It is obvious that the eikonal and transport equations for the quasipotential do not differ from the equations presented in Sec. 1. The leading term of the short-wave asymptotics of the field of a point source in a small neighborhood of it and the excitation coefficient also have the form (2.4), (2.10).

In the case of oceanic flows we deal precisely with small vortices and $\varepsilon \leqslant 2\cdot 10^{-3}$. Hence, to describe the propagation of sound in an ocean with flows it is possible to use the equation for the quasipotential. Moreover, from formulas (2.4), (2.10) it follows that in a small neighborhood of the source the phase of the wave contains a correction term connected with the presence of the flow of order ε while the amplitude contains a correction of only order ε^2. The excitation coefficient of the wave in the region where it is possible to use the ray representation also contains only a correction term of order ε^2.

The author is grateful to V. S. Buldyrev for useful discussion of the papers.

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RIGOROUS JUSTIFICATION OF THE FRIEDLANDER-KELLER FORMULAS

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UDC 517.9;535.4

In the paper an asymptotic expansion (as $k\to\infty$, k the wave number) is proved for the Green function of the problem of diffraction by a smooth convex body in two cases: when one of the points of the source or observer lies on the boundary while the other is an arbitrary distance from the boundary and also when both points lie off the boundary but not far from it. The two-dimensional Dirichlet problem is considered.

In the present paper we present a rigorous justification of the formal asymptotics in the shadow zone for the problem of diffraction by a smooth convex body. We consider the case of the Dirichlet problem in the plane, although the results can easily be carried over to the second and third boundary-layer problems. In [1, 2] the asymptotics was justified in the case where one of the points of the source or the observer was located on the boundary of the domain while the second was near to the boundary. In the present work an analogous result is obtained for two cases: when one of the points lies on the boundary while

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the other is located a finite distance from it and also when both points do not belong to the boundary but lie in a region near the boundary.

1. The Main Results

Let $\Phi(M,\xi)$ be the Green function of the diffraction problem

$$\begin{cases} (\Delta+\kappa^2)\Phi(M,\xi)=0 \\ \Phi(\varsigma,\xi)\big|_{\varsigma\in\Gamma}=\delta(\varsigma,\xi) \\ \sqrt{\tau}\left(\frac{\partial\Phi}{\partial\tau}-i\kappa\Phi\right)\longrightarrow 0, \qquad \tau\longrightarrow\infty . \end{cases} \tag{1.1}$$

Let $\widetilde{\Phi}(M,\xi)$ be the approximate Green function satisfying the conditions

$$\begin{cases} (\Delta+\kappa^2)\widetilde{\Phi}(M,\xi)=Q(M,\xi) \\ \widetilde{\Phi}(\varsigma,\xi)\big|_{\varsigma\in\Gamma}=\delta(\varsigma,\xi)+Q_\Gamma(\varsigma,\xi) \\ \sqrt{\tau}\left(\frac{\partial}{\partial\tau}\widetilde{\Phi}-i\kappa\widetilde{\Phi}\right)\longrightarrow 0, \ \tau\longrightarrow\infty . \end{cases} \tag{1.2}$$

Below we use coordinates (n,s), where n is the normal length to Γ and s is arc length on Γ and also the ray coordinates α and τ :

$$AP - \text{tangent to } \Gamma, \qquad \alpha=s_{A\xi}, \qquad \tau=|AP| .$$

For $n\longrightarrow 0\ (\tau\longrightarrow 0)$ we have the relations

$$\begin{aligned} s &= \alpha+\tau+O(\tau^3) \\ n &= \frac{1}{2\rho(0)}\tau^2+O(\tau^3) \end{aligned} \tag{1.3}$$

where $\rho(s)=\varkappa(s)^{-1}$ is the radius of curvature of Γ .

To construct the function $\widetilde{\Phi}(P,\xi)$ we use the following expressions: Φ_1 is the exact solution for the circle of curvature at the point ξ, $\Phi_2^{\pm}$ is the creeping wave (see [3, 4]), and $\Phi_3^{\pm}$ is the Friedlander–Keller wave (see [4])

$$\Phi_3^{\pm}(P,\xi)=\frac{q^{\pm}(\alpha,\tau,\kappa)e^{i\kappa(\alpha+\tau)}}{\sqrt{\tau}} \tag{1.4}$$

$$q^{\pm}(\alpha,\tau,\kappa)=\text{const}\ \kappa^{1/2}\exp\left\{iz_1\left(\frac{\kappa}{2}\right)^{1/3}\int_0^{\alpha}\varkappa^{2/3}d\alpha+O(\kappa^{-1/3})\right\}\left[1+O\left(\kappa^{-2/3}\tau^{-2}\right)\right] \tag{1.5}$$

$$\Phi_2^{\pm}(P,\xi)=\sum_{j=0}^{N}\kappa^{-j/3}V_j^{\pm}(P,\xi)\exp\left\{\pm i\kappa s+i\left(\frac{\kappa}{2}\right)^{1/3}z_1\int_0^{s^{\pm}}\varkappa^{2/3}ds\right\} \tag{1.6}$$

where $\varkappa=1/\rho(s)$ is the curvature, and z_1 is the first root of the Airy function.

It is known (see [4]) that in the region $n\asymp\kappa^{-2/3+\varepsilon}$ the creeping waves go over into Friedlander–Keller waves. (The notation $f\asymp q$ means that $f=O(q)$ and $q=O(f)$ simultaneously.)

Thus, suppose $\widetilde{\Phi}(P,\xi)$ is

$$\begin{aligned} \Phi(P,\xi) &= \Phi_1(P,\xi)\mu_2(\alpha)+\Phi_2^{+}(1-\mu_2^{+}(\alpha))\mu_1(\tau)+ \\ &+\Phi_2^{-}(1-\mu_2^{-}(\alpha))\mu_1(\tau)+\Phi_3^{+}(1-\mu_2^{+}(\alpha))(1-\mu(\tau))+\Phi_3^{+}(1-\mu_2^{-}(\alpha))(1-\mu_1(\tau)) \end{aligned} \tag{1.7}$$

We define the neutralizers realizing the matching in the transition regions as follows:

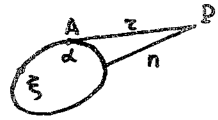

Fig. 1

$$\mu_1(r)=\begin{cases}1, & r\leqslant r_1, \quad r_1, r_2 \asymp \kappa^{-1/3+\varepsilon/2}\\ 0, & r\geqslant r_2, \quad r_2-r_1\asymp \kappa^{-1/3+\varepsilon/2}\end{cases}$$

$$\mu_2(\alpha)=\begin{cases}1, & |\alpha|\leqslant\alpha_1, \quad \alpha_1,\alpha_2\asymp \kappa^{-1/3+\varepsilon/2}\\ 0, & |\alpha|\geqslant\alpha_2, \quad \alpha_2-\alpha_1\asymp\kappa^{-1/3+\varepsilon/2}\end{cases} \tag{1.8}$$

$$\mu_2^{\pm}(\alpha)=\mu_2(\pm\alpha),\ \alpha\gtrless 0.$$

In [2] the asymptotics was justified for $\Phi(M,\xi)$ when the point M lies in the shadow relative to ξ and is not far from the boundary; that is, the following result was proved.

THEOREM 1. If

$$\alpha_M>\mathrm{const}\ \kappa^{-1/3+\varepsilon/2},\quad n_M=O(\kappa^{-2/3+\varepsilon}), \tag{1.9}$$

$\varepsilon<1/6$, then there is the asymptotic expansion

$$\Phi(M,\xi)=\{\Phi_2^+(M,\xi)+\Phi_2^-(M,\xi)+O(\kappa^{-N_1}\lambda_0)\} \tag{1.10}$$

where the curvature $\varkappa(s)$ is a continuously differentiable function $\varkappa(s)\in C^N[0,\ell]$, ℓ is the length of the contour Γ, $N_1=\frac{N}{3}-\varepsilon_0$, ε_0 is proportional to ε, and

$$\lambda_0(M,\xi,\kappa)=\exp\Big\{-\mathrm{Im}\, z_1\Big(\frac{\kappa}{2}\Big)^{1/3}\int_0^{\alpha_M}\varkappa^{-2/3}\,d\alpha\Big\}. \tag{1.11}$$

The estimate (1.10) will now be extended to the shadow zone outside the boundary layer; namely, we shall prove the following result.

THEOREM 2. Suppose

$$\alpha_M>\mathrm{const}\ \kappa^{-1/3+\varepsilon/2}\ \text{and}\ n_M>\mathrm{const}\ \kappa^{-2/3+\varepsilon}, \tag{1.12}$$

$\varepsilon<1/6$, ; then the following asymptotic estimate holds:

$$\Phi(M,\xi)=\{\Phi_3^+(M,\xi)+\Phi_3^-(M,\xi)+O(\kappa^{-N_1}\lambda_0)\} \tag{1.13}$$

under the assumption that $\varkappa(s)\in C^N[0,\ell]$.

In the present work a theorem analogous to Theorem 1 is also proved for the Green function $G(M,M_0)$ with source $M_0\bar{\in}\Gamma$.

Suppose $G(M,M_0)$ is the Green function for the problem of diffraction by Γ:

$$\begin{cases}(\Delta+\kappa^2)G(M,M_0)=\delta(M,M_0)\\ \quad G(\xi, M_0)|_{\xi\in\Gamma}=0\\ \sqrt{r}\Big(\dfrac{\partial G}{\partial r}-i\kappa G\Big)\to 0 \quad \text{as} \quad r\to\infty.\end{cases} \tag{1.14}$$

We introduce the notation

$$\begin{cases}\sigma=\Big(\dfrac{\kappa}{2}\Big)^{1/3}\displaystyle\int_0^s\varkappa^{2/3}\,ds\\ \nu=\kappa^{2/3}(2\varkappa(s))^{1/3}n.\end{cases} \tag{1.15}$$

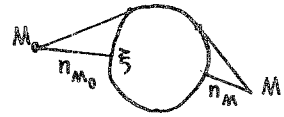

Fig. 2

We shall prove the following result.

THEOREM 3. Suppose the points M and M_0 lie in the boundary layer and are located in the shadow relative to one another:

$$n_M = O(\kappa^{-2/3+\varepsilon}), \quad n_{M_0} = O(\kappa^{-2/3+\varepsilon}) \tag{1.16}$$

$$\sigma_M - \sqrt{\nu_M} - \sqrt{\nu_{M_0}} > \mathrm{const}\ \kappa^{\varepsilon/2} .$$

Suppose also that $\varkappa(s) \in C^N[0,\ell]$. Then there is the estimate

$$G(M,M_0) = [G_2^+(M,M_0) + G_2^-(M,M_0)][1 + O(\kappa^{-N_1})] \tag{1.17}$$

where

$$G_2^\pm(M,M_0) = \sum_{j=0}^{N} \kappa^{-\frac{j+1}{3}} W_j(M,M_0) \exp\left\{\pm i\kappa s_M + i\left(\frac{\kappa}{2}\right)^{1/3} z_1 \int_0^{\pm s_M} \varkappa^{2/3} ds\right\} \tag{1.18}$$

is the formal asymptotics in the shadow, i.e., the creeping wave.

The proof of the theorems is carried out by the methods of [1, 2]. In addition, in [2] it was shown how to extend the theorems to a contour Γ of finite smoothness if they have been proved for an analytic contour Γ . Here the situation is analogous. We therefore assume that Γ is an analytic contour: $\varkappa(s)$ is an analytic function in the strip

$$\operatorname{Im} s = O(1) \tag{1.19}$$

and $\varkappa(s)$ is bounded there:

$$\varkappa(s) = O(1) \tag{1.20}$$

2. Proof of Theorem 2

As in [2], using Green's formula, we obtain an integral equation for Φ :

$$\Phi(M,\xi) - \tilde{\Phi}(M,\xi) = -K(M,\xi) + \int_\Gamma K_0(\zeta,\xi)\Phi(M,\zeta)\,d\zeta \tag{2.1}$$

where

$$K_0(\zeta,\xi) = K(\zeta,\xi) - Q_\Gamma(\zeta,\xi) , \tag{2.2}$$

$$K(M,\xi) = \frac{i}{4} \iint_\Omega H_0^{(1)}(\kappa|MP|)\, Q(P,\xi)\,dP \tag{2.3}$$

and Ω is the exterior of Γ.

The error $Q(P,\xi)$ in (2.3) can be represented in the form

$$Q = Q_1^+ + Q_1^- + Q_2^+ + Q_2^- \tag{2.4}$$

where

$$Q_1^\pm = 2(1-\mu_2^\pm)\nabla(\Phi_2^\pm - \Phi_3^\pm)\nabla\mu_1 + 2(\Phi_2^\pm - \Phi_3^\pm)\nabla\mu_1\nabla\mu_2^\pm \tag{2.5}$$
$$+ (1-\mu_2^\pm)(\Phi_2^\pm - \Phi_3^\pm)\Delta\mu_1 + \mu_1(\Phi_1 - \Phi_2^\pm)\Delta\mu_2^\pm + (\Delta + \kappa^2)\Phi_2^\pm(1-\mu_2^\pm)\mu_1 ,$$

$$Q_2^\pm = 2(1-\mu_1)\nabla(\Phi_3^\pm - \Phi_2^\pm)\nabla\mu_2^\pm + (1-\mu_1)(\Phi_1 - \Phi_3^\pm)\Delta\mu_2^\pm + (\Delta+\kappa^2)\Phi_3^\pm(1-\mu_1)(1-\mu_2^\pm) . \tag{2.6}$$

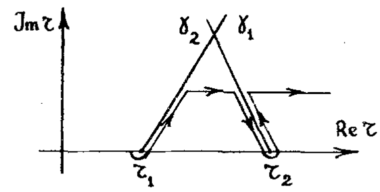

Fig. 3

In correspondence with (2.4), we represent $K(M,\xi)$ in the form

$$K(M,\xi) = K^{+}(M,\xi) + K^{-}(M,\xi) \tag{2.7}$$

$$K^{\pm}(M,\xi) = K_1^{\pm}(M,\xi) + K_2^{\pm}(M,\xi). \tag{2.8}$$

The integrals with the signs $\pm$ are estimated similarly, and below we therefore estimate only $K^{+}(M,\xi)$, omitting the sign "+."

We first consider $K_1(M,\xi)$:

$$K_1(M,\xi) = \frac{i}{4}\iint_{\Omega} H_0^{(1)}(\kappa|MP|)\,Q_1(P,\xi)\,dP \tag{2.9}$$

By (2.5) we have

$$\operatorname{supp} Q_1(P,\xi) = \left\{\Omega_\varepsilon \,\middle|\, |\alpha| > \mathrm{const}\,\kappa^{-1/3+\varepsilon/2},\ \tau = O(\kappa^{-1/3+\varepsilon/2})\right\} \tag{2.10}$$

i.e., the integration in (2.9) goes over the boundary-layer region Ω_ε, located in the shadow relative to ξ. The integral (2.9) was estimated in [2] with the difference that there $n_M = O(\kappa^{-2/3+\varepsilon})$; it is not hard to carry over the same estimates to the case $n_M > \mathrm{const}\,\kappa^{-2/3+\varepsilon}$. We have the estimate

$$|K_1(M,\xi)| < \left\{\kappa^{\varepsilon}\chi(\alpha_M,\alpha^*) + \kappa^{-N_1}\left[1-\chi(\alpha_M,\alpha^*)\right]\right\}\lambda_0(M,\xi,\kappa) \tag{2.11}$$

$$\chi(\alpha_M,\alpha^*) = \begin{cases} 1, & \alpha_M \le \alpha^* \\ 0, & \alpha_M > \alpha^* \end{cases}, \quad \alpha^* = \mathrm{const}\,\kappa^{-1/3+\varepsilon/2}. \tag{2.12}$$

For the definition of $\lambda_0(M,\xi,\kappa)$ see (1.11).

We now consider $K_2(M,\xi)$. The region of integration is now

$$\operatorname{supp} Q_2(P,\xi) = \left\{\Omega'_\varepsilon \,\middle|\, |\alpha| > \mathrm{const}\,\kappa^{-1/3+\varepsilon/2},\ \tau > \mathrm{const}\,\kappa^{-1/3+\varepsilon/2}\right\} \tag{2.13}$$

i.e., Ω'_ε is a region in the shadow relative to ξ but lying outside the boundary layer. We shall show how to obtain an estimate for

$$K_2(M,\xi) = \frac{i}{4}\iint_{\Omega'_\varepsilon} H_0^{(1)}(\kappa|MP|)\,Q_2(P,\xi)\,dP \tag{2.14}$$

Because of the piecewise analyticity of the integrand, we choose a deformation of the contour of integration on τ in (2.14) into the complex plane as shown in Fig. 3 where γ_1 and γ_2 are lines of discontinuity of the neutralizer $\mu_1(\tau)$. On the horizontal segments we set

$$\operatorname{Im}\tau = \mathrm{const}\,\kappa^{-1/3}\ln\kappa \tag{2.15}$$

i.e., everywhere on the contour

$$\operatorname{Im}\tau = o(\operatorname{Re}\tau) \tag{2.16}$$

we integrate on α along the real axis.

We shall estimate the part of (2.14) corresponding to horizontal segments of the contour of integration on τ. The integral along the lines of discontinuity γ_1 and γ_2 can be estimated as in [2].

We introduce some useful notation; suppose that for $\kappa \to \infty$:

$$V = \Phi \exp\{\Psi\} \text{ when } \Phi = O(\kappa^{const}).$$

Then Ψ is called the exponent of the rapidly varying factor of V (r.v.f. V).

We shall find the r.v.f. of the integral in (2.14):

$$\Psi = \text{r.v.f. } H_0^{(1)}(\kappa|MP|)Q_2(P,\xi) = -\mathrm{Im}\,\kappa R - \mathrm{Im}\,\kappa(\alpha+\tau) - \left(\frac{\kappa}{2}\right)^{1/3} \mathrm{Im}\, z_1 \int_0^{\alpha} \varkappa^{2/3} d\alpha \tag{2.17}$$

where

$$P = (\alpha, \tau), \quad R = |MP|, \quad \vec{R} = \overrightarrow{MP}. \tag{2.18}$$

We set

$$\Psi_M = -\left(\frac{\kappa}{2}\right)^{1/3} \mathrm{Im}\, z_1 \int_0^{\alpha_M} \varkappa^{2/3} d\alpha. \tag{2.19}$$

Then

$$\Psi = \Psi_M - \mathrm{Im}\,\kappa(R+\tau) - \left(\frac{\kappa}{2}\right)^{1/3} \mathrm{Im}\, z_1 \int_{\alpha_m}^{\alpha} \varkappa^{2/3} d\alpha. \tag{2.20}$$

We have

$$\mathrm{Im}\, R = \frac{\partial R}{\partial \tau}\, \mathrm{Im}\,\tau\,[1+o(1)]. \tag{2.21}$$

It is not hard to find that

$$\frac{\partial R}{\partial \tau} = \left(\frac{\vec{R}}{R}, \vec{e}\right), \quad \vec{e} = \frac{\overrightarrow{AP}}{|AP|} \tag{2.22}$$

and thus

$$\frac{\partial R}{\partial \tau} = \cos\varphi = -\cos\beta. \tag{2.23}$$

We shall estimate Ψ

1) Let $\alpha - \alpha_M \asymp 1$.

a) Suppose first that $\alpha < \alpha_M$. Since $\beta > \beta_0$ (see Fig. 4), we have $\beta > const$, i.e., $1 - \cos\beta_0 > const$, and hence from (2.23), (2.20), and (2.15) we obtain

$$\Psi < \Psi_M - const\ \kappa^{2/3}. \tag{2.24}$$

Estimate (2.24) shows that the region 1) a) makes an exponentially small contribution to $K_2(M,\xi)$.

b) Suppose that $\alpha > \alpha_M$; we then immediately obtain from (2.20)

$$\Psi < \Psi_M - const\ \kappa^{1/3}. \tag{2.25}$$

The region of integration 1) b) again gives an exponentially small contribution (as compared with $\exp \Psi_M$).

2) Suppose $\alpha - \alpha_M = o(1)$

a) If $\alpha < \alpha_M$ and $\alpha_M - \alpha > const\ \kappa^{-1/3} \ln \kappa$, then for $\alpha - \alpha_M = o(1)$, we have

$$1 - \cos\beta \asymp const\,(\alpha - \alpha_M)^2.$$

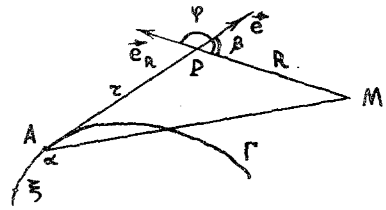

Fig. 4

From (2.20), (2.23), (2.15) we thus obtain

$$\Psi < \Psi_M - \mathrm{const}\,\kappa(\alpha-\alpha_M)^2\,\mathrm{Im}\,\tau + \mathrm{const}\,\kappa^{1/3}(\alpha_M-\alpha). \tag{2.26}$$

We have

$$\kappa(\alpha-\alpha_M)\,\mathrm{Im}\,\tau \gg \kappa^{1/3}(\alpha_M-\alpha). \tag{2.27}$$

Then

$$\Psi < \Psi - \mathrm{const}\,\kappa(\alpha-\alpha_M)^2\,\mathrm{Im}\,\tau < \Psi_M - \mathrm{const}\,\ln^3\kappa \tag{2.28}$$

and hence $e^{\Psi} < e^{\Psi_M}\cdot\kappa^{-c}$ for all $c>0$, i.e., the region 2) a) makes a small contribution to (2.11).

b) Suppose $\alpha > \alpha_M$ and $\alpha-\alpha_M > \mathrm{const}\,\kappa^{-1/3}\ln\kappa$.

From (2.20) we obtain

$\Psi < \Psi_M - c\ln\kappa$, where c is any number.

We again obtain power smallness.

Thus, the essential contribution to the integral $K_2(M,\xi)$ is given by a narrow neighborhood of the ray

$$\alpha = \alpha_M,\ \tau \geqslant \mathrm{const}\ \kappa^{-1/3+\varepsilon/2} \tag{2.29}$$

such that

$$\alpha-\alpha_M = o(\kappa^{-1/3}\ln\kappa). \tag{2.30}$$

Integration over this neighborhood refines only the power part of the estimate. By carrying out similar but involved computations, it is possible to obtain the following estimate:

$$|K_2(M,\xi)| < \left\{\kappa^{\varepsilon}\chi(\alpha_M,\alpha^*) + \kappa^{-N_1}[1-\chi(\alpha_M,\alpha^*)]\right\}\lambda_0(M,\xi,\kappa). \tag{2.31}$$

The estimates (2.11), (2.31) for $K_1(M,\xi)$ and $K_2(M,\xi)$ make it possible to justify the asymptotics of $\Phi(M,\xi)$ just as in [2]. Theorem 2 has thus been proved.

3. Proof of Theorem 3

Suppose the conditions of Theorem 3 hold.

We construct a quasi-Green function $\tilde{G}(M,M_0)$ as follows:

$$\tilde{G}(M,M_0) = G_1(M,M_0)\mu_1(n)\mu_2(s) + G_2^+(M,M_0)\mu(n)[1-\mu_2^+(s)] + G_2^-(M,M_0)\mu_1(n)[1-\mu_2^-(s)], \tag{3.1}$$

where $G_1(M,M_0)$ is the exact Green function for the circle of curvature at the point $\xi\in\Gamma$, $\overrightarrow{M_0\xi}\perp\Gamma$, $G_2^{\pm}(M,M_0)$ are creeping waves (see (1.18)), $\mu_1(n)$ realizes a cut-off with respect to n in the boundary-layer region,

$$\mu_1(n) = \begin{cases} 1, & n\leqslant n_1,\ \ n_1, n_2 \asymp \kappa^{-2/3+\varepsilon} \\ 0, & n\geqslant n_2,\ \ n_2-n_1 \asymp \kappa^{-2/3+\varepsilon} \end{cases} \tag{3.2}$$

and $\mu_2(s)$ realizes the matching in the transition regions:

$$\mu_2(s)=\begin{cases}1, & |s|\leqslant s_1, \quad s_1,s_2\asymp \kappa^{-1/3+\varepsilon/2}\\ 0, & |s|>s_2, \quad s_2-s_1\asymp \kappa^{-1/3+\varepsilon/2}\end{cases} \tag{3.3}$$

$$\mu_2^{\pm}(s)=\mu_2(\pm s),\ s\geqslant 0 .$$

We denote the characteristic exponential factor by $\lambda_0(M,M_0,\kappa)$:

$$\lambda_0(M,M_0,\kappa)=\exp\Big\{-\Big(\frac{\kappa}{2}\Big)^{1/3}\operatorname{Im} z_1\int_0^{s_M}\varkappa^{2/3}ds+\frac{2}{3}\operatorname{Re}(z_1-\nu_M)^{3/2}+\frac{2}{3}\operatorname{Re}(z_1-\nu_{M_0})^{3/2}\Big\}. \tag{3.4}$$

Using Green's formula, it is not hard to obtain for G and $\tilde{G}$ the relation

$$\tilde{G}(M,M_0)-G(M,M_0)=-K(M,M_0)+\int_\Gamma\big[\tilde{G}(\xi,M_0)+K(\xi,M_0)\big]\frac{\partial G}{\partial n}d\xi \tag{3.5}$$

where

$$K(M,M_0)=-\frac{i}{4}\iint_\Omega H_0^{(1)}(\kappa|PM|)\,Q(P,M_0)\,dP \tag{3.6}$$

We note that

$$\frac{\partial}{\partial n}G(\xi,M)=\Phi(M,\xi) \tag{3.7}$$

and for the function $\Phi(M,\xi)$ estimates have already been obtained for the entire shadow region (Theorems 1 and 2).

Thus, if we obtain an exponentially sharp estimate for $K(M,M_0)$ then, just as in [2] using the uniqueness lemma (see [4]) for the formyl asymptotics and also Theorem 1, it is not hard to prove that $\tilde{G}(M,M_0)$ gives the asymptotics of $G(M,M_0)$ in the shadow zone.

The following estimate can be proved for the integral $K(M,M_0)$:

$$|K(M,M_0)|<\mathrm{const}\big\{\kappa^{\varepsilon}\chi(s_M,s^*)+\kappa^{-N_1}[1-\chi(s_M,s^*)]\lambda_0(M,M_0,\kappa), \tag{3.8}$$

where $s^*=\mathrm{const}\ \kappa^{-1/3+\varepsilon/2}$.

As before, the estimate for $K(M,M_0)$ is obtained by choosing a suitable deformation of the contours of integration on s and n into the complex domain and estimates on the contours of the r.v.f. of the integrand in (3.6).

We use the notation

$$\begin{aligned}&\text{r.v.f. } \lambda_0(M,M_0,\kappa)=\Psi_0(M,M_0)\\ &\text{r.v.f. } H_0^{(1)}(\kappa|PM|)\,Q(P,M_0)=\Psi(M,P,M_0).\end{aligned} \tag{3.9}$$

We shall determine the form of $\Psi(M,P,M_0)$. We have

$$Q=Q_1+Q_2^{+}+Q_2^{-}, \tag{3.10}$$

where

$$Q_1(P,M_0)=2\mu_2(s)\nabla G_1\nabla\mu_1(n)+\mu_2 G_1\Delta\mu_1(n), \tag{3.11}$$

$$Q_2^{\pm}(P,M_0)=2\mu_1\nabla(G_1-G_2^{\pm})\nabla\mu_2^{\pm}+(G-G_2^{\pm})\mu_1\Delta\mu_2^{\pm}+(\Delta+\kappa^2)G_2^{\pm}\mu_1(1-\mu_2^{\pm})+2(1-\mu_2^{\pm})\nabla G_2\nabla\mu_1+G_2(1-\mu_2^{\pm})\Delta\mu_1. \tag{3.12}$$

In correspondence with (3.10) we write

$$K(M,M_0)=K_1(M,M_0)+K_2^{+}(M,M_0)+K_2^{-}(M,M_0). \tag{3.13}$$

We shall estimate $K_1(M,M_0)$. By the construction of $\tilde{G}$ we obtain from (3.11)

$$\operatorname{supp} Q_1(P,M_0)=\big\{P(n,s)\,\big|\,n_1\leqslant n\leqslant n_2,\ |s|\leqslant s_2\big\}. \tag{3.14}$$

Let

$$n_1 - n_M \geqslant \text{const}\ \kappa^{-2/3+\varepsilon}, \quad n_1 - n_{M_0} \geqslant \text{const}\ \kappa^{-2/3+\varepsilon}. \tag{3.15}$$

For $G_1(P, M_0)$ we have (see [3])

$$G_1(P,M_0) = \text{const} \sum_{a=1}^{\infty} \frac{H^{(1)}_{\eta a}(\kappa r) H^{(1)}_{\eta a}(\kappa r_0) H^{(2)}_{\eta a}(\kappa \rho(0))}{\frac{\partial}{\partial \eta} H^{(1)}_{\eta}(\kappa \rho(0))\big|_{\eta a}} \left[\frac{e^{i\eta a \varphi} - e^{-i\eta a(\varphi - 2\pi)}}{1 - e^{i2\pi\eta a}}\right], \tag{3.16}$$

where ηa are roots of the equation

$$H^{(1)}_{\eta a}(\kappa \rho(0)) = 0,$$

n, φ are the polar coordinates of the point P (at the point $M_0 : r = r_0, \varphi = 0$), and the pole of the polar coordinate systems is the center of the circle of curvature for the point ξ.

Since $\operatorname{Im} r_0 = \operatorname{Im} \rho(0) = 0$, for Ψ it is possible to take

$$\begin{gathered} \Psi(M,P,M_0) = -A \operatorname{Im}(\eta a \varphi) + A \operatorname{Re} F\left(\frac{\eta a}{r}\right) - \operatorname{Im} \kappa R, \\ F(t) = t \ln\left(t + \sqrt{t^2 - 1}\right) - \sqrt{t^2 - 1}, \quad R = |PM|. \end{gathered} \tag{3.17}$$

Formula (3.17) is obtained by considering the form of the asymptotics of the Hankel functions contained in (3.16).

We remark that $\Psi(M, P, M_0)$ has the same form as the r.v.f. estimated in [2]. By choosing the same deformation of the contour for n and leaving the contour of integration on S unchanged, we find similarly

$$\Psi < \Psi_0 - \text{const}\ \kappa^{\varepsilon} \tag{3.18}$$

The estimate (3.18) shows that an estimate holds for $K_1(M, M_0)$ which is exponentially small as compared with (3.8).

We shall estimate $K_2^{\pm}(M, M_0)$. The estimates are similar for the signs $\pm$, and we therefore omit these indices.

From (3.12) we obtain

$$\operatorname{supp} Q_2(P, M_0) = \left\{P(n,s) \,\middle|\, |s| \geqslant s_2,\ n = O\left(\kappa^{-2/3+\varepsilon}\right)\right\} \tag{3.19}$$

i.e., Q_2 is nonzero only in the zone of deep shadow and there $\tilde{G}(P, M)$ can be expressed in terms of the creeping waves $G_2^{\pm}$; therefore, for the region of integration (3.19) we can take the following r.v.f:

$$\Psi(M,P,M_0) = -\operatorname{Im} \kappa R + \Psi_0(P,M_0) = -\operatorname{Im} \kappa R + \Psi_0(P,\xi) + \frac{2}{3} \operatorname{Re}(z_1 - \nu_{M_0})^{3/2}. \tag{3.20}$$

It is evident that Ψ differs from the r.v.f. used in [2] in estimating $K(M, \xi)$ only by the constant term $\frac{2}{3} \operatorname{Re}(z_1 - \nu_{M_0})^{3/2}$. Therefore, by using the same contours of integration, we obtain an estimate for $K(M, M_0)$ which differs from the estimate for $K(M, \xi)$ by the factor $\exp\left\{\frac{2}{3} \operatorname{Re}(z_1 - \nu_{M_0})^{3/2}\right\}$, i.e., the estimate (3.8) which was required. From this we immediately obtain (1.17).

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SPACE–TIME RAY METHOD FOR WAVES OF SMALL DEFORMATION IN A NONLINEAR ELASTIC MEDIUM

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Space–time ray solutions for longitudinal waves of small deformation in a nonlinear elastic medium are constructed.

We consider the system of equations of a nonlinear elastic medium in a Lagrangian coordinate system

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_k}\left(\frac{\partial A}{\partial u_{i,k}}\right), \tag{1}$$

where x_i are Cartesian coordinates, t is time, u_i is the displacement vector, $u_{i,k} = \frac{\partial u_i}{\partial x_k}$, $\rho_0 = \rho_0(x_k)$ is the initial density of the medium, $A = A(I_1, I_2, I_3), x_k)$ is the free-energy density of the medium (ρ_0 and A are smooth functions of their arguments), I_1, I_2, I_3 are the invariants of the deformation tensor

$$D_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i} + u_{s,i} \cdot u_{s,k}), \tag{2}$$

$$I_1 = \mathrm{tr}\, D, \quad I_2 = \frac{1}{2}[(\mathrm{tr}\, D)^2 - \mathrm{tr}\, D^2], \quad I_3 = \det D. \tag{3}$$

Equations (1) are the Euler equations for the Lagrangian equal to the difference of the density of kinetic and free-energy of the medium:

$$\mathcal{L} = \frac{1}{2}\rho_0\left(\frac{\partial u_i}{\partial t}\right)^2 - A(I_1, I_2, I_3). \tag{4}$$

Before proceeding to the construction of space–time ray (STR) solutions, we must clarify the question of scales. In describing wave processes in a nonlinear elastic medium there are three characteristic scales of length l, λ, L , where l is the magnitude of displacement of particles of the medium, λ is the wavelength, and L is a characteristic parameter of variation of properties of the medium. As is known, for ray solutions it is necessary that the relation $\lambda \ll L$ be satisfied and that $\varepsilon = \frac{\lambda}{L}$ be a small parameter of the problem. Moreover, for a nonlinear elastic medium, in contrast to a linear medium, there is still another characteristic dimensionless parameter $\mu = l/\lambda = O(1)$. This parameter characterizes the magnitude of the deformation tensor. The study of the short-wave wave fields for the case $\mu \ll 1$ (waves of small deformation) and $\mu \sim 1$ (waves of large deformation) is completely different. The case of waves of large deformation present greater difficulties for investigation.

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