_2$, if t_1 and t_2 are congruent (they may only differ in choice of tied variables);

$$\frac{t \equiv s_1 \quad t \equiv s_2}{s_1 \equiv s_2}, \qquad \frac{s_1 \equiv s_2}{t[s_1] \equiv t[s_2]},$$
$$t^T \equiv 0.$$

Two proofs f_1 and f_2 in HCC of the same sequent $A \to B$ are equivalent if and only if their deductive terms are equivalent. The equivalence of deductive terms is effectively recognizable. Consider the following conversions:

$$\begin{array}{lll}
(1) & t \mapsto \lambda x^A(t, x^A) & \text{, if } t \in (A \supset B),\ t \neq \lambda y t' \\
(2) & t \mapsto \langle \ell t, r t \rangle & \text{, if } t \in (A \& B) \text{ , where } t \neq \langle t_1, t_2 \rangle \\
(3) & (\lambda x^A t, s^A) \mapsto t_{x_A}[s^A] & \\
(4) & \delta \langle t_\ell, t_r \rangle \mapsto t_\delta & \text{where } \delta = \ell, r \\
(5) & t^T \mapsto 0 &
\end{array}$$

subject to an additional condition: A substituted occurrence of t as a subterm in a larger term is not caused by an occurrence of the form† in (1) or lt, rt in (2).

A term t is normal (or is in the normal form) if no conversion is applicable to it or its subterms. It is proved in [6] that each term t has a unique (up to a congruence) normal form (we will denote it by $\bar{\bar{t}}$) and each sufficiently long sequence of conversions stops at this normal form. Terms t_1 and t_2 are equivalent if and only if $\bar{\bar{t}}_1$ and $\bar{\bar{t}}_2$ are congruent.

In the conclusion of this section, we introduce several notions pertinent to the connection between HCC and the category M of finite sets.

By an *interpretation* I:HCC $\to$ M will be meant an arbitrary functor preserving the structure of a Cartesian closed category, i.e., such that $I(A \& B) = I(A) \times I(B)$, $I(A \supset B) = I(B)^{I(A)}$ (for the category M we will use the standard notation of the product and the Hom-functor), $I(T) = \{*\}$ (the standard one-element set), and such that l_{AB}, r_{AB} are mapped to the corresponding projections of the product, ε_{AB} to the operation of evaluating a function at its argument, and the operation + to the usual λ-abstraction. Of course, if $f \equiv g$ in HCC, then $I(f) = I(g)$ in M for each I. It is readily seen that the following holds:

LEMMA 2. Each map of the set $\{a_1, \ldots, a_n, \ldots\}$ of propositional variables into M is uniquely extended to an interpretation.

If an interpretation I is given and t^B is a deductive term with free variables $x_1, \ldots, x_n$, then the expression $\lambda x_1 \ldots \lambda x_n t^B$ can be viewed as a record of some map with values in I(B) depending on n arguments; in particular, the record of the map I(f), where $f: A \to B$, will be $\lambda x^A \tau(f)$. For each formula A the variable x^A can be viewed as the variable for elements of the set I(A); 0 as *; < > as the operation of forming ordered pairs; ( ) as the operation of evaluating a function at its argument; l and r as projections; and λ as the symbol of the usual λ-abstraction. Interpreting the symbols in this way, the term t itself can be viewed as a parametric definition of values of the corresponding map. Sometimes, when a possibility of confusion arises, we will use the notation I(t) to emphasize that the symbols occurring in t should be read in the manner just described.

2. Let $\mathbb{N}^+$ denote the set of positive integers. In this section, we will consider numerical terms (n-terms for short) which are constructed from variables $p_1, \ldots, p_n$ assuming values in $\mathbb{N}^+$, and natural numbers, using the operations of multiplication and raising to a power.

To be precise, p_i is an n-term $(1 \leq i \leq \infty)$; $c \in \mathbb{N}^+$ is an n-term; if P and Q are n-terms, then $(P \cdot Q)$ and (P^Q) are n-terms. We will omit parentheses where it does not lead to confusion.

†Something is apparently missing in Russian original — Translator.

We will use the letters P, Q, and R (with or without subscripts) to denote n-terms.

The expression $P \cong Q$ will be used as an abbreviation for $\forall p_1,\ldots,p_n\ (P = Q)$, where P and Q contain only variables from the list $p_1,\ldots,p_n$.

Definition 1. We assign to each formula in HCC some n-term. Namely, to the formula α_i is assigned the n-term p_i; to the formula T the n-term $1 \in \mathbb{N}^+$; if n-terms P and Q are assigned to formulas A and B, then the n-term $(P\cdot Q)$ corresponds to the formula A & B and the n-term (Q^P) to the formula $A \supset B$.

P_A denotes the n-term assigned to the formula A.

For each n-term P and any $c_1,\ldots c_m \in \mathbb{N}^+$, let the expression $P\downarrow^{p_1,\ldots,p_m}_{c_1,\ldots,c_m}$ denote the result of substituting c_1 for all occurrences of $p_1;\ldots;c_m$ for all occurrences of p_m.

Let I be some interpretation

$$c_1 \leftrightharpoons Card(I(\alpha_1)),\ldots, c_n \leftrightharpoons Card(I(\alpha_n))\ldots$$

LEMMA 1. Suppose that all variables appearing in a formula A are contained among $\alpha_1,\ldots,\alpha_m$. Then

$$Card(I(A)) = (P_A)\downarrow^{p_1,\ldots,p_m}_{c_1,\ldots,c_m}$$

This lemma is proved by induction on the construction of A. The proof of the induction passage uses the equalities

$$Card(I(A_1 \& A_2)) = Card(I(A_1))\cdot Card(I(A_2))$$
$$Card(I(A_1 \supset A_2)) = Card(I(A_2))^{Card(I(A_1))} \quad \square$$

The notation $A \simeq B$ means that formulas A and B are isomorphic in HCC.

LEMMA 2. If $A \simeq B$, then $P_A \cong P_B$.

Proof. If $\alpha: A \to B$ is an isomorphism, then for each interpretation the map $I(\alpha)$ is an isomorphism in M. Two sets are isomorphic in M if and only if their cardinalities coincide. Let m be such that all variables appearing in A or B are contained among $\alpha_1,\ldots,\alpha_m$. If Card × (I(A)) = Card (I(B)), then, by Lemma 1, $I(P_A)\downarrow^{p_1,\ldots,p_m}_{c_1,\ldots,c_m} = (P_B)\downarrow^{p_1,\ldots,p_m}_{c_1,\ldots,c_m}$. Since under all interpretations $c_1,\ldots,c_m$ can assume any values (see the appropriate lemma in Sec. 1), the lemma is proved.

Since HCC is a Cartesian closed category, the following holds:

LEMMA 3. For arbitrary formulas A, B, and C

(1) $A \& B \simeq B \& A$	(2) $A \& (B \& C) \simeq (A \& B) \& C$
(3) $(A \& B) \supset C \simeq A \supset (B \supset C)$	(4) $A \supset (B \& C) \simeq (A \supset B) \& (A \supset C)$
(5) $A \& T \simeq A$	(6) $T \& A \simeq A$
(7) $A \supset T \simeq T$	(8) $T \supset A \simeq A$

LEMMA 4 (on replacement). Let Φ be an arbitrary formula in HCC, v some occurrence of the formula A in Φ as a subformula. If $A \simeq B$, then $\Phi \simeq \Phi'$, where Φ' is obtained by replacing A by B in v.

The *proof* is done by induction on the depth of the occurrence v. $\square$

We will now describe how, for each formula A, a formula isomorphic to it can be constructed (it will be called the reduced form of A, denoted by $\tilde{A}$) such that $A \simeq B$ if and only if $\tilde{A} \doteqdot \tilde{B}$.

Transformations of formulas studied below are reduced to consecutive replacements of some of their subformulas by other subformulas isomorphic to them. By Lemma 4, formulas obtained as a result of these transformations are isomorphic to the original ones. We will not state it every time.

The construction of A from A is done in several steps. First, using conditions (5)-(8) of Lemma 3, we consecutively replace the extreme left occurrences of subformulas of the suitable form containing T by occurrences of shorter subformulas. As a result, we obtain either T or a formula not containing T; we denote the obtained formula by A^0.

A formula will be called *quasireduced* if it does not contain the constant T and logical connectives do not occur in the conclusion of an implication. Obviously, the conclusions of all implications in a quasireduced formula are variables. Subformulas of a quasireduced formula are quasireduced formulas.

Let $\theta(A)$ denote the number of occurrences of the variables and of T in a formula A and $\Psi(A)$ the number of occurrences of logical connectives belonging to the conclusion of at least one implication in A.

For each formula A not containing the constant T, a quasireduced formula isomorphic to it can be constructed which we denote by A^+ (the quasireduced form of A). A^+ is defined by induction on $\theta(A) + \Psi(A)$.

If A is a variable, then $A^+ \mathrel{\underset{\cdot}{=}} A$;

$$(A_1 \& A_2) \mathrel{\underset{\cdot}{=}} A_1^+ \& A_2^+ ;$$
$$(A_1 \supset a_1)^+ \mathrel{\underset{\cdot}{=}} A_1^+ \supset a_i ; \ (A_1 \supset (A_2 \supset A_3))^+ \mathrel{\underset{\cdot}{=}} ((A_1 \& A_2) \supset A_3)^+ ;$$
$$(A_1 \supset (A_2 \& A_3))^+ \mathrel{\underset{\cdot}{=}} (A_1 \supset A_2)^+ \& (A_1 \supset A_3)^+ .$$

LEMMA 5. $\theta(A^+) \leqslant 2^{\theta(A)}$.

The proof is done by induction on $\theta(A) + \Psi(A)$ taking into account that θ grows only at the passage from a formula of the form $(A_1 \supset (A_2 \& A_3))$ to a formula of the form $(A_1 \supset A_2) \& (A_1 \supset A_3)$. □

By the associativity of conjunction [condition (2) of Lemma 3], we will write multiple conjunctions without parentheses until the end of this section. As usual, by the multiplicity of a conjunction $A_0 \& \ldots \& A_k$ is meant k. Expanding conjunctions in the antecedents of implications in a quasireduced formula into multiple implications, using condition (3) of Lemma 3, one can, starting with any quasireduced formula B, pass to a formula of the form $B' \mathrel{\underset{\cdot}{=}} B_0 \& \ldots \& B_m$, where B_i, $0 \leqslant i \leqslant m$, do not contain conjunction and T. This implies a lemma which we will need in Sec. 3.

LEMMA 6. Each formula A not isomorphic to T is isomorphic to the multiple conjunction of formulas not containing conjunction and T.

Proof. If $A \not\approx T$, then $A^0 \mathrel{\underset{\cdot}{\neq}} T$. Pass from A to the quasireduced formula $(A^0)^+$ and expand conjunctions in the antecedents of implications. □

In the work of the decision algorithm for isomorphism, the quasireduced form of formulas will be used, while the reduced form will only be needed to show its correctness.

The reduced form is obtained from the quasireduced one by ordering the conjunction terms.

By the implicative depth $l(v)$ of an occurrence v of a subformula is meant the number of antecedents of implications in which v appears. By the implicative depth $l(A)$ of a formula A is meant the maximum of implicative depths of occurrences of its subformulas.

Let U be a set well-ordered by some relation >. Let $>_n$ denote the lexicographic order relation on the set of finite sequences of elements of U; then $>_n$ will also be a relation of well-ordering.

We will define the reduced form for quasireduced formulas and the relation > of well-ordering on the set of reduced forms of quasireduced formulas by induction on $l(A)$ and max × $(l(A), l(B))$, respectively.

(1) Let $l(A) = 0$. Then $A \mathrel{\underset{\cdot}{=}} a_{i_0} \& \ldots \& a_{i_n}$ for some n, where a_{i_k}, $0 \leqslant k \leqslant n$, are variables. Put $\tilde{A} \mathrel{\underset{\cdot}{=}} a_{i_{\sigma(0)}} \& \ldots a_{i_{\sigma(n)}}$, where σ is a permutation of the numbers $0, \ldots, n$ such that $i_{\sigma(0)} \geqslant \cdots \geqslant i_{\sigma(n)}$. $a_i > a_j$ if and only if $i > j$. $A > B$, where $A \mathrel{\underset{\cdot}{=}} a_{i_0} \& \ldots \& a_{i_n}$, $B \mathrel{\underset{\cdot}{=}} a_{j_0} \& \ldots \& a_{j_m}$, $i_0 \geqslant \cdots \geqslant i_n$, $j_0 \geqslant \cdots \geqslant j_m$ if and only if $a_{i_1}, \ldots, a_{i_n} >_n a_{j_1}, \ldots a_{j_m}$.

(2) Suppose the relation $>$ and the reduced form have been defined for all quasireduced formulas A and B such that $0 \leqslant l(A), l(B) < k$. Let $l(B) \leqslant l(A) = k$. First, we define the reduced form and the relation $>$ for those formulas whose outermost connective is an implication. If $A \doteqdot (A_1 \supset \tilde{u}_i)$, then put $\tilde{A} = (\tilde{A}_1 \supset \tilde{u}_i)$. If $A = A_1 \supset \alpha_i$, $B \doteqdot B_1 \supset \alpha_j$ are reduced formulas, then put $A > B$ if and only if $A_1, \alpha_i >_n B_1, \alpha_j$. Now let the outermost connective of A be a conjunction. Then $A = A_0 \& \ldots \& A_n$ for some n, where $A_0, \ldots, A_n$ are quasireduced formulas whose outermost connective is implication. Put $\tilde{A} \doteqdot \tilde{A}_{\sigma(0)} \& \ldots \& \tilde{A}_{\sigma(n)}$, where σ is a permutation such that $\tilde{A}_{\sigma(0)} \geqq \cdots \geqq \tilde{A}_{\sigma(n)}$. If $A \doteqdot A_0 \& \ldots \& A_n$, $B \doteqdot B_0 \& \cdots \& B_m$ are reduced formulas and $\max(l(A), l(B)) = k$, then put $A > B$ if and only if $A_0, \ldots, A_n >_n B_0, \ldots, B_m$.

Let A be an arbitrary formula. If $A^0 \doteqdot T$, then put $\tilde{A} \doteqdot T$, otherwise put $\tilde{A} = ((\tilde{A}^0)^{\div})$. A formula A is said to be reduced if $\tilde{A} \doteqdot A$. To the definition of $>$, we add this condition: $A > T$ for each reduced formula A other than T.

Note that we pass from a quasireduced formula to a reduced one using only conditions (1) and (2) of Lemma 3.

Note also that each subformula of a reduced formula is again a reduced formula.

Let A and B be reduced formulas and let n be such that all variables appearing in A or B are contained among $\alpha_1, \ldots, \alpha_n$. Let k denote that maximal multiplicity of conjunction in B.

<u>LEMMA 7.</u> If $A > B$, then for each constant $\delta > 0$ there exists a number ρ such that if $c \in \mathbb{N}^+$ and $c > \rho$, then

$$(*) \quad (\rho_A)\big\downarrow^{\rho_1,\ldots,\rho_n}_{c_1,\ldots,c_n} > \delta \cdot (\rho_B)\big\downarrow^{\rho_1,\ldots,\rho_n}_{c_1,\ldots,c_n}$$

where $c_1 \leftrightharpoons c,\ c_2 \leftrightharpoons c^{L},\ldots, c_n \leftrightharpoons c^{L^{n-1}},\ L > K+2$.

Considering L fixed, we introduce the notation

$$\rho[\vec{c}] \leftrightharpoons \rho\big\downarrow^{\rho_1,\ldots,\rho_n}_{c_1,\ldots,c_n}.$$

<u>Proof.</u> It suffices to prove the lemma for the case when either $A \doteqdot \alpha_i$ or $A = (A \supset \alpha_i)$ for some i. Indeed, if $A \doteqdot A_0 \& \ldots \& A_r$, $B \doteqdot B_0 \& \cdots \& B_s$, then we can assume that $A_p \not\doteqdot B_q$ for any p and q $(0 \leqslant p \leqslant r,\ 0 \leqslant q \leqslant s)$; otherwise, in both parts of the inequality $(*)$ common factors can be cancelled, producing an equivalent inequality with $P_{A'}$ and $P_{B'}$, where A' and B' are obtained from A and B by removing common conjunctive terms. If A and B have no common conjunctive terms, then $A > B$ implies $A_0 > B$. The inequality $(*)$ for A_0 and B is obviously no weaker than the original one.

The proof of this case will be conducted by induction on $l = \max(l(A), l(B))$.

<u>Basis:</u> If $l = 0$, then $A \doteqdot \alpha_i$, and $B \doteqdot T$ or $B \doteqdot \alpha_{j_0} \& \ldots \alpha_{j_s}$. In the latter case, $i > j_0 \geqslant \ldots \geqslant j_s$; $k \geqslant s$. In both cases, it is enough to take $c > \delta$. The first case is obvious; in the second we have: $\rho_{\alpha_i}[\vec{c}] = c^{L^{(i-1)}} \geqslant (c^{L^{(j_0-1)}})^{L} > \delta (c^{L^{(j_0-1)}})^{s} \geqslant \delta \cdot \rho_B[\vec{c}]$.

<u>Induction passage:</u> Let $l > 0$. First, let $B \doteqdot B' \supset \alpha_j$ for some $j > 0$. Since $A > B$, $A \doteqdot A' \supset \alpha_i$, and either $A' > B'$ or $A' \doteqdot B'$ but $i > j$. Suppose $A' > B'$. By the induction hypothesis, there exists ρ such that for $c > \rho_0$

$$\rho_{A'}[\vec{c}] > (L^{(j-1)}+1) \cdot \rho_{B'}[c].$$

Now, whenever $c > \rho = \max(\delta, \rho_0, 1)$ we have:

$$\rho_A[\vec{c}] = (c^{L^{(i-1)}})^{\rho_{A'}[\vec{c}]} \geqslant c^{\rho_{A'}[\vec{c}]} > c^{(L^{(j-1)}+1)\cdot \rho_{B'}[\vec{c}]} \geqslant \delta (c^{L^{(j-1)}})^{\rho_{B'}[\vec{c}]} = \delta \rho_B[\vec{c}].$$

If $A' \underset{\cdot}{=} B'$ but $i > j$, then whenever $c > \max(\delta, 1)$, we have:

$$\rho_A[\vec{c}] = \left(c^{b^{(i-1)}}\right)^{\rho_{A'}[\vec{c}]} \geqslant \left(c^{(b^{(j-1)}+1)}\right)^{\rho_{A'}[\vec{c}]} > \delta\left(c^{b^{(j-1)}}\right)^{\rho_{A'}[\vec{c}]} = \delta \cdot \rho_B[\vec{c}].$$

Now let $B = B_0 \& \ldots \& B_s$, $K \geqslant S > 0$. Since B is a reduced formula $B_0 \underset{\cdot}{\geqq} \ldots \underset{\cdot}{\geqq} B_s$ and the external connective of $B_0, \ldots, B_s$ is an implication. As has just been proved, there exist constants $\rho_1, \ldots, \rho_s$ such that for $c > \rho_q$, $1 \leqslant q \leqslant s$, we have $P_{B_0}[\vec{c}] \geqslant P_{B_q}[\vec{c}]$. Let $B \underset{\cdot}{=} B' > a_j$ for some $j > 0$. If $A' > B'$, then let ρ_0 be a number such that for $c > \rho_0$, $\rho_{A'}[\vec{c}] > (S+2) b^{(j-1)} \cdot \rho_{B'}[\vec{c}]$. Then for $c > \rho = \max(\delta, \rho_0, \ldots, \rho_s, 1)$ we have:

$$\rho_A[\vec{c}] = c^{b^{(i-1)} \cdot \rho_{A'}[\vec{c}]} \geqslant c^{\rho_{A'}[\vec{c}]} > c^{(S+2)\cdot b^{(j-1)} \cdot \rho_{B'}[\vec{c}]}$$

$$> \delta \cdot c^{(S+1)\cdot b^{(j-1)} \cdot \rho_{B'}[\vec{c}]} \geqslant \delta \cdot (\rho_{B_0}[\vec{c}])^{S+1} \geqslant \delta \cdot \rho_{B_0}[\vec{c}] \cdot \ldots \cdot \rho_{B_S}[\vec{c}] = \delta \cdot \rho_B[\vec{c}].$$

If $A' \underset{\cdot}{=} B'$ but $i > j$, then one can take $\max(\delta, \rho_1, \ldots, \rho_s, 1)$ as ρ. For $c > \rho$

$$\rho_A[\vec{c}] = c^{b^{(i-1)} \cdot \rho_{A'}[\vec{c}]} > c^{(S+2)\cdot b^{(j-1)} \cdot \rho_{A'}[\vec{c}]} = c \cdot c^{(S+1)\cdot b(j-1) \cdot \rho_{A'}[\vec{c}]} >$$
$$> \delta \cdot c_1^{(S+1)\cdot b^{(j-1)} \cdot \rho_{A'}[\vec{c}]} = \delta \cdot (\rho_{B_0}[\vec{c}])^{(S+1)} \geqslant \delta \cdot \rho_B[\vec{c}]. \quad \square$$

From Lemmas 2 and 7, taking into account that $A \simeq \tilde{A}$, it follows that:

THEOREM. $A \simeq B$ if and only if $\tilde{A} \underset{\cdot}{=} \tilde{B}$.

COROLLARY 1. If A and B are quasireduced formulas, then $A \simeq B$ if and only if one can pass from A to B as a result of replacing subformulas by isomorphic subformulas, using only conditions (1) and (2) of Lemma 3 (commutativity and associativity of conjunction).

Proof. Only these conditions are used in the passage from the quasireduced to the reduced form. □

COROLLARY 2. If A and B are conjunctions of variables, then $A \simeq B$ if and only if A and B contain the same variables the same number of times. If $A \underset{\cdot}{=} A' \supset a_i$, where A' is a conjunction of variables, then $A \simeq B$ if and only if $B \underset{\cdot}{=} B' \supset a_j$, B' is a conjunction of variables, $B' \simeq A'$, and $i = j$.

In conclusion, we will describe a recognition algorithm for isomorphism of formulas in HCC, which does not use the complex passage to the reduced form. To this end, we will need one lemma.

Let A, B, and C be quasireduced formulas, $C \underset{\cdot}{=} C' \supset a_i$. Let A* and B* denote the result of replacement of all occurrences of subformulas isomorphic to C in A and B by an occurrence of a variable a_k not appearing in A and B (obviously, A* and B* are quasireduced formulas, too).

LEMMA 8. $A^* \simeq B^*$ if and only if $A \simeq B$.

The proof is based on Corollary 1 and uses the fact that permutations of conjunctive terms and rearrangement of parentheses in conjunctions either appear within an occurrence of a subformula whose outermost connective is an implication or do not change the form of this subformula. □

Let A and B be arbitrary formulas. We will describe the steps of work of the algorithm.

(1) Pass to A^0 and B^0. If $A^0 \underset{\cdot}{=} T$ or $B^0 \underset{\cdot}{=} T$, then $A \simeq B$ if and only if $A^0 \underset{\cdot}{=} T = B$. If $A^0 \underset{\cdot}{\neq} T$ and $B^0 \underset{\cdot}{\neq} T$, then pass to the next step.

(2) Pass to $(A^0)^+$ and $(B^0)^+$. If $l((A^0)^+)$ and $l((B^0)^+) \leqslant 1$, then check whether they are isomorphic (see Corollary 2). Otherwise, pass to the next step.

(3) Find the extreme left occurrence v of a variable for which $l(v) = l(A) \geqslant 1$. This occurrence is generated by an occurrence of a subformula C of the form $C' \supset a_i$, where C' contains no implications (v is contained in C'). Write all subformulas $C_1, \ldots, C_n$ of the formula A and B containing one implication. Choose among them those isomorphic to C (on the basis of

Corollary 2). Pass to Step (2) with formulas $((A^0)^+)^*$ and $((B^0)^+)^*$.

The algorithm will complete its work, since

$$\theta(((A^0)^+)^*) + \theta((((B^0)^+)^*)^*) < \theta((A^0)^+) + \theta((B^0)^+).$$

Some estimates:

For each formula A, $\theta(A^0) \leqslant \theta(A)$; $\theta((A^0)^+) \leqslant 2^{\theta(A)}$ (see Lemma 5). Actually, when one passes from A^0 to $(A^0)^+$, θ can grow exponentially. This happens, e.g., in formulas of the form

$$\underbrace{(\dots(a_1 \& a_1 \supset a_1 \& a_1) \supset \dots a_1 \& a_1) \supset a_1 \& a_1}_{n \text{ times}}$$

In recognizing isomorphism of quasireduced formulas A and B, the main operation is comparing symbols. If $l(A)$, $l(B) \leqslant 1$, then, to recognize isomorphism, $\frac{(\theta(A)+\theta(B))(\theta(A)+\theta(B)+1)}{2}$ comparisons of symbols would suffice. It is possible to prove by induction on $\theta(A) + \theta(B)$ that in the course of recognizing isomorphism of quasireduced formulas by the above algorithm no more than $(\theta(A) + \theta(B))^2(\theta(A) + \theta(B) + 1)$ comparisons of symbols are used.

3. In this section, symbols a, b may denote each of the propositional variables $a_1, \dots, a_k, \dots$.

The notation $I(t_1) \cong I(t_2)$ means that $I(t_1)$ is equal to $I(t_2)$ for all values of free variables; $t_1 \cong t_2$ means that $I(t_1) \cong I(t_2)$ for all interpretations I. If f and g are morphisms of HCC, then $I(\tau(f)) \cong I(\tau(g))$ if and only if $I(f) = I(g)$ in M. $I(t_1) \cong I(t_2)$ if and only if $I(\overline{\overline{t}}_1) \cong I(\overline{\overline{t}}_2)$ since the normalization of a term does not change the corresponding map in M. The relation $\cong$ is transitive, symmetric, and reflexive.

Let $\Gamma \supset A$, where $\Gamma \rightleftharpoons A_1, \dots, A_n$ is a list of formulas, denote the formula $A_1 \supset (A_2 \supset \dots (A_n \supset A)$; $A_1 \& \dots \& A_n$, the formula $(\dots(A_1 \& A_2)\dots) \& A_n$; for a list $\Delta \rightleftharpoons s_1 \dots s_n$ of deductive terms and a term t of suitable types, let the expression (t, Δ) be the abbreviation for $(\dots(t, s_1), \dots, s_n)$ and $\langle\Delta\rangle$ the abbreviation for $\langle\dots\langle s_1, s_2\rangle, \dots s_n\rangle$; let $\lambda\xi t$, where $\xi = x_0, \dots, x_n$ is a list of variables of any types, be the abbreviation for $\lambda x_1 \dots \lambda x_n t$; finally, let I_n^k denote the usual combination of l and r representing the notation of the projection onto the k-th factor of the Cartesian product of n objects.

Henceforth, whenever lists are used in the notation of formulas and terms we assume that the list cannot be extended, i.e., expressions of the form $A \supset (\Gamma \supset B)$, $\lambda\xi\lambda x t$ should not arise.

Definition 1. (1) By a u-formula is meant a formula not containing conjunctions and T; (2) by a u-variable is meant a variable whose type is a u-formula; (3) by u-terms are meant deductive terms defined by induction as follows: (a) a u-variable is a u-term; (b) if x is a u-variable and Δ is a list of u-terms of suitable types, then (x, Δ) is a u-term; (c) if Δ is a list of u-terms none of which has the form $\langle\Sigma\rangle$ for some list of terms Σ, then $\langle\Delta\rangle$ is a u-term; (d) $\lambda\xi t$ is a u-term if ξ is a list of u-variables and t a u-term which does not have the form $\langle\Delta\rangle$ for some list of terms Δ.

Remark 1. Obviously, each subterm of a u-term is a u-term. The type of each u-term is a conjunction of u-formulas; if t does not have the form $\langle\Delta\rangle$, then the type of t is a u-formula. Note also that each u-formula which is not a propositional variable has the form $\Gamma \supset a$, where Γ is a list of u-formulas.

Let $\chi(t)$ denote the number of occurrences of symbols in t; let $\chi_1(t_1, t_2) = \max(\chi(t_1), \chi(t_2))$.

THEOREM 1. For any two morphisms f, $g: A \to B$ in HCC, where $B \not\cong T$, there exist normal u-terms t_1 and t_2 such that for each interpretation I

$$I(f) = I(g) \Longleftrightarrow I(t_1) \cong I(t_2).$$

The proof of this theorem will be given after the proof of the main theorems 2 and 3.

Let t be a normal u-term. Let v be an arbitrary occurrence of some u-variable in t. By the *domain of action* of the occurrence v is meant the occurrence of the smallest subterm τ of the term t having a propositional variable as its type and containing v. If v is an occurrence of $x^{\Gamma\supset a}$, where $\Gamma \mathrel{\dot=} A_1,\dots,A_n$, then $\mathcal{T} \mathrel{\dot=} (x^{\Gamma\supset a}, \Delta)$, where $\Delta = S_1^{A_1},\dots,S_n^{A_n}$.

By the *degree* of an occurrence v of a variable x in t is meant the number of occurrences of x in t, other than v, in whose domain of action v lies. By the degree of a variable x in t is meant the maximum of degrees of its occurrences in t. By the degree of a term t [we denote it by d(t)] is meant the sum of the degrees of variables occurring in this term.

d(t) does not exceed the number of occurrences of variables in t; a fortiori, $d(t) \leqslant \chi(t)$; d(t) does not decrease in the passage to a superterm.

Let $\alpha_k \leftrightharpoons \{\beta_k, \gamma_k\}$ be two-element sets $(k \geqslant 1)$, $\alpha_k \cap \alpha_j = \emptyset$ $(k \neq j)$. For any set α let

$$(\alpha)^i \leftrightharpoons \underbrace{\alpha \times \dots \times \alpha}_{i\text{ times}} \quad (i \geqslant 1), \quad \alpha^0 \leftrightharpoons \{*\}.$$

THEOREM 2. If t_1 and t_2 are normal u-terms and $t_1 \not\equiv t_2$, then $I(t_1) \not\equiv I(t_2)$ for some interpretation $I: a_k \to (\alpha_k)^{i_k}$ $(k \geqslant 1)$, where $i_k \leqslant d(t_1) + d(t_2) + 1$.

Assuming that Theorems 1 and 2 have already been proved, one can easily obtain the following main result:

THEOREM 3. For any two morphisms f, g:A → B in HCC there exists an interpretation I such that $f \equiv g \iff I(f) = I(g)$.

Proof of Theorem 3. $f \equiv g$ implies $I(f) \equiv I(g)$ for all interpretations I. It remains to show that there exists an interpretation I for which $I(f) = I(g)$ implies $f \equiv g$. If $B \not\simeq T$, then apply theorems 1 and 2. □ If $B \simeq T$, then there exists a unique morphism from A to B and, therefore, $f \equiv g$. In this case, one can take any interpretation as I.

Generally, the estimation of cardinality of sets $I(a_k)$ $(k \geqslant 1)$ depends on characteristics of the normal form of some terms associated with τ(f) and τ(g) which, however, does not impede obtaining a decision algorithm for equivalence $f \equiv g$ independent of the usual algorithm [passage to $\bar{\bar{\tau}}(f)$ and $\bar{\bar{\tau}}(g)$ and comparing these terms].

Namely, one should consider interpretations of the form $I: a_k \to (\alpha_k)^{i_k}$ for all i_k starting with 0 (gradually increasing them) and for each of them compare I(τ(f)) with I(τ(g)) for all values of free variables. Actually, we are interested in the sets $I(a_k)$ only for those a_k (there are finitely many of them) which appear in both τ(f) and τ(g). This algorithm is complimentary to the usual one in the sense that it provides the *negation* of $f \equiv g$ (if $f \not\equiv g$) no later than when the upper bound for i_k is reached, whereas the usual algorithm provides the *confirmation* of $f \equiv g$ (if $f \equiv g$) no later than when the normal forms are found.

The estimate of numbers i_k may turn out to be greatly exaggerated. For instance, if

$$t_1 = \underbrace{(x^{a\supset a}, (x^{a\supset a}, \dots (x^{a\supset a}}_{2m}, x^a)\dots)$$

$$t_2 = \underbrace{(x^{a\supset a}, (x^{a\supset a}, \dots (x^{a\supset a}}_{2n+1}, x^a)\dots), \quad a \leftrightharpoons a_k$$

then for each interpretation I for which $I(a_k) = \alpha_k$, $I(t_1) \not\equiv I(t_2)$. It is enough to take $x^a = \beta_k$, $x^{a\supset a} = \lambda y(\bar{y})$, where $\bar{\beta}_k = \gamma_k$ and $\bar{\gamma}_k = \beta_k$, in order to negate $I(t_1) \cong I(t_2)$. The same values of variables negate $I(t_1') \cong I(t_2')$ for any two terms which are reduced to t_1 and t_2 as their respective normal forms, independently of the length of a reduction sequence.

Proof of Theorem 2. Since t_1 and t_2 are normal terms, $t_1 \equiv t_2$, one may assume that $t_1 \neq t_2$ even an arbitrary renaming of tied variables. The proof will be conducted by induction on $\chi_1(t_1, t_2)$. We can assume that t_1 and t_2 have the same type; otherwise, already for the interpretation $I: a_k \to (\alpha_k)$ $(k \geqslant 1)$, the values $I(t_1)$ and $I(t_2)$ lie in distinct sets.

If $\chi_1(t_1, t_2) = 1$, then $t_1 \mathrel{\dot=} x^a \neq t_2 \mathrel{\dot=} y^a$. One can take $I: a_k \to \alpha_k$, $x^a = \beta$, $y^a = \gamma$ (if $a \leftrightharpoons a_k$, then $\beta = \beta_k$, $\gamma \leftrightharpoons \gamma_k$).

Below, we will consider only interpretations of the form $I: a_k \mapsto (\alpha_k)^{l_k}$ $(k \geq 1)$. Let $\chi_1(t_1, t_2) = L > 1$. The following cases are possible:

(1) $t_1 \mp \lambda \xi_1 t_1'$, $t_2 \mp \lambda \xi_2 t_2'$, $\xi_1 \mp x_1^{A_1}, \ldots, x_n^{A_n}$, $\xi_2 \mp y_1^{A_1}, \ldots, y_n^{A_n}$;

(2) $t_1 \mp \langle \Delta_1 \rangle$, $t_2 \mp \langle \Delta_2 \rangle$, $\Delta_1 \mp r_1^{A_1}, \ldots r_n^{A_n}$, $\Delta_2 \mp s_1^{A_1}, \ldots, s_n^{A_n}$;

(3) $t_1 \mp (x_1^{\Sigma_1 \supset a}, \Delta_1)$, $t_2 \mp (x_2^{\Sigma_2 \supset a}, \Delta_2)$, $x_1 \neq x_2$;

(4) $t_1 \mp (x^{\Sigma \supset a}, \Delta_1)$, $t_2 \mp (x^{\Sigma \supset a}, \Delta_2)$, x does not appear in Δ_1 and Δ_2, $\Sigma \mp A_1, \ldots, A_n$, $\Delta_1 \mp r_1^{A_n}, \ldots, r_n^{A_n}$, $\Delta_2 \mp s_1^{A_1}, \ldots, s_n^{A_n}$.

(5) $t_1 \mp (x^{\Sigma \supset a}, \Delta_1)$, $t_2 \mp (x^{\Sigma \supset a}, \Delta_2)$, Σ, Δ_1 and Δ_2 have the same form as in (4) but x appears in Δ_1 or Δ_2.

In all cases, r_i and s_i $(1 \leqslant i \leqslant n)$ are normal u-terms.

All features of the form of terms in (1)-(5) [e.g., the fact that in the lists ξ_1 and ξ_2 in (1) the numbers of variables and the types of x_j and y_j $(1 \leqslant j \leqslant n)$ coincide] follow from the fact that t_1 and t_2 are normal u-terms of the same type.

To justify the induction passage in cases (1)-(4), no change of interpretation is required. In cases (1) and (2) it is not even necessary to define new values of free variables. In case (3) it suffices to take distinct constant functions as values of x_1 and x_2. In case (4) one has to apply the induction hypothesis to terms in Δ_1 and Δ_2 [from $t_1 \not\mp t_2$, it follows that $r_i \not\mp s_i$ for some i $(1 \leqslant i \leqslant n)$ and, by the induction hypothesis, $I(r_i) \neq I(s_i)$ for some values of free variables] and, using the fact that x does not appear in Δ_1 and Δ_2, choose a function Φ separating values of r_i and s_i as a value of x.

The most difficult is the last remaining case (5).

Let $x^{\Sigma \supset a}, x_1^{C_1}, \ldots, x_l^{C_l}$ be the list of all free variables of the terms t_1 and t_2. As in case (4), let $r_i \equiv s_i$. Apply the induction hypothesis. Let I be an interpretation and Φ, $\Phi_1, \ldots, \Phi_l$ be values of variables $x^{\Sigma \supset a}, x_1^{C_1}, \ldots, x_l^{C_l}$, respectively, in the sets $I(\Sigma \supset a)$, $I(C_1), \ldots, I(C_l)$ for which $I(r_i) \neq I(s_i)$. Let $\Psi_1 \in I(A_1), \ldots, \Psi_n \in I(A_n)$, $\varphi_1 \in I(A_1), \ldots, \varphi_n \in I(A_n)$ be the values of the terms $I(r_1), \ldots, I(r_n)$, $I(s_1), \ldots, I(s_n)$, respectively, for these values of free variables.

Since we have already chosen the value of x (it is Φ), it may happen that $(\Phi, \Psi_1, \ldots, \Psi_n) = (\Phi, \varphi_1, \ldots, \varphi_n)$. Consider a new interpretation I', where $I'(b) = I(b)$ if $b \neq a$ and $I'(a) = I(a) \times \{\beta, \gamma\}$. We will try to choose new values of variables $x, x_1, \ldots, x_l$ – $\Phi' \in I'(\Sigma \supset a), \Phi_1' \in I'(C_1), \ldots, \Phi_l' \in I'(C_l)$, respectively, in such a way that among the new values of the terms $I(r_j)$ and $I(s_j)$ $(1 \leqslant j \leqslant n)$, which we will denote by Ψ_j' and φ_j', respectively, Ψ_i' and φ_i' would still be distinct and $(\Phi', \Psi_1', \ldots, \Psi_n')$ would be equal to $\langle \eta_1, \beta \rangle$, while $(\Phi', \varphi_1', \ldots, \varphi_n')$ would be equal to $\langle \eta_2, \gamma \rangle$ [where $\eta_1 = (\Phi, \Psi_1, \ldots, \Psi_n) \in I(a)$, $\eta_2 = (\Phi, \varphi_1, \ldots, \varphi_n) \in I(a)$).

Definition 2. Let B be an arbitrary u-formula and let $\omega \in I(B)$ and $\omega' \in I'(B)$ be arbitrary constants. We define a binary relation $\omega' | \omega$ (ω' extends the definition of ω) by induction on the number of occurrences of propositional variables in B.

Let $B \mp b \not\mp a$. Then $\omega' | \omega$ if $\omega' = \omega$.

Let $B \mp a$. Then $\omega' | \omega$ if $\omega' \in \{\omega\} \times \{\beta, \gamma\}$, i.e., the left component of ω' coincides with ω.

Let $B \mp \Gamma \supset b$, where $\Gamma \mp B_1, \ldots, B_m$. Let $b \not\mp a$. Then $\omega' | \omega$ if for any $\omega_1 \in I(B_1), \ldots, \omega_m \in I(B_m)$ and $\omega_1' \in I'(B_1), \ldots, \omega_m \in I'(B_m)$ such that $\omega_1' | \omega_1, \ldots, \omega_m' | \omega_m$, $(\omega', \omega_1', \ldots, \omega_m') = (\omega, \omega_1, \ldots, \omega_m)$. Let $b \mp a$. Then $\omega' | \omega$ if for any $\omega_1, \ldots, \omega_m$ and $\omega_1', \ldots, \omega_m'$ such that $\omega_1' | \omega_1, \ldots, \omega_m' | \omega_m$, $(\omega', \omega_1', \ldots, \omega_m') \in \{(\omega, \omega_1, \ldots, \omega_m)\} \times \{\beta, \gamma\}$.

LEMMA 1. (a) Let $\omega' \in I(\mathcal{B}')$, ω_0, $\omega_1 \in I(\mathcal{B})$ and $\omega'|\omega_0, \omega'|\omega_1$. Then $\omega_0 = \omega_1$. (b) Let $\omega \in I(\mathcal{B})$. Then there exists $\omega' \in I'(\mathcal{B})$ such that $\omega'|\omega$. (c) If $\mathcal{B} \mp \Sigma \supset \alpha$ (where $\Sigma \mp \mathcal{B}_1, \ldots, \mathcal{B}_m$), then ω' can be chosen in such a way that the function

$$\tau(\lambda\xi(\omega', \xi)) : I'(\mathcal{B}_1) \times \ldots \times I'(\mathcal{B}_m) \to \{\beta, \gamma\}$$

would be equal to any a priori given function.

The proof is easily conducted by induction on the number of occurrences of variables in B. □

LEMMA 2. Let t be a normal u-term which is not of the form <Δ>, $x_1^{B_1}, \ldots, x_m^{B_m}$ be its free variables, I and I', interpretations connected in the way described above; $\omega_1 \in I(\mathcal{B}_1), \ldots \omega_m \in I(\mathcal{B}_m)$; $\omega_1' \in I'(\mathcal{B}_1), \ldots, \omega_m' \in I'(\mathcal{B}_m)$; $\omega_1'|\omega_1, \ldots, \omega_m'|\omega_m$. Then $\omega'|\omega$, where $\omega \Leftrightarrow I(t)\left\{ \begin{smallmatrix} x_1, \ldots, x_m \\ \omega_1, \ldots, \omega_m \end{smallmatrix} \right.$ and $\omega' \Leftrightarrow I(t)\left\{ \begin{smallmatrix} x_1, \ldots, x_m \\ \omega_1', \ldots, \omega_m' \end{smallmatrix} \right.$.

The proof is done by induction on $\chi(t)$. □

We will complete the proof of Theorem 2.

By Lemma 1 (c), there exists Φ' such that $\Phi'|\Phi$. $\Phi' \in I'(\Sigma \supset \alpha)$, $\Phi \in I(\Sigma \supset \alpha)$ and for any Ψ_j' and φ_j', where $\Psi_j'|\Psi_j$ and $\varphi_j|\varphi_j'$ $(1 \leqslant j \leqslant n)$, $(\Phi', \Psi_1', \ldots, \Psi_n') = \langle \eta_1, \beta \rangle \in I'(\alpha)$ $(\Phi', \varphi_1', \ldots, \varphi_n') = \langle \eta_2, \gamma \rangle \in I(\alpha)$ holds.

Let $\Phi_1', \ldots, \Phi_\ell'$ be arbitrary extensions of the definition of old values $\Phi_1 \in I(C_1), \ldots, \Phi_\ell \in I(C_\ell)$ of free variables $x_1^{C_1}, \ldots, x_\ell^{C_\ell}$ of the terms t_1 and t_2 [these extensions exist by Lemma 1 (b)]. Choose as new values of free variables $x^{\Sigma \supset \alpha}$ $x_1^{C_1}, \ldots, x_\ell^{C_\ell}$, respectively, $\Phi' \in I'(\Sigma \supset \alpha), \Phi_1' \in I'(C_1), \ldots, \Phi_\ell' \in I'(C_\ell)$.

For these values of free variables, we introduce the notation $\Psi_j' \Leftrightarrow I'(r_j)$, $\varphi_j' \Leftrightarrow I'(s_j)$ $(1 \leqslant i \leqslant n)$. By Lemma 2, $\Psi_j'|\Psi_j$ and $\varphi_j'|\varphi_j$. Then, by the choice of value of the variable $x^{\Sigma \supset \alpha}$, we have

$$I'(t_1) = \langle \eta_1, \beta \rangle, \; I'(t_2) = \langle \eta_2, \gamma \rangle$$

Since $x^{\Sigma \supset \alpha}$ appears in at least one of the lists Δ_1 and Δ_2, we have $d(t_1) + d(t_2) \geqslant d(r_j) + d(s_j) + 1$ for all j $(1 \leqslant j \leqslant n)$. Therefore, by definition of I', $I(a_k) = (\alpha_k)^{i_k}$ and $i_k \leqslant d(r_j) + d(s_j) + 1$ implies $I'(a_k) = (\alpha_k)^{i_k'}$ and $i_k' \leqslant d(t_1) + d(t_2) + 1$. The proof of Theorem 2 is complete. □

LEMMA 3. Suppose a term t^A is such that: (a) For all subterms of t of the form <Δ> none of the list entries has the form <Γ>; (b) the term t contains no subterms of the form (λξt, Γ), f<Δ>, where f is some combination of l and r; (c) $A \mp A_1 \& \ldots \& A_k$, where $A_1, \ldots, A_k$ are u-formulas; (d) the constant $\mathbb{O}$ does not appear in t and all variables appearing in t are u-variables. Then t is a u-term.

Proof. Note that (c) implies (c'): t does not have the form λξ<Δ> or <...,λξ<Δ>,...>. We will prove by induction on $\chi(t)$ that from (a), (b), (c'), and (d), the assertion of the lemma follows. The base ($\chi(t) = 1$) follows from (d). The conditions (a), (b), and (d) are preserved in the passage to a subterm, so, to justify the induction passage, one has only to check the fulfillment of (c'). Let $\chi(t) > 1$.

The case $t \mp ft'$, where f is a combination of l and r, is impossible if (b) and (d) are fulfilled.

In the case of $t \mp \lambda\xi t'$ and $t \mp \langle s_1, \ldots, s_n \rangle$ $(n > 1)$, the induction passage is easily justified.

Slightly more complicated is the last case when $t \mp (s^{\Sigma \supset A}, \Gamma)$. [We assume that s does not have the form (s_1, Δ)]. Using the induction hypothesis, it is easily shown that in this case s is a u-variable. Then $\Sigma \supset A$ is a u-formula, the types of terms of the list Γ are members of the list Σ and, therefore, u-formulas. Thus, the terms in Γ satisfy (c) and, a fortiori, (c'). By induction, they are u-terms. Then t is also a u-term. □

LEMMA 4. Let t^A be a term and v be an occurrence of its subterm τ^B such that each subterm t_1 of the term t containing v (including t) has the form $\lambda\xi t_1'$ or <Δ> for some list of terms Δ. Then B is a subformula of A.

The proof is easily obtained by induction on the depth of the occurrence v. □

LEMMA 5. Let t^A be a term such that the condition (a), (b), and (c) of Lemma 3 are fulfilled and also (d'): all *free* variables of the term t are u-variables. Then t is a u-term.

Proof. We will show that (d) follows from (a), (c), and (d'). Then the required conclusion is obtained by Lemma 3.

Suppose that (d) is not fulfilled, i.e., $\emptyset$ appears in t or t contains some (tied) variables. Let v be the extreme left occurrence in t of a subterm τ of the form $\emptyset$ or $\lambda\xi\tau'$, where $\xi \doteq x_1^{A_1},\ldots,x_n^{A_n}$ and at least one of the formulas $A_1,\ldots,A_n$ is not a u-formula. Obviously, the type τ is not a u-formula.

Consider the largest subterm t_1 of the term t containing v and such that each subterm of t_1 containing v has the form $\lambda\xi t_1'$ or $\langle\Delta\rangle$. By Lemma 4, the type of t_1 is not a u-formula, so $t_1 \doteq t$ contradicts (c). Let t_2 be the smallest subterm of t containing t_1 properly.

The case $t_2 \doteq \delta t_1$, where $\delta \doteq l_1 r$, is impossible by (b). Then $t_2 \doteq (s^{\Sigma\supset A}, \Gamma)$. $t_1 \doteq s$ contradicts (b); therefore, t_1 is a member of the list Γ. The condition (d) is fulfilled for the term s by the choice of the occurrence v, and the condition (c) because of (b). Therefore, s is a u-term and $\Sigma \supset A$ is a u-formula; the members of the list Σ are also u-formulas. But the type of t_1 is not a u-formula. The obtained contradiction shows that (d) is fulfilled for t. □

LEMMA 6. Let τ_1, τ_2 and t_1, t_2 be terms such that there exists terms $r_1^{C_1},\ldots,r_k^{C_k}$ and $s_1^{\mathfrak{D}_1},\ldots,s_l^{\mathfrak{D}_l}$ and pairwise-distinct variables $x_1^{C_1},\ldots,x_k^{C_k}$ and $y_1^{\mathfrak{D}_1},\ldots,y_l^{\mathfrak{D}_l}$ (where $x_1,\ldots,x_k$ do not appear in any of the terms $r_1,\ldots,r_k$ and $y_1,\ldots,y_l$ do not appear in any of the terms $s_1,\ldots,s_l$) for which

$$(\tau_i)_{x_1,\ldots,x_k}[r_1,\ldots,r_k] \equiv t_i,\ (t_i)_{y_1,\ldots,y_l}[s_1,\ldots,s_l] \equiv \tau_i\ (i=1,2).$$

Then for each interpretation I

$$I(\tau_1) \cong I(\tau_2) \Longleftrightarrow I(t_1) \cong I(t_2).$$

To prove the implication $I(\tau_1) \cong I(\tau_2) \Rightarrow I(t_1) \cong I(t_2)$, it suffices to note that for all values of free variables $I(r_1),\ldots,I(r_k)$ are admissible values of the variables $x_1,\ldots,x_k$. The reverse implication is proved similarly. □

Proof of Theorem 1. Let f, $g{:}A \to B$ be morphisms of HHC, $B \not\doteq T$. Then, by Lemma 6 from Sec. 2, there exist isomorphisms $h_1{:}A' \to A$ and $h_2{:}B \to B'$, where A' and B' are multiple conjunctions of u-formulas. For each interpretation I, $I(f) = I(g) \Longleftrightarrow I(f') = I(g')$, where $f' = h_2(fh_1)$, $g' = h_2(gh_1)$. Consider the terms $\tau_1 \leftrightharpoons \bar{\bar{\tau}}(f')$ and $\bar{\bar{\tau}}_2 \leftrightharpoons \tau(g')$. Their type is B' and they contain a unique free variable x^{A_1}. We will show that $I(\tau_1) \cong I(\tau_2) \Longleftrightarrow I(t_1) \cong I(t_2)$ for some normal u-terms t_1 and t_2. Let $A' \doteq A_1 \& \ldots \& A_k$ $(k \geqslant 1)$. Since the terms τ_1 and τ_2 are normal, each occurrence of the variable $x^{A_1\&\ldots\&A_k}$ in them is contained in a subterm of the form $I_k^q x^{A_1\&\ldots\&A_k}$ for some $q(1 \leqslant q \leqslant k)$. We introduce the notation $r \leftrightharpoons \langle y_1^{A_1},\ldots,y_k^{A_k}\rangle$, where $y_1,\ldots,y_k$ are pairwise distinct new variables, $s_q \leftrightharpoons I_k^q x^{A_1\&\ldots\&A_k}$ $(1 \leqslant q \leqslant k)$. Let $t_i \doteq \overline{\overline{(\tau_i)_x[r]}}$ $(i = 1, 2)$; it is easily shown that t_i is obtained from τ_i by replacing each of the subterms $I_k^q x$ by y_q. The term t_i is normal and its type is the same as that of τ_i; therefore, by Lemma 5, t_i is a u-term. Obviously, $(t_i)_{y_1,\ldots,y_k}[s_1,\ldots,s_k] \doteq \tau_i$. By Lemma 6, $I(t_1) \cong I(t_2) \Longleftrightarrow I(t_1) \cong I(t_2)$. Then also $I(f) = I(g) \Longleftrightarrow I(t_1) \cong I(t_2)$. □

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