INVERSE PROBLEM OF QUANTUM SCATTERING THEORY. II.

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INTRODUCTION

The current survey summarizes the comparatively quiescent development of what is called the inverse problem of quantum scattering theory over the past 15 years. The preceding decade, during which this problem was formulated and subsequently intensively developed, was dealt with in my survey [25]. Its entire 25-yr history demonstrates that it is one of the most intriguing and instructive branches of mathematical physics and reveals in its development new and unexpected aspects, and that it is far from being exhausted.

There exists somewhat self-consistent methods for presenting the formalism of the inverse problem. An elementary approach is based on the study of the properties of solutions of differential and integral equations characteristic for it by methods of classical analysis. The monograph of Z. S. Agranovich and V. A. Marchenko [1] is an example of this presentation. In the current survey, as in [25], we follow a different approach, using wherever possible an operator-theoretic approach. The origin of this method of describing the inverse problem was set forth by Kay and Moses [38, 39]. In this approach the inverse problem of scattering theory does not appear isolated, but finds a natural place within the framework of general scattering theory.

Let us recall the general statements of scattering theory for the Schröedinger operator, with which we will deal henceforth. It is a matter of comparing the spectral properties of two operators H and H_0 defined in the Hilbert space $\mathfrak{H}=L_2(\mathbb{R}^n)$ by the formal differential equations

$$H=-\Delta+v(x);\quad H_0=-\Delta.$$

Here Δ is the Laplace operator

$$\Delta=\frac{\partial^2}{\partial x_1^2}+\ldots+\frac{\partial^2}{\partial x_n^2}$$

and v(x) is a real function sufficiently continuous and rapidly decreasing as $|x|\to\infty$. The operators H and H_0 defined in $\mathfrak{H}$ on the dense domain $\mathfrak{D}=W_2^2(\mathbb{R}^n)$, define self-adjoint operators, which we will denote by these letters. The operator H_0 has absolutely continuous spectrum. Its diagonal representation is realized by means of the Fourier transformation

$$\psi(x)\to T_0\psi=\varphi(k),\quad H_0\psi(x)\to k^2\varphi(k).$$

Here

$$\varphi(k)=\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\int e^{i(k,x)}\psi(x)\,dx.$$

The assertion that the operator H has the same absolutely continuous spectrum as H_0 is the fundamental result of scattering theory. More precisely, there exists an invariant decomposition relative to H of the space $\mathfrak{H}$ in the direct orthogonal sum

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$$\mathfrak{H}=\mathfrak{H}_d+\mathfrak{H}_{a.c.}$$

of natural subspaces corresponding to the discontinuous and absolutely continuous spectrum of this operator. Here the restriction of H to $\mathfrak{H}_{a.c.}$ is unitarily equivalent to H_0.

There exist among the operators isometric in $\mathfrak{H}$ that realize this equivalence two operators $U^{(\pm)}$ distinguished in terms of their physical origin. They are called wave operators and are defined by

$$U^{(\pm)}=\lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t},$$

where the limit is understood in the sense of a strong operator topology. There exists a broad literature that deals with the conditions on v(x) under which these limits exist. Our problem does not include the presentation of this "direct" problem of scattering theory, although several results relative to the existence of operators $U^{(\pm)}$ will also be mentioned in the text. Details on this question may be found, for example, in the monograph of Kato [36]. We note that the fact that the operator $V = H - H_0$ is a function-multiplication operator plays no role in general discussions in scattering theory.

The operators $U^{(\pm)}$ are isometric:

$$U^{(\pm)*}U^{(\pm)}=I; \quad U^{(\pm)}U^{(\pm)*}=I-P.$$

Here P is a projector on the subspace $\mathfrak{H}_d$, which, as a rule, is finite-dimensional. The second relation is said to be a completeness condition. The unitary equivalence spoken of above is realized by the equation

$$HU^{(\pm)}=U^{(\pm)}H_0.$$

The physical meaning of wave operators is based on the following concepts. In quantum mechanics the operator

$$U(t)=e^{-iHt}$$

describes the evolution of a system, which in our case, consists of a particle in the field of a potential center. Over a long period of time a particle with positive energy exits far from the center, becoming sensitive to its influence, and as a result its development over the course of time as $|t|\to\infty$ is actually described by the operator

$$U_0(t)=e^{-iH_0t},$$

corresponding to free motion.

More precisely we may correlate to every single-parameter family of vectors $\psi_-(t)$, describing free motion (wave packet)

$$\psi_-(t)=e^{-iH_0t}\psi_-$$

a solution of the Schröedinger equation

$$\psi(t)=e^{-iHt}\psi$$

such that

$$\|\psi(t)-\psi_-(t)\|\to 0$$

as $t\to-\infty$. The precise equation defining such a solution and following from the existence of wave operators, has the form

$$\psi=U^{(-)}\psi_-.$$

Every solution of the Schröedinger equation $\psi(t)$ from the given class as $t \to \infty$ again reduces to a wave packet, in general differing from $\psi_-(t)$:

$$\| \psi(t) - \psi_+(t) \| \to 0$$

as $t \to \infty$, where

$$\psi_+(t) = e^{-iH_0 t}\psi_+; \quad \psi_+ = U^{(+)*}\psi.$$

The last equation is justified since the operators $U^{(+)}$ and $U^{(-)}$ have a common range of values.

The passage from the wave packet $\psi_-(t)$ describing the initial state of a particle, to the wave packet $\psi_+(t)$ describing its final state is also the process by which a particle is scattered by the center. All information on this process is contained in the operator S, which relates both wave packets by the formula

$$\psi_+ = S\psi_-.$$

Comparing the equations expressing ψ in terms of ψ_- and ψ_+ in terms of ψ, we see that

$$S = U^{(+)*}U^{(-)},$$

where it follows from the properties of $U^{(+)}$ and $U^{(-)}$ that S is unitary and commutes with H_0,

$$S^*S = SS^* = I; \quad [S, H_0] = 0.$$

These relations reflect conservation of particle and energy flux in the course of scattering.

The operator S is said to be the scattering operator. Its representation

$$\hat{S} = T_0 S T_0^*$$

in a diagonal realization of H_0 is defined by the equation

$$\hat{S}\varphi(k) = \varphi(k) - 2\pi i \int f(k, l)\, \delta(k^2 - l^2)\, \varphi(l)\, dl,$$

where δ, the integrand, explicitly takes into account the fact that S and H_0 commute. The function $f(k, l)$ defined for $|k| = |l|$ is called the scattering amplitude.

There exists an alternative approach to the scattering theory, called the stationary approach, based on the study of the asymptotics of the eigenfunctions of H as $|x| \to \infty$. The scattering amplitude $f(k, l)$ here explicitly occurs in the description of these asymptotics. Such an approach and its relation to the nonstationary approach will be illustrated in the text.

The problem of reconstructing the potential $v(x)$ relative to the scattering amplitude $f(k, l)$ is said to be the inverse problem of scattering theory. This problem is not defined if the perturbation $V = H - H_0$ is an arbitrary operator, since an entire set of operators V can easily be selected with respect to an arbitrary unitary operator of the form $\hat{S}$, such that the corresponding operator S is a scattering operator for the pair H_0 and $H_0 + V$. It becomes meaningful only under a further condition, to which V is a function-multiplication operator. Henceforth, this condition will be said to be the locality of the potential.

Over the past 50 years the inverse problem has been solved for the case most interesting for physical applications of a shperically symmetric potential, viz., when $n = 3$, and

$$v(x) = v(|x|) = v(r), \; 0 \leq r < \infty.$$

In this case the scattering amplitude $f(k, l)$ depends only on the lengths of the vector k and l, which are equal by the condition, and on the angle between them so that it is actually a function $f(|k|, \cos\theta)$ of two variables. The partial scattering amplitudes $f_l(|k|)$ arise in decomposing $f(|k|, \cos\theta)$ in Legendre polynomials

$$f(|k|, \cos\theta) = -\frac{1}{2\pi^2|k|} \sum (2l+1) f_l(|k|) P_l(\cos\theta)$$

so characteristic for spherically symmetric problems. The unitarity condition on an operator S can be explicitly borne in mind by setting

$$f_l(|k|) = \exp\{i\eta_l(|k|)\} \sin \eta_l(|k|),$$

where $\eta_l(|k|)$ is a real function, called the asymptotic phase because of its role in the stationary formulation of the scattering problem.

The fundamental result obtained by V. A. Marchenko [16] and M. G. Krein [12] states that the potential v(x) is reconstructed in one of the asymptotic phases $\eta_l(|k|)$, one such phase being an arbitrary real function satisfying the integrality condition

$$\eta_l(\infty) - \eta_l(0) = \pi m, \quad m \in \mathbf{Z}_+.$$

Here m positive numbers associated with the characteristics of a discontinuous spectrum of the radial Schröedinger operator

$$H_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + v(r),$$

whose reconstruction constitutes our problem, must be specified for a unique determination of the potential as must be the phase η_l.

It is precisely this result that was dealt with in the survey [25]. The subsequent development of the inverse problem, about which we shall speak in the current article, is associated with the Schröedinger operator in the general case without spherical symmetry type assumptions. Here, two cases are distinguished, theoretically differing in terms of technical difficulties: $n = 1$ and $n \geq 2$. In the first case, tools developed for the radial Schröedinger operator with $l = 0$ turned out to be applicable. The chief role is played by the existence of a fundamental system of solutions for the corresponding one-dimensional differential equation. In the multidimensional case, when we must deal with a partial differential equation in addition to ordinary differential equations, the concepts of a fundamental system vanishes. It may be that it is precisely this circumstance that constituted the hindrance that has extended the study of the multidimensional inverse problem over such a long period of time.

In spite of this circumstance, which constitutes a technical distinction, it turned out that the one-dimensional and multidimensional inverse problems are to some degree analogous in many ways. This analogy can be particularly seen in an operator-theoretic language, which we have chosen for our presentation for precisely this reason. It is of interest that all these assertions on analogy refer to the one-dimensional, but not radial Schröedinger operator. In this sense the one-dimensional case plays a fortunate role as an intermediate link between the radial Schröedinger operator and the multidimensional operator, being technically close to the former and conceptually anticipatory of the fundamental outlines of the latter.

We will now indicate the principal distinction between the inverse problem considered in this survey and the case of the radial Schrödinger operator. It consists in the overdeterminacy of this problem. In the radial case we must construct a function v(r) decreasing to infinity of a variable r that varies on the half-axis, in terms of a function $\eta_l(|k|)$ of a variable |k| also on the half-axis and also satisfying an asymptotic condition as $|k| \to \infty$. It is therefore not remarkable that the function $\eta_l(|k|)$ can be chosen arbitrarily. A similar simple calculation of parameters demonstrates that the scattering amplitude for more complex problems cannot be arbitrarily selected.

Let us first consider the Schröedinger operator for $n = 1$. An arbitrary unitary scattering amplitude $f(k, l)$, where $k, l \in \mathbf{R}^1$ can be parametricized by four real functions of the variable |k|, running through the half-axis. A symmetry condition, which follows from the realness of the potential and which will be presented in the text, decreases this number down to three. At the same time the potential v(x) can be considered only as two real functions defined on the half-axis. Indeterminacy is present since it is difficult to imagine the physical origin of the problem in which the nondegenerate correspondence of sets of two and three arbitrary functions would be established. In other words these heuristic arguments demonstrate that the scattering amplitude $f(k, l)$ will satisfy the necessary condition and so can be expressed in terms of two real functions of the half-axis. Such a condition in fact arises and is derived in the text.

When $n \geq 2$ this indeterminacy is significantly aggravated. The scattering amplitude here is a function of $2n-1$ variables, while the property of being unitary reduces to this function being real. At the same time the potential is a real function of n variables. As a result of such indeterminacy the problem of determining necessary conditions on the scattering amplitude that follow from the localization condition on the potential, arises and becomes of great importance. It was previously unclear that such conditions can in general be expressed in terms of the scattering amplitude in sufficiently explicit form. Nevertheless, as will be explained in the text, such conditions can be described.

We will now formulate the fundamental statements of the formalism of the inverse problem. We will assume for the sake of definiteness that the operator H has one simple eigenvalue, so that the projector P under an isometricity condition is a projector on the one-dimensional subspace the corresponding eigenvector u spans. The choice of the transformation operator U, i.e., the choice of the solution of the equation

$$HU = UH_0,$$

which differs from wave operators, is the basis of this approach. Every transformation operator is obtained from the wave operators $U^{(\pm)}$ by multiplication on the right by a normalizing operator factor $N^{(\pm)}$ that commutes with H_0,

$$U = U^{(\pm)} N^{(\pm)}, \quad [N^{(\pm)}, H_0] = 0.$$

In particular the scattering operator S is a normalizing factor for $U^{(-)}$ with respect $U^{(+)}$,

$$U^{(-)} = U^{(+)} S.$$

Comparing these two formulas we can see that a factorization of the scattering operator

$$S = N^{(+)} N^{(-)^{-1}} \tag{1}$$

corrresponds to every choice of the transformation operator U.

We now assume that U is invertible in the sense that there exists a vector χ not belonging to the space $\mathfrak{H}$, such that

$$u = U\chi.$$

Then the completeness condition expressed in terms of U will be

$$UWU^* = I, \tag{2}$$

where

$$W = N^{(\pm)^{-1}} N^{(\pm)^{*-1}} + \chi \otimes \chi.$$

The operator W will be called a weight operator.

Equations (1) and (2) constitute the basis for solving the inverse problem. We must find a successful determination of U, such that Eq. (2) uniquely determines it in terms of the weight operator W and that the corresponding factors $N^{(\pm)}$ is uniquely determined by the factorization condition of Eq. (1) in terms of a given operator S. It turns out that similar transformation operators exist and are distinguished by a Volterra property.

Let us clarify in detail what we understand by a Volterra property. In the one-dimensional case this concept is formulated in the most classical fashion. The kernel A(x, y), where $x, y \in R^1$ is said to be triangular if $A(x, y) = 0$ when $x < y$ or $A(x, y) = 0$ when $x > y$. An integral operator with triangular kernel is said to be a triangular operator. Finally an operator of the form "identity element plus triangular operator" is said to be a Volterra operator. We have two possibilities for a Volterra operator in the one-dimensional case:

$$U_1\psi(x)=\psi(x)+\int_x^{\infty}A_1(x,y)\psi(y)\,dy;$$
$$U_2\psi(x)=\psi(x)+\int_{-\infty}^{x}A_2(x,y)\psi(y)\,dy.$$

Both formulas may be described uniquely if a direction γ is introduced, i.e., actually a variable taking two values $\gamma = \pm 1$

$$U_\gamma\psi(x)=\psi(x)+\int_{(y-x)\gamma>0}A_\gamma(x,y)\psi(y)\,dy.$$

In this form the calculation of the Volterra operator is naturally carried over to the case $n \geq 2$. The variable γ in this case is a unit vector and runs through, unlike the one-dimensional case, a connected set, namely the sphere S^{n-1}. An operator of the form

$$U_\gamma\psi(x)=\psi(x)+\int_{(y-x,\gamma)>0}A_\gamma(x,y)\psi(y)\,dy$$

is said to be a Volterra operator with direction γ of Volterra property.

We will now prove that the Volterra property of a transformation operator U_γ reduces the completeness equation (2) to a linear integral equation for the kernel $A_\gamma(x, y)$ occurring in its definition. Suppose γ is some direction and let U_γ be a Volterra transformation operator with direction γ of Volterra property. We consider the operator

$$U_\gamma^{*-1}=I+\widetilde{A}_\gamma.$$

This operator has direction of Volterra property opposite to that of the operator γ, so that

$$A_\gamma(x,y)=0,\ (x-y,\gamma)>0;\ \widetilde{A}_\gamma(x,y)=0,\ (x-y,\gamma)<0.$$

Suppose W_γ is a weight operator for U_γ, and we set

Equation (2) with the notation introduced, can be rewritten in the form

$$A_\gamma+\Omega_\gamma+A_\gamma\Omega_\gamma=\widetilde{A}_\gamma$$

or, in more detail, in terms of the kernels $A_\gamma(x, y)$, $\Omega_\gamma(x, y)$, and $\widetilde{A}_\gamma(x, y)$ in the form

$$A_\gamma(x,y)+\Omega_\gamma(x,y)+\int_{(z-x,\gamma)>0}A_\gamma(x,z)\Omega_\gamma(z,y)\,dz=\widetilde{A}_\gamma(x,y).$$

The right side here vanishes when $(y-x, \gamma) > 0$. If this condition holds, we obtain the linear integral equation

$$A_\gamma(x,y)+\Omega_\gamma(x,y)+\int_{(z-x,\gamma)>0}A_\gamma(x,z)\Omega_\gamma(z,y)\,dz=0$$
$$(y-x,\gamma)>0,$$

which can be used to find the kernel $A_\gamma(x, y)$ in terms of the known kernel $\Omega_\gamma(x, y)$. This equation constitutes a general formulation of the Gel'fand–Levitan equation introduced in [8] for the actual example of a Sturm–Liouville operator on the half-axis.

Thus we will see how to reconstruct the Volterra transformation operator U_γ if the corresponding weight operator W_γ is known. To construct the operator W_γ in terms of a known operator S it is necessary to solve one more problem in the factorization of Eq. (1) to determine the normalizing factors of U_γ.

It is still not entirely understood whether this problem for Volterra U_γ also reduces to a linear equation nor whether it is even solved explicitly in the one-dimensional case. It turns out that normalizing factors for Volterra transformation operators themselves turn out in some sense to be Volterra. We will not clarify this circumstance in more detail here and refer the reader to precise formulations in the text.

Thus, the procedure for solving the inverse problem solving for given scattering amplitude a set of factorization problems (1) to determine the normalizing factors $N_\gamma^{(\pm)}$. A weight operator W_γ is constructed in terms of these data and characteristics of the discontinuous spectrum, if such exists, and then using the Gel'fand–Levitan equation, the transformation operators U_γ are reconstructed. All the stages, in general, can be conceptualized for an arbitrary initial operator S. Moreover if S is unitary, each of the operators

$$H_\gamma = U_\gamma H_0 U_\gamma^{-1}$$

will be self-adjoint. Additional necessary conditions, about which we spoke above, begin to play a role at the next stage, when it is clarified that the operators H_γ in fact are independent of γ. This important statement simultaneously serves for investigating the properties for the reconstructed operator H and for proving that the initial operator S is in fact the scattering operator for the pair H and H_0. The tools for proving the independence of H_γ from γ differ for n = 1 and n ≥ 2, because of the difference between the range of values of the variable γ. When n ≥ 2, i.e., when this set is connected, we can use differentiation with respect to the parameter γ. In the one-dimensional case it is necessary to use more artificial means.

We conclude this description of the tools for solving the inverse problem, since further detail requires a more formal presentation, which will be presented in the text. We note only that, in our opinion, abstract scattering theory can be further developed so that Volterra transformation operators and the existence of separated factorizations of the form (1) of the scattering operator find a natural place within its framework. Apparently the formulation of scattering theory due to Lax and Phillips [45] is the most successful starting point for such a generalization, and a given causality condition will in a reasonable way be the appropriate language.

Let us now indicate on the structure of the survey. Differences in the technique and elaboration of the cases n = 1 and n ≥ 2 forced us to treat them separately. We will finally emphasize the analogies between the corresponding discussions and equations wherever possible.

The one-dimensional case is discussed in Chap. 1. A significant technical simplification for studying the one-dimensional Schröedinger operator lies in the existence of selected fundamental systems of solutions of the correspondinding differential equation. All operator equations are suitably introduced and justified proceeding on the basis of the well-known properties of these solutions. The description of these properties is discussed in Sec. 1, which plays an auxiliary role. In Sec. 2 the fundamental statements of scattering theory for a given concrete example are formulated and proved. Volterra transformation operators are introduced in Sec. 3 and the normalizing factors corresponding to them are obtained in Sec. 4. Gel'fand–Levitan-type equations are formulated in the latter section. Section 5 treats the solvability of these equation. A relation is analyzed there between transformation operators for $\gamma = 1$ and $\gamma = -1$. The general investigation of the inverse problem concludes here. The last Sec. 6 contains a description of an explicit solution of the Gel'fand–Levitan equation for the particular case when the scattering amplitude is a rational function of a parameter k.

Chapter 2 also treats one-dimensional problems. Here a generalization of the formalism developed in Chap. 1 to the case of potentials v(x) having nonzero asymptotic as $x \to -\infty$ (Sec. 1) or to the case of an operator of the form

$$H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

which is a direct generalization of the Schröedinger operator (Sec. 2), is analyzed at an elementary level. In the last section we will describe so-called trace identities, which relate certain functionals of the potential and scattering amplitude. These identities, through not a means for the direct solution of the inverse problem, can indirectly lead to information on the potential according to known properties of the scattering amplitude, and conversely. Section 4 describes an application of the inverse problem of scattering theory to the solution of one-dimensional nonlinear evolutionary equations. The starting point of this application was set forth in the important work of Kruskal et al. [42]. Then P. Lax [44], V. E. Zakharov and A. B. Shabat [11], V. E. Zakharov and the author [10], and others further developed this subject. The inverse problem method of scattering theory for solving nonlinear equations currently draws ever greater attention and is rapidly developing. The region of its applicability is still far from clarified and we will consider in the present survey only two characteristic examples.

In Chap. 3 we return to our fundamental theme and consider the inverse problem for the multidimensional Schröedinger operator. For the sake of definiteness we will deal with the physically interesting case $n = 3$, although all the discussions are trivially carried over to arbitrary $n \geq 2$. In the multidimensional case no fundamental system of solutions nor proof procedure is available due to the far greater cumbersomeness. The scope of the present survey does not allow us to present somewhat instructive estimates needed to make all the constructions of Chap. 3 rigorous. We will therefore limit ourselves to a presentation of only the formal scheme of these constructions. We leave it to the reader to complete the algebraic framework of this scheme by appropriate analytic arguments.

In Sec. 1 we set forth the fundamentals of scattering theory for the three-dimensional Schröedinger operator. General concepts that are of assistance in research on Volterra transformation operators U_γ are described in Sec. 2. The construction of normed factors and the weight operator W_γ are discussed in Secs. 3 and 4. The description of the Gel'fand–Levitan operator and a scheme for studying the inverse problem are presented in Sec. 5, with which Chap. 3 concludes.

No special knowledge is required to read this survey. In particular, it can be read independently of the preceding survey [25], since all the necessary information on the Schrödinger operator are enumerated here once again. We hope that for some mathematicians this survey can serve as an introduction to scattering theory, a branch of functional analysis and mathematical physics which is constantly expanding the domain of its applications.

In concluding this introduction we note trends associated with the inverse problem of scattering theory that are not indicated in this survey. These include: 1. works of B. M. Levitan and M. G. Gasymov, M. G. Krein, and his students on canonical systems and Dirac-type systems on the semi-axis. These works in terms of the formulation of the problem and methods relate to the problems associated with the radial Schröedinger operator. We refer the reader to new studies [7, 13] for references to the literature given there.

2. Works on inverse problems in terms of scattering data for fixed energy. By this problem is understood the reconstruction of the potential v(r) in terms of a known set of asymptotic phases $\eta_l(|k|)$ for all $l = 0, 1, 2, \ldots$ and fixed $|k|$. An operator-theoretic formulation of this problem is not at all evident and the results obtained have yet to reach, in terms of elegance and completeness, the level attained in the spectral formulation of the inverse problem. The most detailed presentation of well-known facts on this problem can be found in Loeffel [47].

3. Works of V. A. Marchenko and his students on the stability of the inverse problem, primarily for the example of the radial Schröedinger equation. The recent monograph of V. A. Marchenko [17] discusses this subject.

We will use no unusual notation. Constants appearing in the limits are denoted by C. An explicit dependence of these constants on parameters is indicated only if this is important. In numbered equations the first digit indicates the number of the section and the second, the number of the equation. The numbering of the sections begins anew within each chapter. A reference to an equation of a different chapter will use a number made of three digits of the type (II.3.14), whose meaning is self-evident.

CHAPTER 1

ONE-DIMENSIONAL SCHRÖEDINGER OPERATOR

In this chapter we will consider the Schröedinger operator

$$H = -\frac{d^2}{dx} + v(x),$$

where the potential v(x) is assumed to be a real measureable function satisfying the condition

$$\int_{-\infty}^{\infty} (1+|x|)\,|v(x)|\,dx < \infty. \qquad \text{(P)}$$

Here the operator H defined on the dense set $\mathfrak{D}=W_2^2(\mathbf{R})$ in the Hilbert space $\mathfrak{H}=L_2(\mathbf{R})$ is a self-adjoint operator. We will introduce and characterize the scattering data corresponding to this operator and describe the procedure for solving the inverse problem for reconstructing the potential v(x) in terms of these data.

1. Fundamental System of Solutions of Schröedinger Equation

This section is auxiliary. Here we will describe two fundamental systems of solutions of the Schröedinger equation

$$H\psi=-\frac{d^2}{dx^2}\psi(x)+v(x)\psi(x)=k^2\psi(x). \tag{1.1}$$

Henceforth k, as a rule, will be a real number, but sometimes we will assume it to be a complex number, particularly when specified.

Condition (P) means that v(x) effectively vanishes as $|x|\to\infty$, so that we may naturally assume that every solution of Eq. (1.1) coincides at infinity with some solution of the equation

$$H_0\psi=-\frac{d^2}{dx^2}\psi(x)=k^2\psi(x),$$

i.e., a linear combination of exponents

$$f_0(x,\ k)=e^{ikx};\ f_0(x,\ -k)=e^{-ikx}.$$

More rigorously, we may prove that there exist solutions $f_1(x, k)$ and $f_2(x, k)$ of Eq. (1.1) which have the asymptotic

$$f_1(x,k)=f_0(x,k)+o(1),\quad x\to\infty, \tag{1.2}$$

$$f_2(x,k)=f_0(x,-k)+o(1),\quad x\to-\infty. \tag{1.3}$$

The proof is based on the fact that the differential equation (1.1) with the boundary conditions (1.2) and (1.3) is equivalent to the equations

$$f_1(x,k)=e^{ikx}+\int_{-\infty}^{\infty}G_1(x-y,k)v(y)f_1(y,k)\,dy; \tag{1.4}$$

$$f_2(x,k)=e^{-ikx}+\int_{-\infty}^{\infty}G_2(x-y,k)v(y)f_2(y,k)\,dy; \tag{1.5}$$

where

$$G_1(x,k)=-\theta(-x)\frac{\sin kx}{k};\qquad G_2(x,k)=\theta(x)\frac{\sin kx}{k}$$

and $\theta(x)$ is the Heaviside function

$$\theta(x)=1,\ x>0;\ \ \theta(x)=0,\ x<0.$$

These equations are Volterra-type integral equations, so that the method of successive approximations always converges for them. Here the parameter k can have complex values from the upper half-plane. As a result of analyzing the successive approximations we will prove that the solutions $f_1(x, k)$ and $f_2(x, k)$ exist and for fixed x are analytic functions of k when Im k> 0 and are continuous when Im k = 0. Here we have the bounds for them

$$|f_1(x,k)-e^{ikx}|\leqslant C\frac{e^{-\mathrm{Im}\,kx}}{1+|k|}\int_x^{\infty}(1+|y|)|v(y)|\,dy; \tag{1.6}$$

$$|f_2(x, k) - e^{-ikx}| \leqslant C\frac{e^{\operatorname{Im} k x}}{1+k}\int\limits_{-\infty}^{x}(1+|y|)|v(y)|dy. \tag{1.7}$$

Such assertions were first obtained by Levinson [46].

It follows from the bound of Eq. (1.6) on the basis of the Jordan lemma that we have for the solution $f_1(x, k)$ the integral representation

$$f_1(x, k) = e^{ikx} + \int\limits_{x}^{\infty} A_1(x, y)\, e^{iky} dy, \tag{1.8}$$

where the kernel $A_1(x, y)$ is quadratically integrable with respect to y for any fixed x. Similarly $f_2(x, k)$ can be represented in the form

$$f_2(x, k) = e^{-ikx} + \int\limits_{-\infty}^{x} A_2(x, y)\, e^{-iky} dy, \tag{1.9}$$

where the kernel $A_2(x, y)$ also is quadratically integrable with respect to y. Such integral representations for solving the Schröedinger equation were introduced by B. Ya. Levin [15].

The detailed properties of the kernels $A_i(x, y)$ can be obtained on the basis of an investigation into integral equations equivalent to the corresponding equations (1.4) and (1.5) for the solutions $f_1(x, k)$ and $f_2(x, k)$

$$A_1(x, y) = \frac{1}{2}\int\limits_{\frac{x+y}{2}}^{\infty} v(z)\, dz - \int\limits_{\frac{x+y}{2}}^{\infty} dt \int\limits_{0}^{\frac{y-x}{2}} dz\, v(t-z)\, A_1(t-z, t+z); \tag{1.10}$$

$$A_2(x, y) = \frac{1}{2}\int\limits_{-\infty}^{\frac{x+y}{2}} v(z)\, dz - \int\limits_{-\infty}^{\frac{x+y}{2}} dt \int\limits_{\frac{y-x}{2}}^{0} dz\, v(t-z)\, A_2(t-z, t+z). \tag{1.11}$$

Such equations were first derived by Z. S. Agranovich and V. A. Marchenko [1]. Agranovich and Marchenko proved the convergence of the method of successive approximations for these equations and obtained bounds on the solutions. To write the bounds it is suitable to introduce the monotone functions

$$\xi_1(x) = \int\limits_{x}^{\infty} |v(y)| dy; \qquad \xi_2(x) = \int\limits_{-\infty}^{x} |v(y)| dy.$$

Bounds on the kernels $A_1(x, y)$ and $A_2(x, y)$ have the form

$$|A(x, y)| \leqslant C\xi_1\left(\frac{x+y}{2}\right); \qquad |A_2(x, y)| \leqslant C\xi_2\left(\frac{x+y}{2}\right). \tag{1.12}$$

Further it is possible to prove using these equations the existence of the first derivatives of $A_1(x, y)$ and $A_2(x, y)$ and to obtain bounds on them. For example,

$$\left|\frac{\partial}{\partial x} A_1(x, y) + \frac{1}{4} v\left(\frac{x+y}{2}\right)\right| \leqslant C\xi_1(x)\, \xi_1\left(\frac{x+y}{2}\right),$$
$$\left|\frac{\partial}{\partial x} A_2(x, y) - \frac{1}{4} v\left(\frac{x+y}{2}\right)\right| \leqslant C\xi_2(x)\, \xi_2\left(\frac{x+y}{2}\right)$$

and similar bounds hold for $\frac{\partial}{\partial y} A_1(x, y)$ and $\frac{\partial}{\partial y} A_2(x, y)$.

Finally, it is evident from the equations that

$$A_1(x, x) = \frac{1}{2}\int\limits_{x}^{\infty} v(y)\, dy; \qquad A_2(x, x) = \frac{1}{2}\int\limits_{-\infty}^{x} v(y)\, dy,$$

so that

$$-2\frac{d}{dx}A_1(x,x)=v(x)=2\frac{d}{dx}A_2(x,x).$$

The pairs $f_1(x,k)$, $f_1(x,-k)=\overline{f_1(x,k)}$ and $f_2(x,k)$, $f_2(x,-k)=\overline{f_2(x,k)}$ for real $k \neq 0$ are fundamental systems of solutions of the fundamental equation (1.1). In fact, since the Wronskian $\{f_1, \bar{f}_1\}=f_1'\bar{f}_1-f_1\bar{f}_1'$ is independent of x, it coincides with its values as $x\to\infty$, which may be calculated using the asymptotic for the solution $f_1(x, k)$ and its derivative. It can be proved that as $x \to \infty$

$$f_1'(x,k)=ike^{ikx}+o(1),$$

so that

$$\begin{aligned}\{f_1(x,k),\ f_1(x,-k)\}&=\lim_{x\to\infty}\{f_1(x,k),\ f_1(x,-k)\}=\\&=ike^{ikx}e^{-ikx}-e^{ikx}(-ik)e^{-ikx}=2ik.\end{aligned} \tag{1.13}$$

We will see that when $k \neq 0$, the Wronskian is nonzero and the solutions $f_1(x, k)$ and $f_1(x, -k)$ are linearly independent. Similarly

$$\{f_2(x,k),\ f_2(x,-k)\}=-2ik, \tag{1.14}$$

so that $f_2(x, k)$ and $f_2(x, -k)$ are also linearly independent when $k \neq 0$.

Any solution of Eq. (1.1) can be represented in the form of a linear combination of the solutions $f_1(x, k)$ and $f_1(x, -k)$ or $f_2(x, k)$ and $f_2(x, -k)$. In particular, we have

$$f_2(x,k)=f_1(x,k)c_{11}(k)+f_1(x,-k)c_{12}(k), \tag{1.15}$$
$$f_1(x,k)=f_2(x,k)c_{22}(k)+f_2(x,-k)c_{21}(k). \tag{1.16}$$

Substituting Eq. (1.15) for $f_2(x, k)$ in Eq. (1.16) and performing the same operation with $f_1(x, k)$ we find that the following equations must hold in Eqs. (1.15) and (1.16) are to be consistent:

$$\begin{aligned}&c_{11}(k)c_{22}(k)+c_{12}(-k)c_{21}(k)=c_{22}(k)c_{11}(k)+c_{21}(-k)c_{12}(k)=1,\\&c_{12}(k)c_{22}(k)+c_{11}(-k)c_{21}(k)=c_{21}(k)c_{11}(k)+c_{22}(-k)c_{12}(k)=0.\end{aligned} \tag{1.17}$$

We may express the coefficients $c_{ij}(k)$, i, j = 1, 2 in terms of the Wronskians of the solutions $f_1(x, k)$ and $f_2(x, k)$. In view of Eqs. (1.13) and (1.14) and also in view of the self-evident equations

$$\{f_1(x,k),\ f_1(x,k)\}=\{f_2(x,k),\ f_2(x,k)\}=0,$$

we find

$$c_{12}(k)=c_{21}(k)=\frac{1}{2ik}\{f_1(x,k),\ f_2(x,k)\}; \tag{1.18}$$
$$c_{11}(k)=\frac{1}{2ik}\{f_2(x,k),\ f_1(x,-k)\}; \tag{1.19}$$
$$c_{22}(k)=\frac{1}{2ik}\{f_2(x,-k),\ f_1(x,k)\}. \tag{1.20}$$

Comparing Eqs. (1.19) and (1.20) we find that

$$c_{11}(k)=-c_{22}(-k),$$

which, incidentally also follows from Eq. (1.17) since $c_{12}(k) = c_{21}(k)$. These equations imply also that

$$|c_{12}(k)|^2=1+|c_{11}(k)|^2=1+|c_{22}(k)|^2.$$

We will see that the four coefficients $c_{ij}(k)$ are in fact expressed in terms of two complex-valued functions

$$a(k)=c_{12}(k);\ b(k)=c_{11}(k),$$

satisfying the condition

$$|a(k)|^2=1+|b(k)|^2. \tag{1.21}$$

Here

$$c_{21}(k)=a(k);\ c_{22}(k)=-b(-k).$$

We will henceforth refer to these functions as conversion factors.

To derive further properties of the functions $a(k)$ and $b(k)$ we express them in terms of the kernel $A_2(x, y)$. For this purpose we note that Eq. (1.5) implies that $f_2(x, k)$ as $x\to\infty$ has the asymptotic

$$f_2(x,\ k)=e^{-ikx}+\frac{1}{2ik}e^{ikx}\int_{-\infty}^{\infty}e^{-iky}v(y)f_2(y,\ k)\,dy-$$
$$-\frac{1}{2ik}e^{-ikx}\int_{-\infty}^{\infty}e^{iky}v(y)f_2(y,\ k)\,dy+o(1).$$

Comparing this equation with the equation

$$f_2(x,\ k)=e^{ikx}b(k)+e^{-ikx}a(k)+o(1),$$

which follows from Eq. (1.15) if we take into account definition (1.2) of the solutions $f_1(x, k)$ we obtain for $a(k)$ and $b(k)$ the equations

$$a(k)=1-\frac{1}{2ik}\int_{-\infty}^{\infty}e^{ikx}v(x)f_2(x,\ k)\,dx, \tag{1.22}$$

$$b(k)=\frac{1}{2ik}\int_{-\infty}^{\infty}e^{-ikx}v(x)f_2(x,\ k)\,dx. \tag{1.23}$$

We now replace $f_2(x, k)$ by the kernel $A_2(x, y)$ using Eq. (1.9). We find

$$a(k)=1-\frac{1}{2ik}\int_{-\infty}^{\infty}v(x)\,dx-\frac{1}{2ik}\int_{0}^{\infty}\Pi_2(x)e^{2ikx}dx,$$

where

$$\Pi_2(x)=2\int_{-\infty}^{\infty}v(y)A_2(y,\ y-2x)\,dy$$

and

$$b(k)=\frac{1}{2ik}\int_{-\infty}^{\infty}\Pi_1(x)e^{-2ikx}dx,$$

where

$$\Pi_1(x)=v(x)+2\int_{x}^{\infty}v(y)A_2(y,\ 2x-y)\,dy.$$

Bounds on $\Pi_1(x)$ and $\Pi_2(x)$ follow from the bounds on $A_2(x, y)$:

$$|\Pi_2(x)| \leqslant C\left(\xi_2\left(-\frac{x}{2}\right) + \xi_1\left(\frac{x}{2}\right)\right),$$
$$|\Pi_1(x)| \leqslant |v(x)| + C\xi_1(x)\,\xi_2(x),$$

which imply that the function $\Pi_2(x)$ is absolutely integrable on the semi-axis $0 \leqslant x < \infty$, while $\Pi_1(x)$ is absolutely integrable on the entire axis. Thus, we have found for the conversion factors $a(k)$ and $b(k)$ an expression in the form of a Fourier transform of absolutely integrable functions. In particular, it follows from the obtained representations that for large k we have for these coefficients the asymptotic

$$b(k) = o\left(\frac{1}{|k|}\right);$$
$$a(k) = 1 + \frac{q}{2ik} + o\left(\frac{1}{|k|}\right); \quad q = \int_{-\infty}^{\infty} v(x)\,dx. \tag{1.24}$$

Moreover, we see that $a(k)$ is the limiting value on the real axis of a function analytic and bounded in the half-plane $\operatorname{Im} k > 0$ and that the asymptotic of Eqs. (1.24) holds for all k with $\operatorname{Im} k \geq 0$.

We will consider the decomposition of the zeroes of $a(k)$ on the complex plane. Because of Eqs. (1.21), $a(k)$ does not vanish on the real axis. Further, it follows from the asymptotic of Eq. (1.24) that $a(k)$ is also nonzero for sufficiently large $|k|$. It therefore follows that $a(k)$ can have only a finite number of zeroes. It follows from the representation of Eq. (1.18) for $a(k)$ by means of the Wronskian of the solutions $f_1(x, k)$ and $f_2(x, k)$ that if $a(k_0) = 0$, these solutions are linearly dependent for $k = k_0$, i.e.,

$$f_1(x, k_0) = cf_2(x, k_0). \tag{1.25}$$

We note that when $\operatorname{Im} k > 0$ the solution $f_1(x, k)$ exponentially decreases as $x \to \infty$ while the solution $f_2(x, k)$ behaves likewise as $x \to \infty$. When $k = k_0$ we may conclude on the basis of Eq. (1.25) that Eq. (1.1) has a solution that is quadratically integrable on the entire axis. The formal self-conjugacy of the equation implies that this is possible only for real k_0^2, i.e., for purely imaginary k_0.

We have thus found that $a(k)$ can have only a finite number of purely imaginary zeroes. We will prove that these zeroes are simple. For this purpose we obtain an expression for $\dot{a}(k_0) = (d/dk)a(k)|_{k=k_0}$. We will proceed on the basis of Eq. (1.1) for $f_1(x, k)$ and $f_2(x, k)$ and the equation

$$\frac{d^2}{dx^2}\dot{\psi} + k^2\dot{\psi} = v(x)\,\dot{\psi} - 2k\psi$$

for $\dot{f}_1(x, k)$ and $\dot{f}_2(x, k)$. We obtain by the standard method the identities

$$\left.\begin{aligned} \{f_1(x, k), \dot{f}_2(x, k)\}\Big|_{-A}^{x} &= 2k \int_{-A}^{x} f_1(x, k)\,f_2(x, k)\,dx, \\ \{\dot{f}_1(x, k), f_2(x, k)\}\Big|_{x}^{A} &= -2k \int_{x}^{A} f_1(x, k)\,f_2(x, k)\,dx. \end{aligned}\right\} \tag{1.26}$$

On the other hand, using Eq. (1.18) we find

$$\begin{aligned} \frac{d}{dk}(2ika(k)) &= 2ia(k) + 2ik\dot{a}(k) = \\ &= \{\dot{f}_1(x, k), f_2(x, k)\} + \{f_1(x, k), \dot{f}_2(x, k)\}. \end{aligned} \tag{1.27}$$

Suppose now that k coincides with one of the zeroes of $a(k)$, which we again denote by k_0. When $k = k_0$ the Wronskians in Eqs. (1.26) taken for $x = \pm A$ vanish and the integrals in the right sides of these equations converge in limit as $A \to \infty$. Comparing Eqs. (1.26) and (1.27) and recalling that $a(k_0) = 0$, we find

$$i\dot{a}(k_0) = \int_{-\infty}^{\infty} f_1(x, k_0)\,f_2(x, k_0)\,dx. \tag{1.28}$$

The solutions $f_1(x, k)$ and $f_2(x, k)$ for imaginary k are real. The integral in the right side of Eq. (1.28) does not vanish because of Eq. (1.25), so that $\dot{a}(k_0) \neq 0$ and, consequently the zeroes of $a(k)$ are simple. These zeroes will henceforth be denoted by $i\varkappa_l$, where $l = 1, \ldots, N$. With this we conclude the study of the properties of the conversion factors $a(k)$ and b(k).

In concluding this section we present an expression for Green's function of Eq. (1.1). Suppose λ is a complex parameter and let us select a branch for $\sqrt{\lambda}$, such that $\operatorname{Im}\sqrt{\lambda} \geqslant 0$. The kernel

$$R(x, y; \lambda) = -\frac{1}{2i\sqrt{\lambda}\,a(\sqrt{\lambda})} f_1(x, \sqrt{\lambda}) f_2(y, \sqrt{\lambda}), \quad y < x;$$
$$R(x, y; \lambda) = R(y, x; \lambda)$$

for fixed x and y is an analytic function of λ on the plane with section on the positive part of the real axis and with simple poles at the points $\lambda = -\varkappa_l^2$. If λ does not coincide with these points and if $\lambda \neq 0$, we have this kernel the bounds

$$|R(x, y; \lambda)| \leqslant Ce^{-\operatorname{Im}\sqrt{\lambda}\,|x-y|}.$$

Here $R(x, y; \lambda)$, as follows from Eq. (1.18), is a solution of the equation

$$-\frac{d^2}{dx^2} R(x, y; \lambda) + v(x) R(x, y; \lambda) - \lambda R(x, y; \lambda) = \delta(x-y).$$

We may prove using these facts that the integral operator $R(\lambda)$ with kernel $R(x, y; \lambda)$ is the resolvent $(H - \lambda I)^{-1}$ of the self-adjoint operator H. Moreover, we may use the properties of this kernel to define the operator H itself, which incidentally we have not done.

Let us now turn our attention to the fact that the function $a(\sqrt{\lambda})$ is in the denominator of the resolvent $R(x, y; \lambda)$ and has zeroes at the eigenvalues of H. In this it reminds us of the characteristic determinant $\det(H - \lambda I)$. We may verify that such an interpretation of it is in fact justified. For example, we have the equation

$$\frac{d}{d\lambda} \ln a(\sqrt{\lambda}) = -\operatorname{Tr}(R(\lambda) - R_0(\lambda)),$$

where $R_0(\lambda)$ is the resolvent of the operator H_0. Subtraction by $R_0(\lambda)$ plays the role of a required regularization for the definition of $\det(H - \lambda I)$.

2. Scattering Theory

Knowledge of the fundamental system of solutions for the Schröedinger equation (1.1) allows us to illustrate in a simple way, using the operator H as an example, the general statements of scattering theory described in the introduction. We will prove how the wave operators $U^{(\pm)}$ for the pair of operators H and H_0 defined in $\mathfrak{H} = L_2(\mathbf{R})$ by the equation

$$H = -\frac{d^2}{dx^2} + v(x); \quad H_0 = -\frac{d^2}{dx^2},$$

can be expressed in terms of appropriate solutions of the stationary Schröedinger equation (1.1). All the properties of the wave operators are subsequently obtained as simple corollaries of this relation.

We begin with a description of the diagonal representation for H_0. We consider the space $\mathfrak{H}_0$ consisting of pairs of functions

$$\varphi(\lambda) = \begin{pmatrix} \varphi_1(\lambda) \\ \varphi_2(\lambda) \end{pmatrix},$$

quadratically integrable on the semi-axis $0 \leqslant \lambda < \infty$ and having the scalar product

$$(\varphi, \varphi')_0 = \int_0^\infty \left(\varphi_1(\lambda)\overline{\varphi_1'(\lambda)} + \varphi_2(\lambda)\overline{\varphi_2'(\lambda)}\right)\frac{d\lambda}{2\sqrt{\lambda}}.$$

A diagonal representation for H_0 can be realized in $\mathfrak{H}_0$. The corresponding isomorphism $\mathfrak{H} \to \mathfrak{H}_0$ is provided by the Fourier transformation

$$\psi(x) \to T_0\psi = \varphi(\lambda),$$

where

$$\varphi_1(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)e^{-i\sqrt{\lambda}x}dx;\varphi_2(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)e^{i\sqrt{\lambda}x}dx.$$

The operator T_0 is unitary:

$$T_0^*T_0=I;\ TT_0^*=I_0.$$

The operator H_0 under the isomorphism T_0 is carried over into an operator for multiplication by the independent variable λ,

$$H_0\psi(x)\to\lambda\varphi(\lambda).$$

We now consider two sets of solutions of the Schröedinger equation:

$$u_1^{(+)}(x,k)=\frac{1}{a(k)}f_2(x,k);\ u_2^{(+)}(x,k)=\frac{1}{a(k)}f_1(x,k); \tag{2.1}$$

$$u_1^{(-)}(x,k)=\overline{u_2^{(+)}(x,k)};\ u_2^{(-)}(x,k)=\overline{u_1^{(+)}(x,k)}. \tag{2.2}$$

For the sake of definiteness, $k>0$ everywhere. The table of the asymptotics of these solutions as $|x|\to\infty$ has the form

$$\begin{aligned}u_1^{(+)}(x,k)&=s_{11}(k)e^{-ikx}+o(1)=e^{-ikx}+s_{12}(k)e^{ikx}+o(1),\\u_2^{(+)}(x,k)&=s_{21}(k)e^{-ikx}+e^{ikx}+o(1)=s_{22}(k)e^{ikx}+o(1)\\u_1^{(-)}(x,k)&=e^{-ikx}+\tilde{s}_{12}(k)e^{ikx}+o(1)=\tilde{s}_{11}(k)e^{-ikx}+o(1),\\u_2^{(-)}(x,k)&=\tilde{s}_{22}(k)e^{ikx}+o(1)=\tilde{s}_{21}(k)e^{-ikx}+e^{ikx}+o(1),\end{aligned}$$

The left column here being referred to $x\to-\infty$, and the right column to $x\to\infty$. The coefficients $s_{ij}(k)$ and $\tilde{s}_{ij}(k)$ occurring in this table are expressed in the following way in terms of the conversion factors $a(k)$ and $b(k)$:

$$s_{11}(k)=\frac{1}{a(k)};\ s_{12}(k)=\frac{b(k)}{a(k)};\ s_{21}(k)=-\frac{b(-k)}{a(k)};\ s_{22}(k)=\frac{1}{a(k)};$$
$$\tilde{s}_{11}(k)=\overline{s_{22}(k)};\ \tilde{s}_{12}(k)=\overline{s_{21}(k)};\ \tilde{s}_{21}(k)=\overline{s_{12}(k)};\ \tilde{s}_{22}(k)=\overline{s_{11}(k)}.$$

These properties follow from relationships of the form of Eqs. (1.15) and (1.16) and the asymptotics of Eqs. (1.2) and (1.3) for the solutions $f_1(x, k)$ and $f_2(x, k)$.

The functions $s_{11}(k)$ and $s_{12}(k)$ have meaning for all $k \neq 0$, since $a(k)$ does not vanish on the real axis. We will prove that if $|a(0)|=\infty$, the coefficients of s_{11} and s_{12} equally have meaning up to $k = 0$. In this case, evidently, $s_{11}(0) = 0$ and only $s_{12}(k)$ is to be considered. It is evident from Eq. (1.22) that $|a(k)|\to\infty$, if

$$\lim_{k\to 0}2ika(k)=\beta=-\int_{-\infty}^{\infty}v(x)f_2(x,0)\,dx\neq 0.$$

Here, as is evident from Eq. (1.23),

$$\lim_{k\to 0}s_{12}(k)=\lim_{k\to 0}\frac{b(k)}{a(k)}=\lim_{k\to 0}\frac{2ikb(k)}{2ika(k)}=\frac{-\beta}{\beta}=-1$$

so that $s_{12}(k)$ is defined by continuity up to $k = 0$ and $s_{12}(0) = -1$. We may similarly prove that in this case $s_{21}(0) = -1$ and $s_{22}(0) = 0$. For larger $|k|$,

$$s_{11}(k)=s_{22}(k)=1+O\left(\frac{1}{|k|}\right);\ s_{12}(k)=O\left(\frac{1}{|k|}\right);\ s_{21}(k)=O\left(\frac{1}{|k|}\right).$$

We can naturally continue $s_{ij}(k)$ to the semi-axis $k<0$ by the equation

$$s_{ij}(-k)=\overline{s_{ij}(k)}.$$

We may easily verify based on the property of Eq. (1.21) that the matrices

$$S(k)=\|s_{ij}(k)\|;\ S^{-1}(k)=\|\widetilde{s}_{ij}(k)\|$$

are inverses of each other, as is indicated by the notation, and are unit matrices, so that for example

$$S^*(k)S(k)=S(k)S^*(k)=I,$$

or, in more detail,

$$|s_{11}|^2+|s_{21}|^2=1=|s_{22}|^2+|s_{12}|^2,$$
$$s_{11}(k)s_{21}(-k)+s_{12}(k)s_{22}(-k)=0. \tag{2.3}$$

We will see, in particular, that for all $k \neq 0$,

$$|s_{12}(k)|<1;\ |s_{21}(k)|<1 \tag{2.4}$$

and we recall that if $|s_{12}(0)|=1$, then

$$s_{12}(0)=s_{21}(0)=-1. \tag{2.5}$$

Finally, a comparison of the asymptotics of the set of solutions $u_i^{(+)}(x, k)$ and $u_i^{(-)}(x, k)$ implies the linear relation

$$u^{(+)}(x,\ k)=S(k)u^{(-)}(x,\ k), \tag{2.6}$$

where natural vector notation is used.

We note that the set of solutions $u_i^{(\pm)}(x, k)$ are naturally interpreted in their asymptotic in terms of a radiation principle. However, we will not use this fact anywhere below.

The solutions $u_i^{(\pm)}(x, k)$ constitute a complete orthonormalized set of eigenfunctions of the continuous spectrum of the operator H. We may verify this fact by calculating the jumps in the resolvent $R(x, y; \lambda)$ through a section in the positive part of the real axis, which corresponds to the continuous spectrum of H. We have the equation

$$R(x,\ y;\ k^2+i0)-R(x,\ y;\ k^2-i0)=$$
$$=\frac{1}{2ik}\left(\overline{u_1^{(\pm)}(x,\ k)}u_1^{(\pm)}(y,\ k)+\overline{u_2^{(\pm)}(x,\ k)}u_2^{(\pm)}(y,\ k)\right),$$

which quite simply verifies the direct substitution of Eqs. (2.1) and (2.2) for the solutions $u_1^{(\pm)}(x,k)$ and $u_2^{(\pm)}(x,k)$ in terms of $f_1(x, k)$ and $f_2(x, k)$ in the right side. The completeness equation which thereby follows has the form

$$\frac{1}{2\pi}\int_0^\infty\left(\overline{u_1^{(\pm)}(x,\ k)}u_1^{(\pm)}(y,\ k)+\overline{u_2^{(\pm)}(x,\ k)}u_2^{(\pm)}(y,\ k)\right)dk+$$
$$+\sum_{l=1}^{N}\overline{u_l(x)}u_l(y)=\delta(x-y).$$

Here the $u_l(x)$, $l=1,\ldots,N$ are orthonormalized eigenfunctions of the discontinuous spectrum of H. The orthogonality relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}u_i^{(\pm)}(x,\ k)\overline{u_j^{(\pm)}(x,\ l)}dx=\delta_{ij}\delta(x-y)$$

can be derived using the identity

$$f(x, k) f(x, l) = \frac{1}{l^2 - k^2} \frac{d}{dx} \{f(x, k) f(x, l)\},$$

which is true for arbitrary solutions of the Schröedinger Eq. (1.1), the asymptotic as $|x| \to \infty$ of the solutions $u_l^{(\pm)}(x, k)$, and the unitary condition on S(k).

We construct using the solutions $u_l^{(\pm)}(x, k)$ two maps $T_\pm : \mathfrak{H} \to \mathfrak{H}_0$ using the equations

$$\psi(x) \to T_\pm \psi = \varphi^{(\pm)}(\lambda); \quad \varphi_l^{(\pm)}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) u_l^{(\pm)}(x, \sqrt{\lambda})\, dx.$$

The completeness and orthogonality relations are written in terms of these operators as follows:

$$T_\pm^* T_\pm = I - P; \quad T_\pm T_\pm^* = I_0.$$

Here P is a projector into $\mathfrak{H}$ on the proper subspace of H spanned by its eigenvectors u_l, $l = 1, \ldots, N$.

We will now prove that the wave vectors can be introduced by the equations

$$U^{(\pm)} = T_\pm^* T_0. \tag{2.7}$$

For the proof it is sufficient to demonstrate that for any vector $\varphi(\lambda) \in \mathfrak{H}_0$ in a dense set the vector

$$\chi^{(\pm)}(t) = e^{-iHt} T_\pm^* \varphi - e^{-iH_0 t} T_0^* \varphi$$

vanishes in norm in $\mathfrak{H}$ as $t \to \pm\infty$. This is in turn easy to verify. In fact, recalling the definition of the operators $T_\pm$ and T_0, we can write the functions $\chi^{(\pm)}(x, t)$ representing the vectors $\chi^{(\pm)}(t)$, in the form

$$\chi^{(\pm)}(x, t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \left[\varphi_1(k^2)\left(\overline{u_1^{(\pm)}(x, k)} - e^{ikx}\right) + \right.$$
$$\left. + \varphi_2(k^2)\left(\overline{u_2^{(\pm)}(x, k)} - e^{-ikx}\right)\right] e^{-ik^2 t} dk.$$

The functions $\varphi_1(k^2)$ and $\varphi_2(k^2)$ can be considered as continuous and finite. By the Riemann–Lebesgue lemma the contribution to the integral

$$\|\chi^{(\pm)}(t)\|^2 = \int_{-\infty}^{\infty} |\chi^{(\pm)}(x, t)|^2 dx$$

from an arbitrary finite interval $|x| \leqslant A$ can be arbitrarily small for sufficiently large $|t|$. Further, the contribution to the integral within the interval $-\infty < x < -A$ and $A < x < \infty$ from terms of type o(1) in the asymptotic as $|x| \to \infty$ of the functions $u_l^{(\pm)}(x, k)$ can be carried out for sufficiently large A uniformly in arbitrarily small t. For this purpose we need only use bounds of the type of Eqs. (1.6) and (1.7). The remaining integrals have the form

$$J_1^{(\pm)}(t) = \int_A^\infty \left|\int_\alpha^\beta G(k) e^{-ik^2 t} e^{\mp ikx} dk\right|^2 dx$$

and

$$J_2^{(\pm)}(t) = \int_{-\infty}^{-A} \left|\int_\alpha^\beta G(k) e^{-ik^2 t} e^{\pm ikx} dk\right|^2 dx,$$

where $[\alpha, \beta]$ is a finite interval on the semi-axis $0 < k < \infty$ and G(k) is a continuous function that vanishes at its endpoints. The assertion according to which

$$J_1^{(+)}, J_2^{(+)} \to 0, \ t \to \infty; \ J_1^{(-)}, J_2^{(-)} \to 0, \ t \to -\infty$$

whose proof we leave to the reader, concludes the proof with

$$\|\chi^{(\pm)}(t)\| \to 0, \ t \to \pm\infty.$$

In fact, we have not only proved the coinciding of Eq. (2.7), but have also given an independent proof for the existence of wave operators $U^{(\pm)}$. Asymptotic completeness, i.e., the relation

$$U^{(\pm)} U^{(\pm)*} = I - P, \tag{2.8}$$

for which there exists in the abstract theory a complex proof, in our case immediately follows from the completeness condition on the functions $u^{(\pm)}(x, k)$. The isometricity condition

$$U^{(\pm)*} U^{(\pm)} = I,$$

which is trivially proved in the abstract theory is equivalent to the orthogonality of the functions $u^{(\pm)}(x, k)$ and can be used for deriving it.

Equation (2.6) can now be written in the form

$$U^{(-)} = U^{(+)} S,$$

where the operator S is defined by the equation

$$S = T_0^* \hat{S} T_0,$$

and the operator $\hat{S}$ is defined in $\mathfrak{H}_0$ by the matrix S(k),

$$\hat{S}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = S(\sqrt{\lambda})\begin{pmatrix}\varphi_1(\lambda)\\ \varphi_2(\lambda)\end{pmatrix}.$$

Evidently, S commutes with H_0,

$$[S, H_0] = 0.$$

We have thus obtained an expression for the scattering operator S in the given case. The matrix S(k) defining it, which yields a representation of it in a diagonal realization of H_0, is said to be the S-matrix. We say that $s_{21}(k)$ and $s_{12}(k)$ are said to be the left and right reflection coefficients, respectively, and the coefficient $s_{11} = s_{22}$, the transmission coefficient in accordance with the interpretation of its matrix elements $s_{ij}(k)$ in the spirit of the radiation principle.

These properties of the S-matrix allow us to reconstruct it if and only if the reflection coefficient is given. In fact, suppose $S_{12}(k)$ is given. We may determine from the unitarity condition of Eq. (2.3) the modulus of the transmission coefficient

$$|s_{11}(k)| = (1 - |s_{12}(k)|^2)^{1/2}.$$

The argument of this coefficient (and thus the entire coefficient) is reconstructed in terms of its modulus based on the analyticity of the coefficient in the upper half-plane. We have the explicit formulas

$$s_{11}(k) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(1 - |s_{12}(l)|^2)}{l - k}\, dl\right\} \prod_{l=1}^{N} \frac{k + i\varkappa_l}{k - i\varkappa_l}, \ \operatorname{Im} k > 0;$$

$$s_{11}(k) = \lim_{\varepsilon \to 0} s_{11}(k + i\varepsilon), \ \operatorname{Im} k = 0. \tag{2.9}$$

The coefficient $s_{21}(k)$ can now be constructed on the basis of the unitarity conditions:

$$s_{21}(k) = -\frac{s_{12}(-k)\, s_{22}(k)}{s_{11}(-k)}. \tag{2.10}$$

This procedure remains meaningful for any functions $s_{12}(k)$ that satisfies the conditions of Eqs. (2.4) and (2.5) and which possesses the asymptotic

$$s_{12}(k)=O\left(\frac{1}{|k|}\right). \tag{2.11}$$

The conditions of Eqs. (2.4) and (2.5) for the resulting $s_{21}(k)$ also hold and for large $|k|$ we have the asymptotic

$$s_{11}(k)=1+O\left(\frac{1}{|k|}\right);\quad s_{21}(k)=O\left(\frac{1}{|k|}\right). \tag{2.12}$$

We note that the analyticity of the transmission coefficient is also the additional necessary condition we spoke of in the introduction in discussing the overdeterminacy of the inverse problem.

Further properties of the Fourier transform of the coefficients $s_{11}(k)$, $s_{21}(k)$, and $s_{12}(k)$ will be found in the next section.

We conclude here with the description of the fundamental objects of scattering theory for the pair of operators H and H_0 and pass to the inverse problem, the problem of reconstructing an operator H, i.e., the potential v(x), in terms of the matrix S(k), i.e., in fact in terms of one of the reflection coefficients.

3. Volterra Transformation Operators

As already noted in the introduction, transformation operators constitute the basis of the technique for solving the inverse problem, i.e., solution of the equation

$$HU=UH_0,$$

which have the structure of Volterra operators

$$U_1\psi(x)=\psi(x)+\int_x^{\infty}A_1(x,y)\psi(y)\,dy; \tag{3.1}$$

$$U_2\psi(x)=\psi(x)+\int_{-\infty}^{x}A_2(x,y)\psi(y)\,dy. \tag{3.2}$$

These operators we have in fact already introduced. Indeed we will define operators V_i, i = 1, 2, operating from $\mathfrak{H}$ into $\mathfrak{H}_0$ by the equations

$$\psi\to V_1\psi=\varphi:\varphi_1(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)\,f_1(x,-\sqrt{\lambda})\,dx;$$

$$\varphi_2(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)\,f_1(x,\sqrt{\lambda})\,dx;$$

$$\psi\to V_2\psi=\varphi:\varphi_1(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)\,f_2(x,\sqrt{\lambda})\,dx;$$

$$\varphi_2(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)\,f_2(x,-\sqrt{\lambda})dx.$$

Then the operators

$$U_i=V_i^{*}T_0,\quad i=1,2, \tag{3.3}$$

are defined by Eqs. (3.1) and (3.2), where the kernels $A_i(x, y)$ are defined in Sec. 1 by Eqs. (1.8) and (1.9).

Let us discuss how the completeness condition of Eq. (2.8) appears in terms of the operators U_1 and U_2. For this purpose we first calculate the normalizing factors, i.e., the operators $N_i^{(\pm)}$, i = 1, 2, realizing the equation

$$U_i = U^{(\pm)} N_i^{(\pm)}, \quad i=1, 2. \tag{3.4}$$

As a consequence of the commutivity condition

$$[H_0, N_i^{(\pm)}]=0,$$

which these operators must satisfy, they can be defined by the matrices $N_i^{(\pm)}(k)$, similar to how the operator S is defined by the matrix S(k). The definition of Eqs. (2.1) and (2.2) imply that

$$f_1 = M_1^{(\pm)}(k)\, u^{(\pm)}; \quad f_2 = M_2^{(\pm)}(k)\, u^{(\pm)}, \tag{3.5}$$

where f_1, f_2, $u^{(\pm)}$ are the columns of the solutions $f_1(x, -k)$, $f_1(x, k)$, $f_2(x, k)$, $f_2(x, -k)$, $u_1^{(\pm)}$, $u_2^{(\pm)}$, respectively, while the matrices $M_i^{(\pm)}(k)$ have the form

$$\left.\begin{aligned} M_1^{(+)}(k) &= \begin{pmatrix} 1 & -b(k) \\ 0 & a(k) \end{pmatrix}; \quad M_1^{(-)}(k) = \begin{pmatrix} a(-k) & 0 \\ -b(-k) & 1 \end{pmatrix}; \\ M_2^{(+)}(k) &= \begin{pmatrix} a(k) & 0 \\ b(-k) & 1 \end{pmatrix}; \quad M_2^{(-)}(k) = \begin{pmatrix} 1 & b(k) \\ 0 & a(-k) \end{pmatrix}. \end{aligned}\right\} \tag{3.6}$$

It therefore follows from the definitions of Eqs. (2.7), (3.3), and (3.4) that the normalizing factors $N_i^{(\pm)}$ are expressed by the equations

$$N_i^{(\pm)} = T_0^* \hat{N}_i^{(\pm)} T_0,$$

where the operators $\hat{N}_i^{(\pm)}$ operate in $\mathfrak{H}_0$ as matrix-multiplication operators:

$$N_i^{(\pm)}(k) = M_i^{(\pm)}(k)^*.$$

Comparing Eqs. (2.6) and (3.5), we also see that the matrices $M_1^{(\pm)}(k)$ and $M_2^{(\pm)}(k)$ factor the matrix S(k):

$$S(k) = M_1^{(+)\,-1}(k)\, M_1^{(-)}(k) = M_2^{(+)-1}(k)\, M_2^{(-)}(k). \tag{3.7}$$

The factorization condition and the triagonal structure of the matrices $M_i^{(\pm)}(k)$, apparent in the explicit equations (3.6), yields a unique determination of them in terms of a given matrix S(k). In fact, if the first equation of Eqs. (3.7) is rewritten in the form

$$\begin{pmatrix} 1 & m_{12} \\ 0 & m_{22} \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} m_{11} & 0 \\ m_{21} & 1 \end{pmatrix},$$

we obtain a linear system of equations for determining the coefficients m_{11}, m_{12}, m_{21}, and m_{22} of the matrices $M_1^{(+)}$ and $M_1^{(-)}$, which yields a unique solution. The second equation of Eqs. (3.7) may be similarly treated. Equations (3.6) yield the desired result. In other words a priori data on the structure of the normalizing factors corresponding to the transformation operators U_1 and U_2 uniquely determines them in terms of a given scattering operator S.

The operators U_i, i = 1, 2, like the wave operators $U^{(\pm)}$, have the proper subspace H corresponding to its absolutely continuous spectrum as range of values. We will show how to expand the domain of definition of the operators by leaving the space $\mathfrak{H}$, so that their range of values subsequently coincides with $\mathfrak{H}$. The discussion is based on the fact that the eigenfunctions of the discontinuous spectrum $u_l(x)$ of H generating the defect subspace for the operators U are proportional to the solution $f_i(x, k)$ when $k = i\varkappa_l$. Thus,

$$u_l(x) = \chi_l^{(1)}(x) + \int_x^\infty A_1(x, y)\, \chi_l^{(1)}(y)\, dy = U_1 \chi_l^{(1)}; \tag{3.8}$$

$$u_l(x) = \chi_l^{(2)}(x) + \int_{-\infty}^x A_2(x, y)\, \chi_l^{(2)}(y)\, dy = U_2 \chi_l^{(2)}, \tag{3.9}$$

where

$$\chi_l^{(1)} = (m_l^{(1)})^{1/2} e^{-\varkappa_l x}; \quad \chi_l^{(2)} = (m_l^{(2)})^{1/2} e^{\varkappa_l x}$$

and then $m_l^{(i)}$ are normalizing factors,

$$m_l^{(i)} = \left(\int_{-\infty}^{\infty} (f_i(x, i\varkappa_l))^2 \, dx \right)^{-1}, \qquad i = 1, 2. \tag{3.10}$$

We note that Eq. (1.28) implies that

$$m_l^{(1)} m_l^{(2)} = \gamma_l^2; \qquad \gamma_l = \frac{i}{\dot{a}(i\varkappa_l)} = i \operatorname{Res} s_{11}|_{k=i\varkappa_l}. \tag{3.11}$$

We now consider the spaces

$$\mathfrak{H}_i = \mathfrak{H} \oplus \mathfrak{B}_i, \tag{3.12}$$

where the finite-dimensional spaces $\mathfrak{B}_i$ are spanned by the function $\chi_l^{(i)}$. The scalar product in $\mathfrak{B}_i$ is induced by the equation

$$(e^{-\varkappa_l x}, e^{-\varkappa_j x})_1 = \delta_{lj}; \quad (e^{\varkappa_l x}, e^{\varkappa_j x})_2 = \delta_{lj}.$$

Equations (3.8) and (3.9) extend the operators U_1 and U_2 to the spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, their range of values then coinciding with $\mathfrak{H}$.

The completeness equation (2.8) in terms of these extended operators then is

$$U_i W_i U_i^* = I. \tag{3.13}$$

The weight operators W_i operate in the spaces $\mathfrak{H}_i$ and the decomposition of Eq. (3.12) reduces them. The operators W_i in the subspaces $\mathfrak{B}_i$ are given in the bases $\{e^{\pm\varkappa_l x}\}$ by the diagonal matrices

$$W_i|_{\mathfrak{B}_i} = \{m_1^{(i)}, \ldots, m_N^{(i)}\}, \; i = 1, 2.$$

The operators W_i in the subspace $\mathfrak{H}$ are given by the equation

$$W_i = N_i^{(\pm)-1} N_i^{(\pm)*-1} = T_0^* \hat{W}_i T_0.$$

We use Eq. (3.6) to express these operators in terms of the matrix elements of S(k). We have on the basis of the unitarity condition,

$$W_1(k) = (M_1^{(\pm)}(k) M_1^{(\pm)*}(k))^{-1} = \begin{pmatrix} 1 & s_{12}(k) \\ s_{12}(-k) & 1 \end{pmatrix}$$

and similarly

$$W_2(k) = (M_2^{(\pm)}(k) M_2^{(\pm)*}(k))^{-1} = \begin{pmatrix} 1 & s_{21}(-k) \\ s_{21}(k) & 1 \end{pmatrix}.$$

Thus, the matrices $W_i(k)$ are expressed in terms of only one of the reflection coefficients. Carrying out the Fourier transformation necessary for the final calculation of the W_i, we find that they are expressed in the form

$$W_i = I + \Omega_i, \quad i = 1, 2,$$

where the Ω_i are integral operators with kernels depending on the sum of the arguments

$$\Omega_i(x, y) = \Omega_i(x + y),$$

where

$$\Omega_1(x) = \sum m_l^{(1)} e^{-\varkappa_l x} + F_1(x); \quad \Omega_2(x) = \sum m_l^{(2)} e^{\varkappa_l x} + F_2(x), \tag{3.14}$$

where

$$F_1(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} s_{12}(k)\,e^{ikx}dk, \quad F_2(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} s_{21}(k)\,e^{-ikx}dk. \tag{3.15}$$

The properties of Eqs. (2.11) and (2.12) imply that the functions $\Omega_1(x)$ and $\Omega_2(x)$ are quadratically integrable on the intervals $a<x<\infty$ and $-\infty<x<b$, respectively, for finite a and b. More detailed information on these kernels will be found in the next section.

4. Gel'fand – Levitan Equations

The heuristic considerations presented in the introduction demonstrate that completeness conditions reduce to linear equations for the kernels $A_1(x, y)$ and $A_2(x, y)$ for the transformation operators U_1 and U_2,

$$A_1(x, y)+\Omega_1(x+y)+\int_x^{\infty} A_1(x, z)\,\Omega_1(z+y)\,dz=0; \quad x<y, \tag{4.1}$$

$$A_2(x, y)+\Omega_2(x+y)+\int_{-\infty}^{x} A_2(x, z)\,\Omega_2(z+y)\,dz=0; \quad x>y. \tag{4.2}$$

These equations were first derived by Kay and Moses [40]. They coincide in out-ward appearance with the equation of V. A. Marchenko [16] from the theory of the radial Schröedinger operator for $l = 0$. One change consists in the range of variation of the variables becoming the entire axis. We will, however, call them the Gel'fand–Levitan equations, since their operator-theoretic content is similar to that for the equations introduced by I. M. Gel'fand and B. M. Levitan in the theory of the inverse Sturm–Liouville inverse spectral problem. To strictly derive these equations we must investigated the operators U_i^{*-1} operating from $\mathfrak{H}_i$ into $\mathfrak{H}$ or replace operator-theoretic concepts by more elementary concepts. One variant of these discussions conceptually similar to [1] and carried out explicitly in [23, 26] will be set forth below.

We will use the equations

$$u_1(x, k)=s_{12}(k)\,f_1(x, k)+f_1(x, -k), \tag{4.3}$$

$$u_2(x, k)=s_{21}(k)\,f_2(x, k)+f_2(x, -k), \tag{4.4}$$

which constitute a variant of Eqs. (1.15) and (1.16). We omit here the index (+) in $u_i^{(+)}(x, k)$, since the functions $u_i^{(-)}(x, k)$ will no longer be used.

We know that the functions $f_1(x, k)$ and $f_2(x, k)$ are analytic and bounded in the upper half-plane and are bounded there

$$f_1(x, k)\,e^{-ikx}=1+O\left(\frac{1}{|k|}\right); \quad f_2(x, k)\,e^{ikx}=1+O\left(\frac{1}{|k|}\right), \tag{4.5}$$

where $O\left(\frac{1}{|k|}\right)$ in general depends on x. The functions $u_1(x, k)$ and $u_2(x, k)$ are also analytic in the upper half-plane except at the points $k = i\varkappa_l$, $l = 1, \ldots, N$, where they have together with $1/a(k)$ simple poles. The corresponding residues are simply associated with the values of the functions $f_1(x, k)$ and $f_2(x, k)$ at these points. For example

$$\operatorname{Res} u_1(x, k)|_{k=i\varkappa_l}=\operatorname{Res} s_{11}(k)|_{k=i\varkappa_l} f_2(x, i\varkappa_l)=\frac{f_2(x, i\varkappa_l)}{-i\int_{-\infty}^{\infty} f_1(x, i\varkappa_l)\,f_2(x, i\varkappa_l)\,dx}=im_l^{(1)}f_1(x, i\varkappa_l), \tag{4.6}$$

where $m_l^{(1)}$ is defined in Eq. (3.10). Similarly,

$$\operatorname{Res} u_2(x, k)|_{k=i\varkappa_l}=im_l^{(2)}f_2(x, i\varkappa_l).$$

At large $|k|$ the functions $u_i(x, k)$ have the asymptotic

$$u_1(x, k)\, e^{ikx} = 1 + O\left(\frac{1}{|k|}\right);\quad u_2(x, k)\, e^{-ikx} = 1 + O\left(\frac{1}{|k|}\right), \tag{4.7}$$

where $O\left(\frac{1}{|k|}\right)$ may also depend on x nonuniformly.

Functions with only a single number occur in each row in Eqs. (4.3) and (4.4) and both equations are completely identical to within the substitutions $1 \leftrightarrow 2$ and $e^{ikx} \leftrightarrow e^{-ikx}$. In each of them two functions f(x, k) and u(x, k) possessing definite analytic properties on the complex plane are related on the real axis in terms of the function s(k) given only for Im k = 0. It turns out that we can reconstruct on the basis of these equations both functions f(x, k) and u(x, k) in terms of given s(k).

For this purpose we pass to the Fourier transformation participating in these equations of the functions. The Fourier transformations for $f_1(x, k)$, $f_2(x, k)$, $s_{12}(k)$, and $s_{21}(k)$ have already been used many times above [cf. Eqs. (1.8), (1.9), and (3.15)]. Suppose

$$u_1(x, k) = e^{-ikx} + \int_{-\infty}^{\infty} B_1(x, y)\, e^{-iky}\, dy. \tag{4.8}$$

Based on the analyticity properties of $u_1(x, k)$ described above,

$$B_1(x, y) = i \sum_{l=1}^{N} \operatorname{Res} u_1(x, k)\big|_{k=i\varkappa_l} e^{-\varkappa_l y} = $$
$$= -\sum_{l=1}^{N} m_l^{(1)} \left(e^{-\varkappa_l x} + \int_x^{\infty} A_1(x, z)\, e^{-\varkappa_l z}\, dz \right) e^{-\varkappa_l y},\quad x < y. \tag{4.9}$$

By the convolution theorem, Eq. (4.3) takes the form, following a Fourier transformation,

$$F_1(x+y) + \int_{-\infty}^{\infty} A_1(x, z)\, F_1(z+y)\, dz + A_1(x, y) = B_1(x, y) \tag{4.10}$$

and when $x<y$ we arrive at Eq. (4.1) on the basis of Eq. (4.9). Equation (4.2) is similarly derived.

The discussions we have presented on the basis of Eqs. (4.3) and (4.4) together with analyticity-type conditions occurring in them for functions of the form of Eqs. (4.1) and (4.2) can be reversed. More precisely, suppose $A_1(x, y)$ is a solution of Eq. (4.1), such that the function $f_1(x, y)$ analytic for Im k > 0 constructed in terms of it by Eq. (1.8) satisfies the condition of Eq. (4.5). We consider the kernel $B_1(x, y)$ defined by Eq. (4.10) and construct using it a function $u_1(x, k)$ by Eq. (4.8). Carrying out a Fourier transformation, we find that $f_1(x, k)$ and $u_1(x, k)$ are related by an equation of the form Eq. (4.3), so that, in particular $u_1(x, k)$ when Im k = 0 satisfies the condition of Eq. (4.7), Equation (4.1) implies that Eq. (4.9) is ture for $B_1(x, y)$ when $x<y$, so that $u_1(x, k)$ has an analytic continuation into the upper half-plane Im k > 0 with poles at the points $i\varkappa_l$, while Eq. (4.6) holds for the corresponding residues. The proof of this equivalence concludes with this fact. In the next section we will study the solvability of the Gel'fand–Levitan equation and will formulate more precisely the corresponding assertion for the existence and uniqueness of a pair of functions u(x, k) and f(x, k) that satisfy such analyticity conditions and are related by an equation of the type of Eq. (4.3).

In concluding this section we will use the Gel'fand–Levitan equation to refine the properties of the reflection coefficients, that is, we will study more precisely the behavior of the functions $F_1(t)$ and $F_2(t)$, about which we so far only know are quadratically integrable. We consider for the sake of definiteness the function $F_1(t)$. We rewrite Eq. (4.1), setting $x = y$:

$$\Omega_1(2x) + A_1(x, x) + 2\int_x^{\infty} A_1(x, 2y-x)\, \Omega_1(2y)\, dy = 0, \tag{4.11}$$

and consider this equation as the function for $\Omega_1(2y)$. This is a Volterra-type equation and the method of successive approximations always converges for it. We obtain based on the bound of Eq. (1.12) for the kernel $A_1(x, y)$ a bound for $\Omega_1(2x)$,

$$\Omega_1(2x) \leqslant C(x)\,\xi_1(x).$$

Here and below we denote by C(x) a monotonically nondecreasing function bounded as $x \to \infty$ and, in general, increasing as $x \to -\infty$. We conclude as a consequence of the differentiability of $A_1(x, y)$ that $\Omega_1(x)$ is also differentiable and using Eq. (4.11), we find the bound

$$\left|\frac{d}{dx}\Omega_1(2x) - \frac{1}{2}v(x)\right| \leqslant C(x)\,\xi_1^2(x).$$

We can similarly prove using Eq. (4.2) the differentiability of $\Omega_2(x)$ and find the bounds

$$|\Omega_2(2x)| \leqslant D(x)\,\xi_2(x);\; \left|\frac{d}{dx}\Omega_2(2x) + \frac{1}{2}v(x)\right| \leqslant D(x)\,\xi_2^2(x).$$

Here and below the function D(x) is a monotonically nondecreasing function bounded as $x \to -\infty$ and increasing, in general, as $x \to \infty$.

The resulting estimates and the properties (P) of the potential imply that

$$\int_a^\infty (1+|x|)\left|\frac{d}{dx}\Omega_1(x)\right| dx \leqslant C(a), \tag{4.12}$$

$$\int_{-\infty}^b (1+|x|)\left|\frac{d}{dx}\Omega_2(x)\right| dx \leqslant D(b). \tag{4.13}$$

The functions $F_1(x)$ and $F_2(x)$ differ from $\Omega_1(x)$ and $\Omega_2(x)$ by a continuous term with decreases as $x \to \infty$ and $x \to -\infty$, respectively. Consequently, inequalities of the type of Eqs. (4.12) and (4.13) are true also for $F_1(x)$ and $F_2(x)$.

We will see that the functions $\frac{d}{dx}F_1(x)$ and $\frac{d}{dx}F_2(x)$ behave similar to v(x) as $x \to \infty$ and $x \to -\infty$, respectively. If v(x) is differentiable, it can be proved using Eqs. (1.10), (1.11), and (4.11) that this analogy extends also to the succeeding derivatives of $\Omega_1(x)$ and $\Omega_2(x)$.

5. Investigation of Inverse Problem

In the preceding sections we explained how the scattering matrix S(k) corresponding to a potential v(x) satisfying property (P) possesses the properties:

1. Unitarity:

$$s_{11}\bar{s}_{12} + s_{21}\bar{s}_{22} = 0;$$
$$|s_{11}|^2 + |s_{12}|^2 = 1 = |s_{21}|^2 + |s_{22}|^2;$$
$$s_{12}(0) = s_{21}(0) = -1, \text{ if } \quad s(0) = 0.$$

2. Realness:

$$s_{ij}(-k) = \overline{s_{ij}(k)};$$

3. Symmetry:

$$s_{11}(k) = s_{22}(k);$$

4. Asymptotic behavior:

$$s_{12} = O\left(\frac{1}{|k|}\right);\; s_{21} = O\left(\frac{1}{|k|}\right);\; s_{11} = 1 + O\left(\frac{1}{|k|}\right);$$

and the Fourier transforms $F_1(x)$ and $F_2(x)$ of the coefficients $s_{12}(k)$ and $s_{21}(k)$ satisfy the condition

$$\int_a^\infty (1+|x|)\left|\frac{d}{dx}F_1(x)\right| dx \leqslant C(a);\; \int_b^\infty (1+|x|)\left|\frac{d}{dx}F_2(x)\right| dx \leqslant D(b).$$

5. Analyticity: the function $s_{11}(k)$ is the limiting value of a function analytic in the half-plane $\mathrm{Im}\,k > 0$, having there the asymptotic $1+O\left(\frac{1}{|k|}\right)$ and a finite number of poles on the imaginary axis.

It is possible that some of these properties are consequences of others, but we will not bother ourselves about this matter. In the current section we will prove that these necessary properties are also sufficient conditions under which a potential $v(x)$ satisfying (P) corresponds to such a matrix $S(k)$. Here we must specify N more positive numbers, where N is the number of poles of $s_{11}(k)$, for a unique definition of $v(x)$, in addition to $S(k)$. This result was formulated in [23] and proved in detail in [26].

The proof of this assertion will be found using the investigation of Gel'fand–Levitan-type equations, which we now begin to describe.

We begin with an investigation of the solvability of Eqs. (4.1) and (4.2). Suppose we are given

1) The function

$$s_{12}(k) = \int_{-\infty}^{\infty} F_1(x) e^{-ikx} dx$$

such that

$$s_{12}(-k) = \overline{s_{12}(k)}; \quad |s_{12}(k)| \leqslant 1$$

and

$$\int_a^{\infty} (1+|x|) \left| \frac{d}{dx} F_1(x) \right| dx < C(a).$$

2) distinct arbitrary positive numbers $\varkappa_l$, $l = 1, \ldots, N$.

3) the same number of positive numbers $m_l^{(1)}$, $l = 1, \ldots, N$. We construct using these data the function $\Omega_1(x)$ in terms of Eq. (3.14) and consider the equation

$$A_1(x, y) + \Omega_1(x+y) + \int_x^{\infty} A_1(x, z) \Omega_1(z+y) dz = 0, \; x < y,$$

as the equation for $A_1(x, y)$. This is an equation in terms of the second independent variable of this function, where x occurs only as a parameter. Setting

$$a_x(y) = A_1(x, y); \quad \omega_x(y) = \Omega_1(x+y);$$
$$\Omega_x g(y) = \int_x^{\infty} g(z) \Omega_1(z+y) dz,$$

we rewrite Eq. (4.1) in the form of the operator equation

$$a_x(y) + \omega_x(y) + \Omega_x a_x(y) = 0. \tag{5.1}$$

The free term $\omega_x(y)$ is absolutely integrable and bounded, and, consequently, is also a quadratically integrable function on the interval $x \leqslant y < \infty$, i.e., $\omega_x(y) \in L_{12}(x, \infty)$. We will find a solution also from $L_{12}(x, \infty)$ and prove that it exists and is unique for any x, $-\infty < x < \infty$.

For this purpose we first verify that we are dealing with an equation possessing a completely continuous operator in $L_1(x, \infty)$ and $L_2(x, \infty)$. Suppose

$$\eta_1(x) = \int_x^{\infty} \left| \frac{d}{dy} \Omega_1(y) \right| dy.$$

The function $\eta_1(x)$ on the basis of Eq. (4.12) is absolutely integrable on the intervals $[a, \infty)$ for any $a > -\infty$ and we have the inequalities

$$\int_a^\infty \eta_1(x)\,dx \leqslant C(a); \quad \int_a^\infty (1+|x|)\,\eta_1^2(x)\,dx \leqslant C(a).$$

Using the bound

$$|\Omega_1(x)| \leqslant \int_x^\infty \left|\frac{d}{dx}\Omega_1(x)\right| dx \leqslant \eta_1(x)$$

we find that

$$\int_x^\infty dy \int_x^\infty dz\, |\Omega_1(y+z)|^2 \leqslant \left(\int_x^\infty \eta_1(x+y)\,dy\right)^2 < \infty,$$

i.e., the operator Ω_x is a Hilbert–Schmidt type operator and its norm approaches zero as $x \to \infty$. The complete continuousness of Ω_x in $L_1(x, \infty)$ is also a well-known corollary of the absolute integrability of $\tau_1(x)$.

We now note that the operator $I + \Omega_x$ is positively defined for any x. In fact it is obtained by limiting the positive operator W from Sec. 3 to $L_2(x, \infty)$. In particular, this implies that the homogeneous equation

$$h(y) + \Omega_x h(y) = 0 \tag{5.2}$$

has no nontrivial solutions in $L_2(x, \infty)$. We now prove that every solution of Eq. (5.2) in $L_1(x, \infty)$ also belongs to $L_2(x, \infty)$. We have

$$|h(y)| \leqslant \int_x^\infty |\Omega_1(y+z)|\,|h(z)|\,dz \leqslant \eta_1(x+y)\int_x^\infty |h(z)|\,dz$$

so that $h_1(y)$ is quadratically integrable on the interval $[x, \infty)$. We conclude that Eq. (5.2) has no nontrivial solutions in $L_1(x, \infty)$, so that Eq. (5.1) is uniquely solvable in $L_1(x, \infty)$ for any x. We will now consider how the bounds for the solution $A_1(x, y)$ follow from this fact.

The operator $(I + \Omega_X)^{-1}$ is uniformly bounded for all x from the interval $[a, \infty)$, since the norm of Ω_X approaches zero as $x \to \infty$

$$\|(I + \Omega_x)^{-1}\|_{L_1} \leqslant C(a),$$

so that

$$\int_x^\infty |A_1(x, y)|\,dy \leqslant C(x).$$

Substituting this bound in the integral occurring in Eq. (4.1), we find that

$$|A_1(x, y)| \leqslant C(x)\,\eta_1(x+y). \tag{5.3}$$

Using Eq. (4.1) we may also verify that the solution $A_1(x, y)$ is singly differentiable, and we may find bounds on the derivatives. Let us estimate the function $\frac{\partial}{\partial x}A_1(x, y)$. Differentiating Eq. (4.1) with respect to x, we arrive at the equation for $b_x(y) = \frac{\partial}{\partial x}A_1(x, y) + \frac{\partial}{\partial x}\Omega_1(x+y)$ of

$$b_x(y) + \mu_x(y) + \Omega_x b_x(y) = 0.$$

Here the free term

$$\mu_x(y) = A(x, x)\,\Omega(x+y)$$

has the bound

$$|\mu_x(y)| \leqslant C(x)\eta_1(2x)\eta_1(x+y).$$

We therefore find for the solution $b_x(y)$ that

$$\left|\frac{\partial}{\partial x}A_1(x, y)+\frac{\partial}{\partial x}\Omega_1(x+y)\right| \leqslant C(x)\eta_1(2x)\eta_1(x+y). \tag{5.4}$$

We may similarly estimate the derivative $\frac{\partial}{\partial y}A_1(x, y)$ and the result is given by

$$\left|\frac{\partial}{\partial y}A_1(x, y)+\frac{\partial}{\partial y}\Omega_1(x+y)\right| \leqslant C(x)\eta_1(2x)\eta_1(x+y). \tag{5.5}$$

The integral equation (4.2) is similarly investigated. If

$$s_{21}(-k)=\overline{s_{21}(k)};\ |s_{21}(k)| \leqslant 1;\ \int_{-\infty}^{b}(1+|x|)\left|\frac{d}{dx}F_2(x)\right|dx \leqslant D(b),$$

This equation is uniquely solvable and we have for the solution the bounds

$$|A_2(x, y)| \leqslant D(x)\eta_2(x+y); \tag{5.6}$$

$$\left|\frac{\partial}{\partial x}A_2(x, y)+\frac{\partial}{\partial y}\Omega_2(x+y)\right| \leqslant D(x)\eta_2(2x)\eta_2(x+y); \tag{5.7}$$

$$\left|\frac{\partial}{\partial y}A_2(x, y)+\frac{\partial}{\partial y}\Omega_2(x+y)\right| \leqslant D(x)\eta_2(2x)\eta_2(x+y), \tag{5.8}$$

where

$$\eta_2(x)=\int_{-\infty}^{x}\left|\frac{d}{dy}\Omega_2(y)\right|dy.$$

If the functions $F_1(x)$ and $F_2(x)$ have more than one derivative, the kernels $A_1(x, y)$ and $A_2(x, y)$ are also multiply differentiable. Bounds on the corresponding derivatives are found in a way similar to what was done for $b_x(y)$.

Returning to the pair of analytic functions u(x, k) and f(x, k), we verify that if we are given a function s(k) satisfying the conditions repeatedly formulated for $s_{12}(k)$ and N unequal positive numbers $\varkappa_l$, $l = 1, \ldots, N$, and further N positive numbers m_l, there exists a unique pair of functions u(x, k) and f(x, k), such that

1) the function f(x, k) and u(x, k) are analytically continued in the upper half-plane $\operatorname{Im} k \geq 0$, where $f(x, k)e^{-ikx}$ is bounded for all k, $\operatorname{Im} k \geq 0$, and u(x, k) has simple poles at the given points $k = i\varkappa_l$, $l = 1, \ldots, N$;

2) the residues u(x, k) are connected to the values of f(x, k) when $k = i\varkappa_l$ by the equation

$$\operatorname{Res} u(x, k)|_{k=i\varkappa_l}=im_l f(x, i\varkappa_l);$$

3) on the real axis by

$$f(x, k)=\overline{f(x, -k)};\ u(x, k)=\overline{u(x, -k)};$$

4) for large |k| by

$$f(x, k)e^{-ikx}=1+O\left(\frac{1}{|k|}\right);\ u(x, k)e^{ikx}=1+O\left(\frac{1}{|k|}\right);$$

5) for real k we have the equation

$$s(k)f(x, k)+f(x, -k)=u(x, k).$$

We now construct using the solutions for the Gel'fand–Levitan equations $A_1(x, y)$ and $A_2(x, y)$ found, operators U_1 and U_2 using Eqs. (3.1) and (3.2) and consider the operators

$$H_i = U_i H_0 U_i^{-1}, \; i = 1, 2. \tag{5.9}$$

We carry out the investigation of these operators at the formal level, without going too deeply into justifications. A rigorous justification of the results obtained here can be derived more simply by following the quite elementary, but laborious discussions of [1].

We first prove that the operators H_i are self-adjoint. For the proof we note that the Gel'fand–Levitan equations derived from the completeness equation (3.13) are in fact equivalent to it, i.e., in other words the operators U_1 and U_2 obtained by us satisfy this equation. We consider for the sake of definiteness the case i = 1. Suppose an operator $\tilde{A}_1$ has kernel $\tilde{A}_1(x, y)$, where

$$\tilde{A}_1(x, y) = \begin{cases} 0, & x < y; \\ \Omega_1(x+y) + \int\limits_x^\infty A_1(x, z)\Omega_1(z+y)\,dz, & x > y. \end{cases}$$

We construct the Volterra operator

$$\tilde{U}_1 = I + \tilde{A}_1.$$

The Gel'fand–Levitan equation can now be written in the form

$$U_1 W_1 = \tilde{U}_1.$$

The operator

$$\tilde{U}_1 U_1^* = U_1 W_1 U_1^*$$

is also self-adjoint since W_1 is self-adjoint. But it is simultaneously a Volterra operator, since the operators $\tilde{U}_1$ and U_1^* are Volterra with identical Volterra direction. These two properties are consistent only if

$$\tilde{U}_1 U_1^* = I,$$

which implies that U_1 satisfies the completeness equation

$$U_1 W_1 U_1^* = I.$$

Using this equation we can rewrite the definition of the operator H_1 in the form

$$H_1 = U_1 H_0 W_1 U_1^*,$$

which implies that H_1 is self-adjoint since H_0 and W_1 commute. The case i = 2 is similarly considered.

We now prove that the H_i are represented in the form

$$H_i = H_0 + V_i,$$

where the V_i are for multiplication by the functions

$$v_1(x) = -2\frac{d}{dx}A_1(x, x); \quad v_2(x) = 2\frac{d}{dx}A_2(x, x). \tag{5.10}$$

For this purpose, assuming again for the sake of definiteness that i = 1, we rewrite Eq. (5.9) in the form

$$H_1 U_1 = U_1 H_0 \tag{5.11}$$

and take into account that the operator $U_1 = I + A_1$ is Volterra, so that

$$A_1(x,y)=\theta(y-x)A_1(x,y),$$

where $\theta(x)$ is the Heaviside function. Equation (5.11) is rewritten in the form

$$-\left[\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)A_1(x,y)\right]\theta(y-x)+2\delta(x-y)\frac{d}{dx}A_1(x,x)+ \\ +(V_1U_1)(x,y)=0,$$

where we set $V_1 = H_1 - H_0$ and assume V_1 to be an integral operator whose kernel can be a generalized function. The resulting equation is consistent with V_1 being self-adjoint and U_1 being a Volterra only if

$$V_1(x,y)=-2\delta(x-y)\frac{d}{dx}A_1(x,x)=\delta(x-y)v_1(x).$$

Moreover, we will see from this fact that $A_1(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2}{\partial x^2}A_1(x,y)-\frac{\partial^2}{\partial y^2}A_1(x,y)-v_1(x)A_1(x,y)=0,$$

which, incidentally, we will not use.

The operator H_2 is analogously investigated. Our result is that the H_i are differential operators of the form

$$H_i=-\frac{d^2}{dx^2}+v_i(x),\quad i=1,2,$$

where the functions $v_i(x)$ are given by Eqs. (5.10).

Equation (5.11) also implies that the functions $f_1(x, k)$ and $f_2(x, k)$ constructed in terms of the kernels of $A_1(x, y)$ and $A_2(x, y)$ by means of Eqs. (1.8) and (1.9) are solutions of the differential equations

$$f_i''(x,k)+k^2f_i(x,k)=v_i(x)f_i(x,k),\quad i=1,2,$$

and have the asymptotics of Eqs. (1.2) and (1.3). Finally, the bounds of Eqs. (5.4), (5.5), (5.7), and (5.8) demonstrate that $v_1(x)$ and $v_2(x)$ satisfy bounds of the form

$$\int_a^\infty(1+|x|)|v_1(x)|\,dx\leqslant C(a);$$
$$\int_{-\infty}^b(1+|x|)|v_2(x)|\,dx\leqslant D(b). \tag{5.12}$$

Moreover, if $F_1(x)$ and $F_2(x)$ are n times differentiable, the potentials $v_1(x)$ and $v_2(x)$ have $n-1$ derivatives. Here $v_1^{[m]}$ and $v_2^{[m]}$ as $x\to\infty$ and as $x\to-\infty$, respectively, behave as $F_1^{[m+1]}$ and $F_2^{[m+1]}$.

We conclude the study of the inverse problem by proving if $\Omega_1(x)$ and $\Omega_2(x)$ are consistent, i.e., $F_1(x)$ and $F_2(x)$ are constructed in terms of the given matrix S(k) satisfying the necessary properties given at the beginning of the section and $m_l^{(1)}$ and $m_l^{(2)}$ are related by Eq. (3.11), $f_1(x, k)$ and $f_2(x, k)$ satisfy the equations

$$s_{11}(k)f_2(x,k)=s_{12}(k)f_1(x,k)+f_1(x,-k);$$
$$s_{22}(k)f_1(x,k)=s_{21}(k)f_2(x,k)+f_2(x,-k). \tag{5.13}$$

It therefore follows that $v_1(x)$ and $v_2(x)$ coincide, and we obtain from the bounds of Eqs. (5.12) that

$$\int(1+|x|)v(x)\,dx<\infty$$

together with the corresponding refinements on differentiability in the case of the differentiability of $F_1(x)$ and $F_2(x)$.

We have thereby found that the operator $H = H_0 + V$ constructed by us belongs to the initial class of the Schröedinger operators. Further, these equations (5.13) imply that H has the set of eigenfunctions $u_i^{(+)}(x, k)$ with asymptotic formulated in the table in Sec. 2, this asymptotic containing as the coefficient $s_{ij}(k)$ the matrix elements of the initial scattering matrix S(k). The construction of the operator H and thereby the proof of the sufficiency of the properties of the scattering matrix stated at the beginning of this section concludes with this fact.

To prove Eqs. (5.13) we use the uniqueness theorem obtained above for the pair of functions u(x, k) and f(x, k), proving that the functions $u_i(x, k)$ and $f_i(x, k)$ constructed using the Gel'fand–Levitan equations satisfy the equations

$$u_1(x, k) = s_{11}(k) f_2(x, k),$$
$$u_2(x, k) = s_{22}(k) f_1(x, k).$$

Equations (5.13) are thereby derived by using Eqs. (4.3) and (4.4).

Suppose $u_2(x, k)$ and $f_2(x, k)$ are obtained using the equation

$$s_{21}(k) f_2(x, k) + f_2(x, -k) = u_2(x, k) \tag{5.14}$$

and the analyticity properties formulated above.

We multiply Eq. (5.14) by $s_{21}(-k)$. We obtain

$$|s_{21}(k)|^2 f_2(x, k) + s_{21}(-k) f_2(x, -k) = s_{21}(-k) u_2(x, k).$$

The second term in the left side is again replaced on the basis of Eq. (5.14),

$$-(1 - |s_{21}(k)|^2) f_2(x, k) + u_2(x, -k) = s_{21}(-k) u_2(x, k).$$

In view of the unitarity condition of Eq. (2.3), the latter equation is rewritten in the form

$$-s_{21}(-k) u_2(x, k) + u_2(x, -k) = |s_{11}(k)|^2 f_2(x, k). \tag{5.15}$$

We introduce the functions

$$u(x, k) = s_{11}(k) f_2(x, k); \quad f(x, k) = u_2(x, k)/s_{11}(k). \tag{5.16}$$

The function u(x, k) is analytic everywhere in the upper half-plane except for the points $k = i\varkappa_l$, where together with $s_{11}(k)$ it has simple poles. The function f(x, k) has no singularities at $k = i\varkappa_l$, since the singularities of $u_2(x, k)$ and $s_{11}(k)$ compensate each other. If $s_{11}(0) = 0$, f(x, k) thereby lacks a singularity at $k = 0$. In fact if $s_{11}(0) = 0$, $s_{12}(0) = -1$, and using Eq. (5.14) we find that $u_2(x, 0) = 0$, so that the ratio u_2/s_{11} lacks a singularity at $k = 0$. Evidently, for real k,

$$f(x, -k) = \overline{f(x, k)}; \quad u(x, -k) = \overline{u(x, k)}.$$

We easily verify that the residues u(x, k) are related to f(x, k) by the equation

$$\operatorname{Res} u(x, k)|_{k=i\varkappa_l} = i m_l^{(1)} f(x, i\varkappa_l).$$

We need only use the condition of Eq. (3.11). Finally, Eq. (5.15) has the form in terms of f(x, k) and u(x, k)

$$s_{12}(k) f(x, k) + f(x, -k) = u(x, k).$$

Based on the uniqueness of this pair of functions as formulated above, we conclude that

$$f(x, k) = f_1(x, k); \quad u(x, k) = u_1(x, k),$$

which implies, by Eq. (5.16), Eq. (5.13).

We conclude with this fact for the general study of the inverse problem for the one-dimensional Schröedinger operator.

6. Particular Cases of the Solution of the Inverse Problem

Here we will consider two examples where the inverse problem has an explicit solution:

1. Absence of reflection, i.e., the coefficients $s_{12}(k)$ and $s_{21}(k)$ identically vanish and the entire nontrivial contribution to the Gel'fand–Levitan equation is provided by the discontinuous spectrum of H.

2. Rational reflection coefficient.

Both examples are combined through the common property of the kernel $\Omega(x+y)$ in the Gel'fand–Levitan equation; it becomes degenerate and the solution reduces to quadratures. However, we will consider these examples separately. Here we will assume in the second example, in order to simplify the equations, that the discontinuous spectrum of H is absent. This will be sufficient for the reader himself to analyze, by combining methods for the first and second examples, how alternations to the equations arise in the general case. These examples make it possible to solve the inverse problem for a dense set of scattering data.

Thus, let us consider the Gel'fand–Levitan equation (4.1) and assume that $s_{12}(k) = 0$, so that the kernel $\Omega_1(x+y)$ has the form

$$\Omega_1(x+y)=\sum_{l=1}^{N} m_l^{(1)} e^{-\varkappa_l (x+y)}.$$

The solution $A_1(x, y)$ in this case is naturally found in the form

$$A_1(x,y)=\sum_{l=1}^{N} g_l(x)\, e^{-\varkappa_l y},$$

an algebraic system of equations naturally written in vector notation

$$g(x)+g_0(x)+W_1(x)\,g(x)=0$$

arising for the function $g_l(x)$. Here $g(x)$ is the desired column vector with components $g_l(x)$, $l = 1, \ldots, N$, $g_0(x)$ is a column of the functions $m_l^{(1)} e^{-\varkappa_l x}$, $l = 1, \ldots, N$ and $W_1(x)$ is a matrix with elements

$$W_{lj}^{(1)}(x)=\frac{m_l^{(1)}}{\varkappa_l+\varkappa_j}\, e^{-(\varkappa_l+\varkappa_j)x}.$$

The solvability of the resulting system is guaranteed by the general results of the preceding section and, solving it, it is possible to find $g_l(x)$ and, together with them, the kernel $A_1(x, y)$. In particular, it can be easily verified that an expression is obtained for $A_1(x, y)$ which in our notation can be written as follows:

$$A_1(x,x)=\mathrm{tr}\left(\frac{d}{dx}W_1(x)\,(I+W_1(x)^{-1}\right)=\frac{d}{dx}\ln\det(I+W_1(x)).$$

We thereby find an expression for the desired potential,

$$v(x)=-2\frac{d^2}{dx^2}\ln\det(I+W_1(x)). \tag{6.1}$$

We can similarly consider Eq. (4.2). We obtain for the potential $v(x)$ the equation

$$v(x)=-2\frac{d^2}{dx^2}\ln\det(I+W_2(x)), \tag{6.2}$$

where $W_2(x)$ is a matrix with elements

$$W_{lj}^{(2)}(x)=\frac{m_l^{(2)}}{\varkappa_l+\varkappa_j}\, e^{(\varkappa_l+\varkappa_j)x}.$$

We note that the transmission coefficient in our case has the form

$$s_{11}(k)=\prod_{l=1}^{N}\frac{k+i\varkappa_l}{k-i\varkappa_l},$$

so that

$$\gamma_l=i\,\mathrm{Res}\,s_{11}(k)\,|_{k=i\varkappa_l}=-2\varkappa_l\prod_{j\neq l}\frac{\varkappa_l+\varkappa_j}{\varkappa_l-\varkappa_j},$$

and we recall that the constants $m_l^{(1)}$ and $m_l^{(2)}$ are related by the equation

$$m_l^{(1)}m_l^{(2)}=\gamma_l^2.$$

It follows from the general considerations of Sec. 5 that Eqs. (6.1) and (6.2) for v(x) coincide under these conditions. Direct verification of this identity constitutes a nontrivial combinatorial problem.

We now pass to the second example. Suppose $s_{12}(k)$ is a rational function of the variable k,

$$s_{12}(k)=r\frac{P_m(k)}{Q_n(k)},$$

where $P_m(k)$ and $Q_n(k)$ are polynomials of degree m and n, $m<n$, having the identity element as coefficient for the highest degree k, and r is a constant. The realness condition

$$s_{12}(-k)=\overline{s_{12}(k)}$$

will hold if

$$r=(i)^{m-n}r_0,$$

where r_0 is a real number and the zeroes of the polynomials $P_m(k)$ and $Q_n(k)$ are located symmetrically about the imaginary axis. The constant r_0 must be sufficiently small in order that

$$|s_{11}(k)|<1.$$

For this purpose it is necessary that all the zeroes of $Q_n(k)$ have nonvanishing imaginary part. For simplicity we will assume that all these zeroes are simple. The case of multiple zeroes can be considered by the corresponding passage to the limit.

The procedure described in Sec. 3 for reconstructing the transmission coefficient $s_{11}(k)$ in terms of $s_{12}(k)$ can be explicitly performed. We recall that we have assumed that the discontinuous spectrum of H is absent, so that $s_{11}(k)$ has no poles in the upper half-plane. The explicit equation for $s_{11}(k)$ has the form

$$s_{11}(k)=\prod_{l=1}^{n}(k+\beta_l)\prod_{l=1}^{n_+}(k+\alpha_l^{(+)})^{-1}\prod_{l=1}^{n_-}(k-\alpha_l^{(-)})^{-1},\quad n_++n_-=n,$$

where $\alpha_l^{(+)}$ and $\alpha_l^{(-)}$ are the zeroes of $Q_n(k)$ in the upper and lower half-plane, respectively, and these are roots of the equation

$$1=s_{12}(k)s_{12}(-k)$$

in the upper half-plane. There are precisely n such roots, since the equation is invariant relative to the substitution $k\to -k$. Using Eq. (2.10), we can see that the second reflection coefficient $s_{21}(k)$ is also a rational function and is represented in the form

$$s_{21}(k)=r'\frac{P'_{m'}(k)}{Q'_{n'}(k)},\quad m'<n',$$

where the polynomials $P'_{m'}(k)$ and $Q'_{n'}(k)$ and the constant r' possess properties similar to that for P_m, Q_n, and r. We will also assume that all the zeroes of $Q'_{n'}$ are simple.

The Fourier transforms $F_1(x)$ and $F_2(x)$ of the reflection coefficients $s_{12}(k)$ and $s_{21}(k)$ are calculated using the Jordan lemma. In particular,

$$F_1(x)=\sum_{l=1}^{n_+}\rho_l e^{i\alpha_l^{(+)}x},\quad x>0;$$

$$F_2(x)=\sum_{l=1}^{n'_-}\rho'_l e^{-i\alpha_l^{'(+)}x},\quad x<0,$$

where the sums are taken over the zeroes of $Q_n(k)$ and $Q'_{n'}(k)$ in the upper half-plane. The coefficients ρ_l and ρ'_l coincide to within a factor i with the residues $s_{12}(k)$ and $s_{21}(k)$ at the poles located at these zeroes.

We will see that the kernels $F_1(x + y)$ and $F_2(x + y)$ of the Gel'fand–Levitan equations (4.1) and (4.2) are degenerate when $x > 0$ and $x < 0$, respectively. We can use a method in these regions for solving them already mentioned in the investigation of the first example. As a result we find the expression for the desired potential

$$v(x)=-2\frac{d^2}{dx^2}\ln\det(I+Z_1(x)),\ x>0; \tag{6.3}$$

$$v(x)=-2\frac{d^2}{dx^2}\ln\det(I+Z_2(x)),\ x<0, \tag{6.4}$$

where the matrices Z_1 and Z_2 have as matrix elements the expressions

$$Z_{lj}^{(1)}=\frac{i\rho_l}{\alpha_l^{(+)}+\alpha_j^{(+)}}\,e^{i\left(\alpha_l^{(+)}+\alpha_j^{(+)}\right)x};$$

$$Z_{lj}^{(2)}=\frac{i\rho_l'}{\alpha_l^{'(+)}+\alpha_j^{'(+)}}\,e^{-i\left(\alpha_l^{'(+)}+\alpha_j^{'(+)}\right)x}.$$

The potential must be a continuous function if $s_{12}(k)$ is to sufficiently decrease as $|k|\to\infty$. In particular, when $m\leqslant n-2$ Eqs. (6.3) and (6.4) are continuous and will coincide at x = 0. A direct proof of this assertion is far from simple. One particular case of the resulting equations, when v(x) = 0 at x < 0 was presented in [37].

We now note that the explicit equations obtained by us for v(x) contain a logarithmic derivative of the determinant of a matrix. It turns out that this fact is not accidental. In can be proved that the potential v(x) is expressed by the equation

$$v(x)=-2\frac{d^2}{dx^2}\ln\ \Delta_1(x)=-2\frac{d^2}{dx^2}\ln\ \Delta_2(x) \tag{6.5}$$

in terms of the Fredholm determinants $\Delta_1(x)$ and $\Delta_2(x)$ of the Gel'fand–Levitan equations (4.1) and (4.2). The appearance of the finite-dimensional matrices W(x) and Z(x) in the examples we are considering is explained by the degeneracy of the corresponding kernels in these equations. We present a brief and formal derivation of Eq. (6.5). The rigorously justified method is too long to be presented here.

We will prove that

$$A_1(x,x)=\frac{d}{dx}\ln\Delta_1(x);\quad A_2(x,x)=-\frac{d}{dx}\ln\Delta_2(x),$$

after which Eq. (6.3) is implied by Eq. (5.10). Thus, suppose

$$\Delta_1(x)=\det(I+\Omega_x),$$

where $I+\Omega_x$ was introduced in Sec. 5 and has the form $P_XW_1P_X$, where P_X is a projector into $L_2(\mathbf{R})$.

$$P_x\psi(y)=\theta(y-x)\psi(y).$$

We now note that

$$\ln \Delta_1(x) = \mathrm{Tr} \ln (I + \Omega_x) = \mathrm{Tr} \ln (I + P_x \Omega_1),$$

where Tr is defined on the left in the space $L_2(x, \infty)$, and on the right, in $L_2(\mathbf{R})$. The operator $P_x\Omega_1 = Q_x$ is an integral operator in $L_2(\mathbf{R})$ with kernel

$$Q_x(y, z) = \theta(y-x)\, \Omega_1(y+z).$$

The last equation implies that

$$\frac{d}{dx} \ln \Delta_1(x) = \mathrm{Tr}\left((I + \Gamma_x) \frac{d}{dx} P_x \Omega_1\right).$$

Here we have introduced the resolvent Γ_x of the operator Q_x, i.e., an integral operator whose kernel satisfies the equation

$$\Gamma_x(y, z) + \theta(y-x)\left[\Omega_1(y+z) + \int_x^\infty \Gamma_x(y, t)\, \Omega_1(t+z)\, dt\right] = 0.$$

Comparing this equation to the Gel'fand–Levitan equation (4.1) we may verify that

$$\Gamma_x(x, x) = A_1(x, x).$$

The trace desired by us in this notation is expressed by

$$\frac{d}{dx} \ln \Delta_1(x) = -\int dy\, dz\, (\delta(y-z) + \Gamma_x(y, z))\; \delta(y-x)\, \Omega_1(z+y) = \\ = \Gamma_x(x, x) = A_1(x, x),$$

which also proves the first equation in Eqs. (6.5). The second equation is proved entirely analogously.

CHAPTER 2

SIMPLE GENERALIZATIONS AND APPLICATIONS

The technical apparatus described in the first chapter is carried over without significant variations to a number of one-dimensional problems in scattering theory. In this section we will consider several examples, limiting ourselves basically to only the formulation of the results, generalizing or modifying the assertions stated in Chap. 1. Moreover, we will consider in Sec. 4 an application of developed methods from scattering theory for solving nonlinear equations in the theory of one-dimensional continuous media.

1. Potentials with Distinct Asymptotics at Infinity

Here we will consider two examples of the Schröedinger operator

$$H\psi(x) = -\frac{d^2}{dx^2}\psi(x) + v(x)\psi(x),$$

where the potential $v(x)$, $x \in \mathbf{R}$, behaves differently as $x \to -\infty$ versus $x \to \infty$:

Example 1.

$$v(x) \to c^2, \quad x \to -\infty; \quad v(x) \to 0, \quad x \to \infty$$

was considered by V. S. Buslaev and V. L. Fomin [6].

Example 2.

$$v(x) \to \infty, \quad x \to -\infty; \quad v(x) \to 0, \quad x \to \infty$$

was studied by P. P. Kulish [14].

We will not exceed the limits of elementary stationary scattering theory. All the treatments can be embedded in an abstract scheme of scattering theory, but not very instructively. We will also not present any proofs, referring instead to the original works.

Let us pass to a description of the first example. We assume that

$$\int_{-\infty}^{0}(1+|x|)|v(x)-c^2|\,dx<\infty;$$
$$\int_{0}^{\infty}(1+|x|)|v(x)|\,dx<\infty. \tag{1.1}$$

Suppose $k_1=\sqrt{k^2-c^2}$ is defined so that $\operatorname{Im} k_1 \geqslant 0$ when $\operatorname{Im} k \geqslant 0$. The solutions $f_1(x, k)$ and $f_2(x, k)$ of the Schröedinger equation

$$\psi''(x)+k^2\psi(x)=v(x)\psi(x) \tag{1.2}$$

are distinguished by the asymptotic conditions

$$f_1(x, k)=e^{ik_1x}+o(1),\ x\to\infty;$$
$$f_2(x, k)=e^{-ik_1x}+o(1),\ x\to-\infty;$$

and are analytic functions for fixed x of the parameters k and k_1 in the upper half-plane. We have the integral representations

$$f_1(x, k)=e^{ikx}+\int_{x}^{\infty}A_1(x, y)\,e^{iky}dy,$$
$$f_2(x, k)=e^{-ik_1x}+\int_{-\infty}^{x}A_2(x, y)\,e^{-ik_1y}dy,$$

bounds similar to that presented in Sec. 1 Chap. 1 holding for the kernels $A_1(x, y)$ and $A_2(x, y)$. The solutions $u_1(x, k)$ and $u_2(x, k)$ of Eq. (1.2) are uniquely determined by the radiation principle and have the form

$$u_1(x, k)=s_{11}(k)\,f_2(x, k)=s_{12}(k)\,f_1(x, k)+f_1(x, -k);\quad k>0,$$

$$u_2(x, k)=s_{22}(k)\,f_1(x, k)=s_{21}(k)\,f_2(x, k)+f_2(x, -k);\quad k\geqslant c.$$

The coefficients $s_{ij}(k)$ determining the scattering matrix possess the following properties:

1. Unitarity:

$$\frac{k}{k_1}\overline{s_{22}(k)}s_{12}(k)+\overline{s_{21}(k)}\,s_{11}(k)=0,\quad k>c;$$
$$\frac{k}{k_1}|s_{22}(k)|^2+|s_{21}(k)|^2=1;\quad \frac{k_1}{k}|s_{11}(k)|^2+|s_{12}(k)|^2=1,\ k>c;$$
$$|s_{12}(k)|=1,\ 0<k<c;$$
$$s_{22}(c)=0\Rightarrow s_{21}(c)=-1;$$
$$s_{11}(0)=0\Rightarrow s_{12}(0)=-1.$$

2. Symmetry:

$$k_1s_{11}(k)=ks_{22}(k).$$

3. Analyticity: the coefficient $s_{11}(k)$ is the limiting value of a function analytic in the upper half-plane and having there simple poles on the imaginary axis at the points $k = i\varkappa_l$ with residues

$$\operatorname{Res} s_{11}(k)|_{k=i\varkappa_l} = i\gamma_l; \quad \gamma_l = \left(\int_{-\infty}^{\infty} f_1(x, i\varkappa_l) f_2(x, i\varkappa_l)\, dx \right)^{-1}.$$

4. Behavior as $|k| \to \infty$: there exist Fourier transformations

$$F_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s_{12}(k) e^{ikx} dk; \quad F_2(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s_{21}(\sqrt{c^2 + k^2}) e^{-ikx} dk$$

such that

$$\int_{a}^{\infty} \left| \frac{d}{dx} F_1(x) \right| (1 + |x|)\, dx \leqslant C(a);$$

$$\int_{-\infty}^{b} \left| \frac{d}{dx} F_2(x) \right| (1 + |x|)\, dx \leqslant C(b).$$

We shall bear in mind the modification of the unitarity and symmetry conditions.

These properties reconstruct the entire matrix S(k) in terms of the reflection coefficient $s_{12}(k)$. At the same time knowledge of s_{21} at $k > c$ is not sufficient for this purpose.

The Gel'fand–Levitan equation for the kernel $A_1(x, y)$ is unchanged:

$$A_1(x, y) + \Omega_1(x + y) + \int_{x}^{\infty} A_1(x, z) \Omega_1(z + y)\, dz = 0, \quad x < y.$$

Here

$$\Omega_1(x) = F_1(x) + \sum_{l=1}^{N} m_l^{(1)} e^{-\varkappa_l x},$$

where the function $F_1(x)$ has already been introduced, while

$$m_l^{(1)} = \left(\int_{-\infty}^{\infty} (f_1(x, i\varkappa_l))^2 dx \right)^{-1}.$$

The Gel'fand–Levitan equation for the kernel $A_2(x, y)$,

$$A_2(x, y) + \Omega_2(x + y) + \int A_2(x, z) \Omega_2(z + y)\, dz = 0, \quad x > y,$$

contains a new term

$$\Omega_2(x) = F_2(x) + \sum_{l=1}^{N} m_l^{(2)} e^{\varkappa_l x} + \frac{1}{2\pi} \int_{0}^{c} dk e^{-x\sqrt{c^2 - k^2}} |s_{11}(k)|^2.$$

Here the $m_l^{(2)}$ are related to the $m_l^{(1)}$ by the equation

$$m_l^{(1)} m_l^{(2)} = \frac{\sqrt{c^2 + \varkappa_l^2}}{\varkappa_l} \gamma_l^2.$$

An investigation of these equations and a proof for the necessity and sufficiency of these properties of S(k) corresponding to the potential v(x) satisfying the condition of Eq. (1.1) will be presented analogous to what was done in Chap. 1. We refer the reader to [6] for details.

Let us now pass to a description of the second example. In this case the Schröedinger equation (1.2) has a solution u(x, k) which possesses the asymptotic

$$u(x, k)=o(1), \quad x \to -\infty;$$
$$u(x, k)=e^{-ikx}+s(k)e^{ikx}+o(1), \quad x \to \infty,$$

where the reflection coefficient s(k) satisfies the unitarity condition $|s(k)| = 1$. At large k, s(k) rapidly oscillates. The inverse problem consists in reconstructing v(x) in terms of given s(k).

Let us refine the conditions on v(x) for which the results formulated below are true. The nature of the increase in v(x) as $x \to -\infty$ is difficult to express explicitly in terms of the asymptotic behavior of ln s(k) as $k \to \infty$. It is, however, possible to expand the class of potentials v(x) so that the necessary and sufficient properties of the corresponding reflection coefficients can be described. We will assume that for finite a,

$$\int_a^\infty (1+|x|)v(x)\,dx < C(a),$$

and the spectrum of the Sturm–Liouville problem

$$-\frac{d^2}{dx^2}\psi+v(x)\psi(x)=\lambda\psi(x), -\infty < x < x_0, \frac{d}{dx}\psi(x)\Big|_{x=x_0}=0$$

for some x_0 is semi-bounded from below and discontinuous. We will also assume that this operator H lacks a discontinuous spectrum.

The existence of the solution f(x, k), such that

$$f(x, k)=e^{ikx}+o(1), \quad x \to \infty,$$

its analytic properties in terms of a parameter k, and, in particular, the integral representation

$$f(x, k)=e^{ikx}+\int_x^\infty A(x, y)e^{iky}dy$$

have already been proved by methods known to us. The coefficient s(k) is uniquely determined by the fact that

$$u(x, k)=f(x, -k)+s(k)f(x, k)$$

is integrable in square in a neighborhood of $x=-\infty$. The properties of s(k) are as follows.

1. The Fourier transformation

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} s(k)e^{ikx}dk,$$

defined as a generalized function, coincides with an absolutely continuous function, such that

$$\int_a^\infty (1+|x|)\left|\frac{d}{dx}F(x)\right|dx < C(a).$$

2. s(k) can be represented in the form

$$s(k)=s_0(k)+O\left(\frac{1}{|k|}\right),$$

where $|s_0(k)|=1$ when $\operatorname{Im} k=0$ and $s_0(k)$ has meromorphic continuation onto the entire complex plane of the variable k, while when $\operatorname{Im} k>0$,

$$|s_0(k)| \leq e^{2\operatorname{Im} k x_0}.$$

3. The function $u(x, k) = f(x, -k) + f(x, k)s(k)$ will have an analytic continuation in the upper half-plane, such that the function $\rho(\lambda)$ defined by the equation

$$\int_{-\infty}^{\infty}\frac{d\rho(\lambda)}{\lambda-k^2}=\frac{u(x_0, k)}{\frac{d}{dx}u(x, k)\Big|_{x=x_0}}$$

is the spectral function of the Sturm–Liouville problem on the semi-axis $(-\infty, x_0)$ with boundary condition $\frac{d}{dx}\psi(x)=0$ when $x = x_0$.

We note that the statement of the latter condition includes, in addition to s(k), the solution f(x, k) of the Schröedinger equation (1.2), so that this property at first glance is not formulated solely in terms of s(k). In fact, the situation is not that bad. It is possible to solve a Gel'fand–Levitan–Marchenko equation for every function s(k) satisfying the first and second conditions and also the unitary condition

$$A(x, y)=F(x+y)+\int_{x}^{\infty}A(x, z)F(z+y)\,dz=0,\quad x<y,$$

at $x = x_0$ and to find the solution f(x, k) and its derivative at $x = x_0$. We next verify the third condition. Reconstruction of the potential v(x) at $x < x_0$ must be carried out using the Gel'fand–Levitan procedure for solving the inverse Sturm–Liouville problem with discontinuous spectrum. Here we conclude the description of the second example and refer the reader for further details to [14].

2. Canonical System

Many methods developed for the one-dimensional Schröedinger operator are naturally carried over (and sometimes even simplified) to the differential operator of a canonical linear system, which we write in matrix notation

$$H=J\frac{d}{dx}+Q(x).$$

Here J is a simplicial matrix and Q is a real symmetric matrix function with zero trace

$$J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};\quad Q(x)=\begin{pmatrix}p(x) & q(x)\\ q(x) & -p(x)\end{pmatrix}.$$

The technical tools for studying the spectral properties of such an operator were developed by M. G. Krein and his students and also by M. G. Gasymov and B. M. Levitan. Corresponding references can be found in [13, 7]. In both these works H is considered on the semi-axis $0\leqslant x<\infty$. In this section we will present the fundamental equations and results of the stationary scattering theory and inverse problem for an operator H on the entire axis, following the thesis of Takhtadzhyan [20].

We consider the system of differential equations

$$J\frac{d}{dx}\psi(x)+Q(x)\psi(x)=k\psi(x),\quad -\infty<x<\infty, \tag{2.1}$$

which plays in our case the role of the Schröedinger equation. The system (2.1) has solutions $f_1(x, k)$ and $f_2(x, k)$ which are column vectors and have the asymptotic

$$f_1(x, k)=\begin{pmatrix}i\\ 1\end{pmatrix}e^{ikx}+o(1),\quad x\to\infty,$$
$$f_2(x, k)=\begin{pmatrix}-i\\ 1\end{pmatrix}e^{-ikx}+o(1),\quad x\to-\infty.$$

To prove the existence of these solutions it is sufficient to assume that the coefficients p(x) and q(x) of Q(x) are absolutely integrable functions,

$$\int_{-\infty}^{\infty}|p(x)|\,dx<\infty;\quad \int_{-\infty}^{\infty}|q(x)|\,dx<\infty. \tag{2.2}$$

The components $f_i^{(1)}(x, k)$ and $f_i^{(2)}(x, k)$ of $f_1(x, k)$ and $f_2(x, k)$ for fixed x have an analytic continuation in the upper half-plane of the parameter k. For large k

$$f_1(x, k)\, e^{-ikx} = \binom{i}{1} + o(1); \quad f_2(x, k)\, e^{ikx} = \binom{-i}{1} + o(1).$$

For real k the pairs of solutions $f_1(x, k)$, $\overline{f_1(x, k)}$ and $f_2(x, k)$, $\overline{f_2(x, k)}$ form two fundamental systems of solutions of Eq. (2.1). The conversion factors $a(k)$ and $b(k)$ are introduced by the equations

$$f_2(x, k) = a(k)\overline{f_1(x, k)} + b(k) f_1(x, k);$$
$$f_1(x, k) = -\overline{b(k)} f_2(x, k) + a(k)\overline{f_2(x, k)}.$$

These coefficients satisfy the identity

$$1 + |b(k)|^2 = |a(k)|^2$$

and can be expressed in terms of $f_1(x, k)$ and $f_2(x, k)$ by means of the equation

$$a(k) = \frac{1}{2i}\{f_1(x, k), f_2(x, k)\}.$$

Here $\{f, g\}$, an analog of the Wronskian, is defined as a bilinear form of the matrix

$$\{f, g\} = f^{(1)} g^2 - f^{(2)} g^{(1)}.$$

We will see that $a(k)$ has an analytic continuation on the upper half-plane and is bounded there as $k \to \infty$,

$$a(k) = 1 + o(1).$$

The function $a(k)$ does not have zeroes in the upper half-plane since such zeroes would correspond to complex eigenvalues of the formally self-adjoint system (2.1).

Knowledge of the fundamental system of solutions of the system of equations allows us to introduce solutions $u(x, k)$ satisfying the radiation principle and to develop scattering theory similar to what was done in Sec. 2 Chap. 1 for the example of the Schröedinger equation with decreasing potential. We will not carry this out here, limiting ourselves to stating that the matrix $S(k)$ is determined by the conversion factors $a(k)$ and $b(k)$ by means of equations that coincide letter for letter with those presented here. The sole difference is that the parameter k now runs through the entire real axis. In particular, the entire matrix $S(k)$ can be reconstructed in terms of one of the reflection coefficients

$$s_{12}(k) = \frac{b(k)}{a(k)}; \quad s_{21}(k) = -\frac{\overline{b(k)}}{a(k)}.$$

We have for the solutions $f_1(x, k)$ and $f_2(x, k)$ the integral representations

$$f_1(x, k) = f_0(x, k) + \int_x^\infty A_1(x, y) f_0(y, k)\, dy;$$

$$f_2(x, k) = \overline{f_0(x, k)} + \int_{-\infty}^x A_2(x, y) \overline{f_0(y, k)}\, dy.$$

Here the column vector $f_0(x, k)$ has the form

$$f_0(x, k) = \binom{i}{1} e^{ikx},$$

and the kernels $A_1(x, y)$ and $A_2(x, y)$ are real matrix functions absolutely integrable with respect to y for fixed x.

The kernels $A_1(x, y)$ and $A_2(x, y)$ can be used to construct the Volterra transformation operators and, in particular, to express in terms of them the completeness condition. We will not carry this out here and present only equations expressing $Q(x)$ in terms of these kernels

$$Q(x) = -[A_1(x, x), J] = [A_2(x, x), J]$$

and a formulation of one of the Gel'fand–Levitan equations

$$A_1(x, y) + F_1(x+y) + \int_x^\infty A_1(x, z) F_1(z+y)\,dz = 0, \quad x < y.$$

Here the matrix function $F_1(x)$ is introduced by the equation

$$F_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re}\left(s_{12}(k)\begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} e^{ikx}\right) dk.$$

It is possible to prove, based on a study of this equation and its analog for the kernel $A_2(x, y)$, that the properties of $S(k)$, viz., its unitarity and symmetry, the analyticity of the transmission coefficient, and the absolute integrability of the Fourier transformations $F_1(x)$ and $F_2(x)$ of the reflection coefficients $s_{12}(k)$ and $s_{21}(k)$

$$\int_a^\infty |F_1(x)|\,dx < C(a), \quad \int_{-\infty}^{b} |F_2(x)|\,dx < C(b)$$

are necessary and sufficient properties of the scattering matrix for a canonical system with matrix $Q(x)$ satisfying the condition of Eq. (2.2).

We will now show how the canonical system is a generalization of the Schröedinger equation. If the matrix $Q(x)$ has the form

$$Q(x) = \begin{pmatrix} 0 & q(x) \\ q(x) & 0 \end{pmatrix}, \tag{2.3}$$

the system of Eqs. (2.1) is equivalent to the Schröedinger equation with operator H of the form

$$H = \left(\frac{d}{dx} + q(x)\right)\left(-\frac{d}{dx} + q(x)\right) = -\frac{d^2}{dx^2} + q^2(x) + q'(x),$$

which is evidently positively defined. That the equation

$$s_{ij}(-k) = \overline{s_{ij}(k)},$$

which also in fact reduces the range of variation of the parameter k to the semi-axis, is real constitutes a necessary and sufficient condition on the scattering data corresponding to the matrix $Q(x)$ of the form of Eq. (2.3).

We note in conclusion that the case of the system (2.1), when the matrix $Q(x)$ has the form

$$Q(x) = \begin{pmatrix} m+p & q \\ q & -m-p \end{pmatrix},$$

where m is a positive constant (system of Dirac equations with mass), was recently considered by I. S. Frolov [30].

3. Trace Formula

In this section we will derive identities relating the momenta, the function $\ln|a(k)|$, where $a(k)$ is one of the conversion factors, with the integral of polynomials formed by derivatives of the potentials in the one-dimensional Schröedinger operator or in the operator of the canonical system. These equations first appeared apparently in [5], which in turn developed a paper by I. M. Gel'fand and V. M. Levitan [9]. In these works it was shown why such identities can naturally be called trace formulas. The derivation which we present will not use general operator-theoretic concepts and is taken from [10].

We first consider the case of the Schröedinger operator. We will assume that the potential $v(x)$ is a Schwartz-type function. Then the reflection coefficients $s_{12}(k)$ and $s_{21}(k)$ decrease as $|k| \to \infty$ more rapidly than $|k|^{-n}$ for any $n > 0$. The equation

$$\ln a(k)=\frac{1}{\pi i}\int\frac{\ln|a(k')|}{k'-k}\,dk'+2i\sum_{l=1}^{N}2\operatorname{arctg}\frac{\varkappa_l}{k},$$

which constitutes one variant of Eq. (I.2.9) implies that ln a(k) has a decomposition in negative degrees of k,

$$\ln a(k)=\sum_{n=1}^{\infty}\frac{c_n}{k^n}, \tag{3.1}$$

where

$$c_{2j}=0;\ c_{2j+1}=-\frac{1}{\pi i}\int_{-\infty}^{\infty}k^{2j}\ln|a(k)|\,dk-\frac{2}{2j+1}\sum_{l=1}^{N}(i\varkappa_l)^{2j+1}.$$

Even degrees vanish as a consequence of the evenness of the integrand $\ln|a(k)|$. Bearing in mind the interpretation of $a(\sqrt{\lambda})$ as a regularized characteristic determinant of H, we can say that the c_{2j+1} are proportional to regularized traces of the half-integer degrees $H^{\frac{2j+1}{2}}$ of H, while such traces vanish for integral degrees.

We now calculate the coefficients c_n directly in terms of v(x). This calculation can be interpreted as the definition of the matrix trace of the operators $H^{j+\frac{1}{2}}$. Identities which are obtained following the setting equal of the thus obtained expressions for c_n are also called the trace formulas. This interpretation will not be evident in the elementary calculations following.

We consider the function

$$\chi(x,k)=\ln f_1(x,k)-ikx.$$

It can be proved that this function is analytic in k when Im k > 0 and sufficiently large |k| for any fixed x. For large |x| it has the asymptotics

$$\chi(x,k)=o(1),\ x\to\infty,\ \chi(x,k)=\ln a(k)+o(1),\ x\to-\infty,$$

so that

$$\ln a(k)=-\int_{-\infty}^{\infty}\sigma(x,k)\,dx,$$

where

$$\sigma(x,k)=\frac{d}{dx}\chi(x,k).$$

The Schröedinger equation implies that σ(x, k) satisfies the Riccati equation

$$\frac{d}{dx}\sigma+\sigma^2-v+2ik\sigma=0,$$

which can be used to determine the asymptotic decomposition

$$\sigma(x,k)=\sum_{n=1}^{\infty}\frac{\sigma_n(x)}{(2ik)^n}. \tag{3.2}$$

We find for the coefficients σ_n(x) the recursion relations

$$\sigma_n(x)=-\frac{d}{dx}\sigma_{n-1}(x)-\sum_{k=1}^{n-1}\sigma_{n-k-1}(x)\sigma_k(x),\ n=2,\ldots;\ \sigma_1(x)=v(x).$$

The first few coefficients have the form

$$\sigma_2(x) = -\frac{d}{dx} v(x);\ \sigma_3(x) = -v^2(x) + \frac{d^2}{dx^2} v(x);$$

$$\sigma_4(x) = -\frac{d^3}{dx^3} v(x) + 4v(x) \frac{d}{dx} v(x);$$

$$\sigma_5(x) = \frac{d^4}{dx^4} v(x) - 6v(x) \frac{d^2}{dx^2} v(x) - 5\left(\frac{d}{dx} v(x)\right)^2 + 2v^3(x).$$

We will see that $\sigma_2(x)$ and $\sigma_4(x)$ are total derivatives. This property is also preserved for all $\sigma_n(x)$ for even n. Returning to $\ln a(k)$ we will verify that the coefficients c_n in the decomposition of Eq. (3.1) are written in the form

$$c_{2j} = 0;\ c_{2j+1} = -\left(\frac{1}{2i}\right)^{2j+1} \int_{-\infty}^{\infty} \sigma_{2j+1}(x)\, dx,$$

so that, for example

$$c_1 = -\frac{1}{2i} \int_{-\infty}^{\infty} v(x)\, dx;\ c_3 = -\frac{1}{8i} \int_{-\infty}^{\infty} v^2(x)\, dx;$$

$$c_5 = -\frac{1}{32i} \int_{-\infty}^{\infty} \left[2v^3(x) + \left(\frac{d}{dx} v(x)\right)^2\right] dx.$$

We have thus found the set of identites

$$\int_{-\infty}^{\infty} \sigma_{2j+1}(x)\, dx = \frac{-(2i)^{2j+2}}{\pi} \int_{0}^{\infty} k^{2j} \ln|a(k)|\, dk - 2\, \frac{(2)^{2j+1}}{2j+1} \sum_{l=1}^{N} \varkappa_l^{2j+1}$$

called the trace formulas. Their interpretation in terms of the traces of the half-integer degrees of H is carried out in a particularly self-evident way due to the presence in the right side of the sum of half-integer degrees of its discrete eigenvalues.

The differential operator of the canonical system is considered analogously. We will use as the function (x, k) following [11], the equation

$$\sigma(x, k) = \frac{d}{dx} \ln\left(\frac{1}{2i} (f_1^{(1)}(x, k) + i f_1^{(2)}(x, k))\right) - ik.$$

It can be proved that

$$\ln a(k) = -\int_{-\infty}^{\infty} \sigma(x, k)\, dx.$$

The canonical system (2.1) for $\sigma(x, k)$ implies the equation

$$2ik\sigma = \sigma^2 + w(x) \frac{d}{dx}\left(\frac{1}{w(x)} \sigma\right) - |w|^2,\ w = p + iq,$$

which can be used for the asymptotic decomposition of the type of Eq. (3.2). Here the first coefficients $\sigma_n(x)$ have the form

$$\sigma_1(x) = -|w|^2;\ \sigma_2 = -w\bar{w}';\ \sigma_3 = -w\bar{w}'' + |w|^4,$$

$$\sigma_4 = -w\bar{w}''' + ww'|w|^2 + 4w^2\bar{w}\bar{w}'.$$

As a result an asymptotic decomposition of the form of Eq. (3.1) is obtained for $\ln a(k)$, where all the coefficients c_n are in general nonzero, the two different methods for calculating them leading to the identities

$$\int_{-\infty}^{\infty} \sigma_n(x)\,dx = (2i)^n \int_{-\infty}^{\infty} k^{n-1} \ln |a(k)| dk.$$

These formulas can be interpreted in terms of regularized traces of the degrees of H of the canonical system.

If the condition for infinite differentiability of v(x) or of p(x) and q(x), which we have used in deriving the identities, does not hold, the number of true trace formulas is determined by the number of the contiuous derivatives. Here the absolute convergence of one integral in these identities guarantees the convergence of the second integral. Thus, the trace formulas are an indirect means for obtaining information on the bahavior of scattering data in terms of properties of the potential.

Identities similar to those presented here for the radial Schröedinger operators and their relativistic analogs were also used in a number of works [31-33] by Italian physicists for more meaningful assertions on the inverse problem.

4. Nonlinear Evolutionary Equations

The work of Kruskal et al. [42] and the succeeding work of Lax [44] opened a new range of application of scattering theory. That is, it turned out that it is possible to describe using the scattering problem the general solution of certain nonlinear evolutionary equations with a single spatial variable. Here we will describe two characteristic examples of such applications.

1. Korteweg-de Vries equation

$$\frac{\partial v(x, t)}{\partial t} = 6\, v(x, t) \frac{\partial}{\partial x}\, v\,(x, t) - \frac{\partial^3}{\partial x^3}\, v\,(x, t). \tag{4.1}$$

2. Nonlinear Schröedinger equation

$$-i\frac{\partial w(x, t)}{\partial t} = \frac{\partial^2 w(x, t)}{\partial x^2} - 2w(x, t)\,|\,w(x, t)\,|^2. \tag{4.2}$$

In the first example v = v(x, t) is a real function while w(x, t) in the second example is complex, w(x, t) = p(x, t) + iq(x, t). We will assume that they rapidly decrease for large |x| and fixed t.

We will prove that both functions constitute a motion equation for an infinite-dimensional Hamiltonian system. We will recall that the definition of such systems includes the simplicial manifold (M, Ω), where M is a differentiable even-dimensional manifold and Ω is a closed nondegenerate 2-form on it, and the distinguished function (Hamiltonian) is $h: M \to R$. The trajectories of the Hamiltonian system are given by the differential equations

$$\dot{\xi} = J\,(dh\,(\xi)), \tag{4.3}$$

where ξ is a point of the manifold, $\dot{\xi}$ is a tangential vector to the trajectory at the point ξ, and J is the map of the 1-form in the vector fields defined by the form Ω (cf., for example, [2]).

We now note that Eq. (4.1) can be written in the form

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\frac{\delta h\,[v]}{\delta v\,(x)}, \tag{4.4}$$

where the functional h[v] has the form

$$h\,[v] = \int_{-\infty}^{\infty} \left[v^3(x) + \frac{1}{2}\left(\frac{dv\,(x)}{dx}\right)^2\right] dx.$$

Comparing Eqs. (4.3) and (4.4) we see that the latter equation in fact is of Hamiltonian form, where h[v] plays the role of the Hamiltonian, while the antisymmetric operator $J = \partial/\partial x$ generates a simplicial structure in the set of functions v(x). The corresponding 2-form, which we write as a bilinear form of the variations $\delta_1 v$ and $\delta_2 v$ of v(x) is determined by an operator inverse to J and having the form

$$\Omega(\delta_1 v, \delta_2 v) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dy\, [\delta_1 v(x)\, \delta_2 v(y) - \delta_1 v(y)\, \delta_2 v(x)]. \tag{4.5}$$

The similar objects for Eq. (4.2) have the form

$$h[w] = \frac{1}{2} \int_{-\infty}^{\infty} \left(\left|\frac{d}{dx} w\right|^2 + |w|^4\right) dx$$

and

$$\Omega(\delta_1 w, \delta_2 w) = \frac{1}{2i} \int_{-\infty}^{\infty} (\delta_1 w(x)\, \overline{\delta_2 w(x)} - \overline{\delta_1 w(x)}\, \delta_2 w(x))\, dx,$$

correspondingly. We will see that the simplicial structure in the space of complex functions w(x) is induced by the natural complex structure of the real space of functions p(x) and q(x), where w = p + iq.

The scattering problem is used to describe substitution of variables in Eqs. (4.1) and (4.2), under which they become explicitly solvable. We describe the corresponding scheme first for the example of Korteweg-de Vries equation. We consider the function v(x) as a potential in the Schröedinger operator H studied in the preceding chapter. Suppose $(s_{12}(k), \varkappa_l, m_l^{(1)})$ are the corresponding scattering data which in turn uniquely define v(x). It turns out that the functional h[v] and the form Ω cannot be expressed explicitly in terms of the scattering data and their variations. The corresponding formulas have the form

$$h[v] = 16 \int^{\infty} k^3 P(k)\, dk - \frac{32}{5} \sum_{l=1}^{N} p_l^{5/2} \tag{4.6}$$

and

$$\Omega = \int_0^{\infty} \delta_1 P(k)\, \delta_2 Q(k)\, dk + \sum_{l=1}^{N} \delta_1 p_l \delta_2 q_l - (1 \leftrightarrow 2), \tag{4.7}$$

where

$$\begin{gathered} P(k) = \frac{2k}{\pi} \ln |a(k)|;\ Q(k) = 2 \arg b(k); \\ p_l = \varkappa_l^2;\ q_l = 2 \ln b_l;\ b_l = i m_l^{(1)} \frac{d}{dk} a(k)\Big|_{k=i\varkappa_l}, \end{gathered} \tag{4.8}$$

the conversion factors $a(k)$ and b(k) being constructed in terms of the initial scattering data.

We will see that the new variables are explicitly canonical, where P(k), p_l, $l = 1, \ldots,$ N play the role of canonical momenta, while Q(k), q_l, $l = 1, \ldots,$ N play the role of canonical coordinates. Moreover, the Hamiltonian h[v] turns out to be a function only of the momenta, so that the Hamiltonians of the equation in new variables is trivially solved. Turning to the scattering data, the corresponding solution can be written in the form

$$s_{12}(k, t) = s_{12}(k, 0)\, e^{8ik^3 t};\ \varkappa_l(t) = \varkappa_l(0);\ m_l^{(1)}(t) = e^{8\varkappa_l^3 t} m_l^{(1)}(0). \tag{4.9}$$

These facts also make it possible to solve the Korteweg-de Vries equation using an auxiliary scattering problem. Suppose v(x, 0) = v(x) is a Schwartz-type function defining the Cauchy data for this equation. The sequence of maps

$$v(x, 0) \to (s_{12}(k, 0), m_l^{(1)}(0), \varkappa_l(0)) \to (s_{12}(k, t), m_l^{(1)}(t), \varkappa_l(t)) \to v(x, t)$$

defines the corresponding solution. Here the first left arrow denotes the solution of the direct scattering problem for the Schröedinger equation with potential v(x) = v(x, 0), the next arrow is defined in Eq. (4.9), and finally, the last arrow constitutes the solution of the inverse problem.

It is precisely this proposition which is presented in the first work of Kruskal et al. [42]. His Hamiltonian interpretation presented here was obtained in [10].

Analogous formulas

$$h[w]=4\int_{-\infty}^{\infty} k^2 P(k)\,dk \tag{4.10}$$

and

$$\Omega=\int_{-\infty}^{\infty}[\delta_1 P(k)\delta_2 Q(k)-\delta_1 Q(k)\delta_2 P(k)]\,dk, \tag{4.11}$$

where

$$P(k)=\frac{1}{\pi}\ln a(k);\quad Q(k)=\arg b(k),$$

can be found for the nonlinear equation of the second example. Here $a(k)$ and $b(k)$ are the conversion factors for the canonical system (2.1) constructed in terms of the functions $p(x) = \mathrm{Re}\, w(x)$ and $q(x) = \mathrm{Im}\, w(x)$. We note that no factor k was present in the definition of the momentum P(k) and that the range of variation of the variable k is now the entire axis. These formulas were found in the thesis of L. A. Takhtadzhyan [20]. They imply that the general solution of the boundary-value problem for the equation is provided for by the sequence of maps

$$w(x,0)\to(a(k,0),\ b(k,0))\to(a(k,t),\ b(k,t))\to w(k,\ t),$$

where

$$a(k,t)=a(k,\ 0);\ b(k,\ t)=e^{4ik^2t}b(k,\ 0).$$

The similar equation

$$-i\,\frac{\partial\psi(x,\ t)}{\partial t}=\frac{\partial^2\psi(x,\ t)}{\partial x^2}+\varkappa\psi(x,\ t)\,|\psi(x,\ t)|^2,\ \ \varkappa>0,$$

(bearing in mind the opposite sign in front of the nonlinearity) was previously considered by V. E. Zakharov and A. B. Shabat [11]. The corresponding scattering problem is non-self-adjoint and has yet to be mathematically investigated.

Let us now dwell briefly on the derivation of Eq. (4.6), (4.7), (4.10), and (4.11). We consider only the Korteweg-de Vries equation, since the second example is considered analogously. Equation (4.6) has already been derived by us in the preceding section and constitutes the identity for the third trace. In fact the coefficient c_5 is proportional to the functional h[v]. The right side of Eq. (4.6) arises if we bear in mind the definition of Eq. (4.8). We note that the preceding results imply that all the functional c_{2j+1} of the function v(x) preserve their values for the Korteweg-de Vries equation. In fact the trace identities demonstrate that all these functionals depend only on momentum-type variables which do not vary with time. This observation provides a simple and exhaustive approach to the description and completeness problem of the motion integrals for the Korteweg-de Vries equation, which has been dealt with in a broad literature. References can be found for example in [43].

To derive Eq. (4.7) for the form Ω we note that it is possible to obtain using the Gel'fand–Levitan equation an expression for the variation of the potential v(x) in terms of the variations of the scattering data,

$$\delta v(x)=\frac{d}{dx}\Bigg[\frac{1}{\pi}\int_{-\infty}^{\infty}\delta s_{12}(k)\,f_1^2(x,\ k)\,dk+ \\ +2\sum_{l=1}^{N}\left(f_1^2(x,\ i\varkappa_l)\,\delta m_l^{(1)}+2im_l^{(1)}f(x,\ i\varkappa_l)\,\dot f(x,\ i\varkappa_l)\,\delta\varkappa_l\right)\Bigg].$$

Here $s_{12}(k)$, $f_1(x, k)$, $\varkappa_l$, and $m_l^{(1)}$ are objects corresponding to the potential v(x) and $\dot{f}_1(x, k)$ is the derivative of $f_1(x, k)$ with respect to k. Calculation of the form Ω is subsequently found by substituting this expression into Eq. (4.5) and calculating the integral with respect to x and y. For this purpose the following equation turns out to be useful

$$\{f^2(x, k), f^2(x, l)\} = \frac{1}{l^2 - k^2} \frac{d}{dx} (\{f(x, k), f(x, l)\})^2,$$

which follows simply from the Schröedinger equation. Details on the calculations can be found in [10], to which we refer the reader. By carrying out the calculations presented there, they can be easily carried over to the second example considered by us and Eq. (4.11) can be obtained. Equation (4.10) constitutes an identity for the traces No. 3 for the canonical system.

With this we conclude the description of an unexpected application of the formalism of scattering theory for solving one-dimensional nonlinear equations. The inherent reasons why this scheme works as well as its range of application has yet to be clarified. One "experimental" approach towards its use is to consider a one-dimensional differential operator for which the direct and inverse scattering problems are investigated and to describe the trace identities. We find a simplicial structure on the set of coefficients of this operator which can be explicitly expressed in terms of the scattering data. Then, the equations generated by this structure and by some functional of the trace formulas as a Hamiltonian can turn out to be exactly solvable. Finally, this scheme has not been studied to any great extent, but it must be used for lack of a better scheme. Equation (4.2) was found precisely by this method.

Searches for new scattering problems that can be studied are of particular interest. Generalizations of the Schröedinger equation or of the canonical system to the case of vector functions are obvious candidates. The monograph [1] demonstrated that most results known for scalar equations can be carried over without any difficulties in the case of vector equations. Equations of higher orders however are more promising. For example, V. E. Zakharov has recently proved that the third-order differential operator

$$Hu = i\frac{d^3}{dx^3} + 2ip(x)\frac{d}{dx} + i\frac{dp(x)}{dx} + q(x) \tag{4.12}$$

generates the nonlinear equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial^2 u(x, t)}{\partial x^2} + 2\left(\frac{\partial u(x, t)}{\partial x}\right)^2 + 2u(x, t)\frac{\partial^2 u(x, t)}{\partial x^2} + \\ + \frac{1}{4}\frac{\partial^4 u(x, t)}{\partial x^4},$$

which plays the role of a continuum analog of the nonlinear Fermi–Pasta–Ulam problem. The inverse problem of scattering theory for the operator of Eq. (4.12) has yet to be solved.

CHAPTER 3

THREE-DIMENSIONAL SCHRÖEDINGER OPERATOR

In this chapter we will consider the Schröedinger operator in a space of functions depending on the three variables

$$H = -\Delta + v(x) = H_0 + V.$$

Here $x \in \mathbf{R}^3$, Δ is the Laplace operator, and v(x) is a function which we will assume to be real, bounded, and sufficiently rapidly decreasing at infinity.

Spectral analysis of this operator is significantly more complex than the one-dimensional Schröedinger operator considered in Chap. 1. This particularly relates to the detailed investigation of such properties as continuity and the asymptotic behavior of the scattering amplitudes, which must be carried out in order to discuss sufficient or necessary conditions on this function corresponding to a potential v(x) of a given class. We will, therefore, in this chapter present only a formal scheme for solving the inverse problem, not making explicit each time under which conditions on v(x) do the discussions hold. A traditional condition on v(x) under which most of the bounds presented below hold asserts that

$$|v(x)| \leqslant C(1+|x|)^{-3-\varepsilon},\ \varepsilon>0.$$

Under this condition the operator H defined in $\mathfrak{H}=L_2(\mathbf{R}^3)$ on the dense domain $\mathfrak{D}=W_2^2(\mathbf{R}^3)$ is self-adjoint.

We consider a three-dimensional case only for the sake of definiteness. All equations generalize without difficulty to the case of arbitrary $n \geq 2$. In appropriate places we will note analogies or divergences from the one-dimensional case treated in Chap. 1. The material set forth first appeared in [29].

1. Scattering Theory

The construction of scattering theory for the pair of operators H and H_0 in a stationary variant is based on the existence of a set of solutions $u^{(\pm)}(x, k)$ of the Schröedinger equation

$$\Delta\psi(x)+k^2\psi(x)=v(x)\psi(x) \tag{1.1}$$

with the following asymptotic behavior at infinity, i.e., as $|x| \to \infty$,

$$u^{(\pm)}(x,\ k)=e^{i(k,x)}+\frac{e^{\pm i|k||x|}}{|x|}f^{(\pm)}(k,\ n)+o\left(\frac{1}{|x|}\right) \tag{1.2}$$

(radiation principle). Here

$$k\in\mathbf{R}^3,\ |x|=(x,\ x)^{1/2},\ |k|=(k,\ k)^{1/2},\ n=\frac{x}{|x|}.$$

The solutions $u^{(\pm)}(x, k)$ are similar to the sets of solutions $u_1^{(\pm)}(x, k)$ and $u_2^{(\pm)}(x, k)$ of Sec. 2 Chap. 1. The role of the index $i = 1, 2$ in played by the direction $\alpha = k/|k|$ of the vector k, which runs through the unit sphere S^2.

An existence proof and an investigation of the solutions $u^{(\pm)}(x, k)$ was carried out by A. Ya. Povzner [18, 19]. The range of the discussions presented in Chap. 1 and based on the existence of a fundamental system of solutions of the Schröedinger equation is not carried over to our case. Therefore, it is necessary to study directly the integral equations of scattering theory

$$u^{(\pm)}(x,\ k)=e^{i(k,x)}+\int G^{(\pm)}(x-y,\ |k|)\,v(y)\,u^{(\pm)}(y,\ k)\,dy. \tag{1.3}$$

Here $G^{(\pm)}(x,\ |k|)$ is Green's function for the Helmholtz equation

$$\Delta G^{(\pm)}(x,\ |k|)+k^2G^{(\pm)}(x,\ |k|)=\delta(x),$$

which can be uniquely defined by the radiation principle. The explicit equations

$$G^{(\pm)}(x,|k|)=-\frac{1}{4\pi}\frac{e^{\pm i|k||x|}}{|x|}$$

well known in the three-dimensional case are obtained from the general equation

$$G^{\pm}(x,|k|)=\left(\frac{1}{2\pi}\right)^n\int\frac{e^{i(l,x)}}{k^2-l^2\pm i0}dl \tag{1.4}$$

after calculation of the integral. In the last equation the well-known generalized function $(x \pm i0)^{-1}$ occurs.

The investigation of A. Ya. Povzner is based on Fredholm theory for Eqs. (1.3). An important role is played by the Kato theorem [35], which implies that homogeneous equations corresponding to Eq. (1.3) for real k do not have nontrivial constrained solutions. A. Ya. Povzner proved that the solutions $u^{(\pm)}(x, k)$ form a complete orthonormalized system of eigenfunctions of the continuous spectrum of H, which fills the entire positive semi-axis. This spectrum has uniform infinite multiplicity, so that the eigenfunctions are numbered in addition to the eigenvalue k^2 by the point $\alpha \in S^2$. Besides a continuous spectrum, H can have a finite number of nonpositive eigenvalues of finite multiplicity. To simplify the equations we will assume that the entire discontinuous spectrum of H consists of a single negative eigenvalue, which we denote by $-\varkappa^2$. The corresponding normalized eigenfunction, which can be assumed to be real, is denoted by u(x).

The completeness and orthogonality conditions on the function $u^{(\pm)}(x, k)$ are written in the form

$$\left(\frac{1}{2\pi}\right)^3 \int \overline{u^{(\pm)}(x, k)}\, u^{(\pm)}(y, k)\, dk + u(x)\, u(y) = \delta(x - y); \tag{1.5}$$

$$\left(\frac{1}{2\pi}\right)^3 \int u^{(\pm)}(x, k)\, \overline{u^{(\pm)}(x, l)}\, dx = \delta(k - l). \tag{1.6}$$

The scattering amplitude $f(k, l)$ is simply related to the function $f^{(+)}(k, n)$ describing the asymptotic of the solutions $u^{(+)}(x, k)$

$$f^{(+)}(k, n) = -2\pi^2 f(k, l); \quad k^2 = l^2; \quad n = \frac{l}{|l|},$$

and can be expressed in terms of the solution $u^{(+)}(x, k)$ by the equation

$$f(k, l) = \left(\frac{1}{2\pi}\right)^3 \int e^{-i(l,x)}\, v(x)\, u^{(+)}(x, k)\, dx, \tag{1.7}$$

which is an analog of Eq. (I.1.22) and (I.1.23).

We now present a relation between the functions $u^{(\pm)}(x, k)$ and the wave operators. For this purpose we introduce a diagonal representation for H_0. Suppose the space $\mathfrak{H}_0$ is formed by the functions $\varphi(\lambda, \alpha)$, defined on $R_+ \times S^2$ and having the scalar product

$$(\varphi, \varphi')_0 = \int_0^\infty \frac{\sqrt{\lambda}\, d\lambda}{2} \int_{S^2} d\alpha\, \varphi(\lambda, \alpha)\, \overline{\varphi'(\lambda, \alpha)},$$

where $d\alpha$ is an element of surface of the sphere S^2. We define the isomorphism $T_0 : \mathfrak{H} \to \mathfrak{H}_0$ by the formula

$$\psi(x) \to T_0\psi = \varphi(\lambda, \alpha); \quad \varphi(\lambda, \alpha) = \left(\frac{1}{2\pi}\right)^{3/2} \int e^{i\sqrt{\lambda}(x,\alpha)}\, \psi(x)\, dx.$$

The operator T_0 is unitary,

$$T_0^* T_0 = I; \quad T_0 T_0^* = I_0$$

and carries H_0 into a λ-multiplication operator,

$$H_0\psi(x) \to \lambda\varphi(\lambda, \alpha).$$

We introduce now two more maps, $T_\pm : \mathfrak{H} \to \mathfrak{H}_0$ and

$$\psi \to T_\pm\psi = \varphi_\pm; \quad \varphi_\pm(\lambda, \alpha) = \left(\frac{1}{2\pi}\right)^3 \int u^{(\pm)}(x, \sqrt{\lambda}\alpha)\, \psi(x)\, dx.$$

The completeness and orthogonality equations of Eqs. (1.5) and (1.6) are written in terms of them as follows:

$$T_\pm^* T_\pm = I - P; \quad T_\pm T_\pm^* = I_0.$$

Here P is a projector into $\mathfrak{H}$ on a one-dimensional subspace spanned by the function $u(x)$.

The wave operators $U^{(\pm)}$ are given by

$$U^{(\pm)} = T_\pm^* T_0. \tag{1.8}$$

This fact can be proved using the scheme presented in Sec. 2 of Chap. 1. Detailed discussions as regards this proof can be found in the article of Ikebe [34], which also contains several refinements of works of A. Ya. Povzner.

Let us now relate the scattering operator S defined by the formula

$$U^{(-)} = U^{(+)}S,$$

to the scattering amplitude f(k, l). We note for this purpose that the solutions $u^{(+)}(x, k)$ and $u^{(-)}(x, k)$ are linearly independent

$$u^{(+)}(x, k) = u^{(-)}(x, k) + \int g(k, l)\,\delta(k^2 - l^2)\,u^{(-)}(x, l)\,dl, \tag{1.9}$$

and by comparing their asymptotics we note that the kernel g(k, l) coincides with the scattering amplitude,

$$g(k, l) = -2\pi i f(k, l), \tag{1.10}$$

which in turn must satisfy a unitarity-type relationship. To write the latter it is convenient to introduce on $\mathfrak{H}_0$ an operators $\hat{S}$ by the equation

$$\hat{S}\varphi(\lambda, \alpha) = \varphi(\lambda, \alpha) - i\pi\sqrt{\lambda}\int_{S^2} f(\sqrt{\lambda}\,\alpha, \sqrt{\lambda}\,\beta)\,\varphi(\lambda, \beta)\,d\beta.$$

This operator is also unitary.

A comparison of these definitions demonstrates that the scattering operator can be written in the form

$$S = T_0^* \hat{S} T_0,$$

which also yields the desired relation. We note the analogy of the equations obtained and those in Sec. 2 of Chap. 1.

Henceforth, we will find it convenient to refer to the space $\mathfrak{H}_0$ simply as $L_2(\mathbf{R}^3)$ using the identity

$$\varphi(\lambda, \alpha) = \varphi(k);\ |k| = \sqrt{\lambda};\ \frac{k}{|k|} = \alpha.$$

In this case T_0 is a Fourier transform,

$$\psi(x) \to T_0\psi = \varphi(k);\quad \varphi(k) = \left(\frac{1}{2\pi}\right)^3 \int e^{i(k,x)}\psi(x)\,dx.$$

The operator $\hat{S}$ in this notation is given by

$$\hat{S}\varphi(k) = \varphi(k) - 2\pi i \int f(k, l)\,\delta(k^2 - l^2)\,\varphi(l)\,dl. \tag{1.11}$$

Let us now return to the solutions $u^{(\pm)}(x, k)$ and say a few words about their properties. It can be proved that the functions $u^{(+)}(x, |k|\alpha)$ for fixed x and α have an analytic continuation into the upper half-plane of the parameter $s = |k|$ and have a simple pole at the point $s = i\varkappa$. The principal part in the neighborhood of this pole has the form

$$\operatorname{Res} u^{(+)}(x, s\alpha)|_{s=i\varkappa} = c(\alpha)\,u(x), \tag{1.12}$$

where

$$c(\alpha) = \frac{1}{2i\varkappa}\int e^{-\varkappa(x,\alpha)} v(x)\,u(x)\,dx \tag{1.13}$$

and u(x) is the already-mentioned eigenfunction. It is necessary to study Eq. (1.3) in the complex domain $s = |k|$ for the proof. One variant of these discussions can be found in [24]. Further, for large $|k|$, we have the asymptotic

$$u^{(+)}(x, k) = e^{i(k,x)} + o(1), \tag{1.14}$$

while if v(x) is differentiable, we can replace here o(1) by $O(1/|k|)$. A derivation can be found in [22]. In particular, these results imply that the scattering amplitude is forward, i.e., the function

$$f(|k|, \alpha)=f(k, k)$$

for fixed α has an analytic continuation into the upper half-plane of the parameter $s = |k|$, has a pole when $s = i\varkappa$ with principle part

$$\operatorname{Res} f(s, \alpha)|_{s=i\varkappa}=\frac{2i\varkappa}{(2\pi)^3} c(\alpha) c(-\alpha) \tag{1.15}$$

and at infinity has the asymptotic

$$f(s, \alpha)=q+o(1), \quad q=\left(\frac{1}{2\pi}\right)^3 \int v(x)\, dx.$$

The proof requires the use of the representation of Eq. (1.7) for the scattering amplitude.

The last property first found by the physicists Wong [48] and Khuri [41] is an analog of the analyticity condition on the transmission coefficient $a(k)$ of Chap. 1. However, unlike the one-dimensional case, this condition far from exhausts all the necessary conditions on the scattering amplitude which follows from the locality of the potential. In the next section we will find a profound generalization of this property, formulated in [28].

We will refer to the problem of reconstructing a potential $v(x)$ in terms of given scattering amplitude $f(k, l)$ as the inverse problem. The results presented imply one important distinction of the multidimensional case from the one-dimensional case considered above; when $n \geq 2$ there exists at most one potential that decides the inverse problem. In fact Eq. (1.14) implies that for large $|k| = |l|$,

$$f(k, l)=\left(\frac{1}{2\pi}\right)^3 \int e^{i(k-l, x)} v(x)\, dx+o(1),$$

so that, setting $k - l = m$ and letting $|k|$ tend to infinity, we can reconstruct the Fourier transformation of the potential in this limit. This long established and simple assertion found in [3, 21] was for a long time the only rigorous result on the multidimensional inverse problem. It finally is not of particular interest, though in every case it implies that, unlike the one-dimensional case, all the characteristics of the discontinuous spectrum necessary for solving the inverse problem are calculated in terms of the scattering amplitude itself.

2. Researches on Volterra Transformation Operators

In the one-dimensional case the transformation operators U_1 and U_2 distinguished by the Volterra condition, play an important role. As already noted in the introduction, the set of operators U_γ with Volterra direction $\gamma \in S^2$,

$$U_\gamma \psi(x)=\psi(x)+\int\limits_{(y-x, \gamma)>0} A_\gamma(x, y) \psi(y)\, dy \tag{2.1}$$

is their natural multidimensional analog. In this section we will demonstrate how to prove the existence of such operators.

Chapter 1 demonstrates that Volterra transformation operators are generated by a set of solutions of the Schröedinger equation possessing special analyticity properties. In our case it is necessary to find solutions $f_\gamma(x, k)$ of Eq. (1.1) that have an analytic continuation into the upper half-plane of the variable $s = (k, \gamma)$ for fixed x and $k_\perp=k-(k, \gamma)\gamma$, such that as $|x| \to \infty$,

$$|f_\gamma(x, k) e^{-is(x, \gamma)}| \leqslant C \tag{2.2}$$

when $\operatorname{Im} s \geqslant 0$ and has an asymptotic for large s,

$$f_\gamma(x, k) e^{-is(\gamma, x)}=e^{i(k_\perp, x)}+o(1).$$

In fact if such sets of solutions $f_\gamma(x, k)$ exist, we may verify by introducing maps T_γ from $\mathfrak{H}$ into $\mathfrak{H}_0$ by the equation

$$\psi(x) \to T_\gamma \psi = \varphi_\gamma(\lambda, \alpha), \quad \varphi_\gamma(\lambda, \alpha) = \left(\frac{1}{2\pi}\right)^{3/2} \int f_\gamma(x, \sqrt{\lambda}\alpha)\, \psi(x)\, dx,$$

that the operators

$$U_\gamma = T_\gamma^* T_0 \tag{2.3}$$

have the form of Eq. (2.1), where the kernel $A_\gamma(x, y)$, being perhaps a generalized function of $x_\perp$ and $y_\perp$, will be a classical function of the variables (x, γ) and (y, γ), so that the condition according to which it vanishes when $(x, \gamma) > (y, \gamma)$, is justified.

We may attempt to find solutions of the type of $f_\gamma(x, k)$ using integral equations of the form

$$u_\gamma(x, k) = e^{i(k, x)} + \int G_\gamma(x - y, k)\, v(y)\, u_\gamma(y, k)\, dy \tag{2.4}$$

for an appropriate choice of Green's function $G_\gamma(x, k)$ of the Helmholtz equation. In the one-dimensional case the solutions $f_1(x, k)$ and $f_2(x, k)$ generating Volterra transformation operators have been defined in precisely this way [cf. (I.1.4)]. To satisfy an analyticity condition on Green's function $G_\gamma(x, k)$ they must satisfy the requirement that for fixed γ, x, and $k_\perp$ they must have an analytic continuation into the upper half-plane of the parameter $s = (k, \gamma)$, such that

$$\left| G_\gamma(x, k)\, e^{-is(x, \gamma)} \right| \leqslant C \frac{1}{|x|}.$$

Such functions in fact exist. Guiding concepts for searching for them and an analogy to the functions $G_1(x, k)$ and $G_2(x, k)$ of Sec. 1 Chap. 1 were presented in [27]. Here we will limit ourselves to presenting an expression for G_γ in the form of an integral which, unfortunately, cannot be explicitly calculated:

$$G_\gamma(x, k) = \left(\frac{1}{2\pi}\right)^3 \int \frac{e^{i(l, x)}}{k^2 - l^2 + i0\,(k - l, \gamma)}\, dl, \tag{2.5}$$

where $(x + i0a)^{-1}$ is understood as $(x + i0)^{-1}$ for $a > 0$ and as $(x - i0)^{-1}$ when $a < 0$. It can be easily verified that $G_\gamma(x, k)$ depends on k only in terms of the combinations $s = (k, \gamma)$ and $\mu^2 = k^2 - (k, \gamma)^2$. When Im $s > 0$, Eq. (2.5) can be rewritten in the form

$$G_\gamma(x, k) = G_\gamma(x, \mu, s) = \left(\frac{1}{2\pi}\right)^3 \int \frac{e^{i(l + s\gamma, x)}}{\mu^2 + s^2 - (l + s\gamma)^2}\, dl,$$

which imply the analyticity and boundedness-type properties formulated above. We further note that when Im $s = 0$, we have the realness condition

$$G_\gamma(x, k) = \overline{G_\gamma(x, -k)}. \tag{2.6}$$

One important distinction between $G_\gamma(x, k)$ and one-dimensional $G_1(x, k)$ and $G_2(x, k)$ is that it is not Volterra. Therefore, to study Eq. (2.4) we cannot use the method of successive approximations and will need Fredholm theory. Here we find that the solution $u_\gamma(x, k)$ of this equation exists, is an analytic function of $s = (k, \gamma)$ when Im $s > 0$, and satisfies there a boundedness condition for all s, such that the homogeneous equation

$$h(x) - \int G_\gamma(x - y, \mu, s)\, v(y)\, h(y)\, dy = 0 \tag{2.7}$$

has no nontrivial solutions satisfying the condition

$$|h(x)\, e^{-is(x, \gamma)}| \leqslant C.$$

Singular s such that these solutions exist are located discontinuously, lack accumulation points when Im $s > 0$, and are poles of finite order for $u_\gamma(x, k)$.

Such singular s, in general, exist. In fact a comparison of Eqs. (2.5) and (1.4) makes it clear that when $k \| \gamma$ Green's functions $G^{(+)}(x, |k|)$ and $G_\gamma(x, k)$ coincide. Thereby the solutions $u^{(+)}(x, k)$ and $u_\gamma(x, k)$ for $k \| \gamma$ also coincide. The variable $s = (k, \gamma)$ under this condition is simply $|k|$. At the end of Sec. 1 we mentioned that $u^{(+)}(x, k)$ for fixed x has an analytic continuation into the upper half-plane of the variable $s = |k|$ and has there a simple pole at $s = i\varkappa$. Thus $u_\gamma(x, k)$ for $k_\perp = 0$ has a pole when $s = i\varkappa$. As γ varies or, what is the same thing, a nonvanishing $k_\perp$ appears, this pole will move without vanishing. Thus, we have verified that if an operator H has discontinuous spectrum, the singular values s exist.

We cannot prove an analog of the Kato theorem for Eq. (2.4), i.e., we cannot guarantee that this equation is solvable for all real k nor that the singular values s do not leave the real axis. We, therefore, must require that the potential v(x) be given such that these solutions do not exist. We will refer to this requirement as condition C. Henceforth, we will be able to formulate an equivalent condition in terms of the scattering amplitude. When this condition holds, Eq. (2.6) implies that

$$u_\gamma(x, k) = \overline{u_\gamma(x, -k)}. \tag{2.8}$$

Thus, the solutions $u_\gamma(u, k)$ cannot be used to determine Volterra transformation operators because of these singularities. It is, however, easy to refine them. For this purpose we consider the regularized Fredholm determinant $\Delta_\gamma(k)$ of Eq. (2.4). The formal definition is provided by the equation

$$\ln \Delta_\gamma(k) = \operatorname{Tr}(\ln(I - G_\gamma(k) V) + G_\gamma(k) V), \tag{2.9}$$

where we use obvious notation $G_\gamma(k)$ for an integral operator with kernel $G_\gamma(x-y, k)$. The trace in the right side can be understood in an operator-theoretic sense if the operators under the sign of the trace are symmetrized by the scheme

$$V = |V|^{1/2} J |V|^{1/2}, \quad G_\gamma(k) V \to |V|^{1/2} G_\gamma(k) |V|^{1/2} J.$$

We will bear in mind that this method can be carried out and henceforth will not refer to it. The function $\Delta_\gamma(k)$ depends on k only in terms of the variables μ and s and $\Delta_\gamma(\mu, s)$ for fixed γ and μ is analytic with respect to s in the upper half-plane, there having the asymptotic

$$\Delta_\gamma(\mu, s) = 1 + o(1)$$

and vanishes at singular s. Here the multiplicity of the corresponding zeroes is sufficient for all the poles in the product

$$f_\gamma(x, k) = u_\gamma(x, k) \Delta_\gamma(k) \tag{2.10}$$

to be annihilated, so that the set of solutions $f_\gamma(x, k)$ satisfies all the requirements described at the beginning of this section and can therefore be used to define Volterra transformation operators.

With this we conclude the description of research into multidimensional Volterra transformation operators, which can in fact be far more exciting than can be seen from this presentation. In the next section we will begin a calculation of the normalizing factor corresponding to these operators.

This section we conclude with a few more remarks on the determinant $\Delta_\gamma(k)$. The relation we have noted between the functions $G^{(+)}(x, |k|)$ and $G_\gamma(x, k)$ at $k \| \gamma$ imply that $\Delta_\gamma(k)$ when $k \| \gamma$ is the Fredholm determinant $\Delta^{(+)}(|k|)$ of the integral equation of scattering theory

$$\ln \Delta^{(+)}(|k|) = \operatorname{Tr}(\ln(I - G^{(+)}(|k|) V) + G^{(+)}(|k|) V).$$

We easily obtain from this expression that

$$\frac{d}{d\lambda} \ln \Delta^{(+)}(\sqrt{\lambda}) = -\operatorname{Tr}((H - \lambda I)^{-1} - (H_0 - \lambda I)^{-1}),$$

so that $\Delta^{(+)}(\sqrt{\lambda})$ is a regularized characteristic determinant of H. In this sense it is analogous to the conversion factor $a(\sqrt{\lambda})$ of Chap. 1. Henceforth, we will require the formula

$$\arg \Delta^{(+)}(|k|) = \frac{1}{2i} \ln \det S(k^2), \tag{2.11}$$

where $S(k^2)$ is a set of operators in $L_2(S^2)$ naturally generated by the operator $\hat{S}$. The derivation of this relation, which is characteristic for trace formulas, can be found, for example, in [4].

3. Normalizing Factors for the Solution $u_\gamma(x, k)$

Equations (2.10) and (2.3) demonstrate that we can find normalizing factors for the transformation operators U_γ if we know the determinant $\Delta_\gamma(k)$ and the operators $\hat{Q}_\gamma^{(\pm)}$, by means of which the solutions $u_\gamma(x, k)$ can be expressed as a linear combination of the solutions $u^{(\pm)}(x, k)$. If we assume this to be an integral operator whose kernel $Q_\gamma^{(\pm)}(k, l)$ is a generalized function, the corresponding formula will have the form

$$u_\gamma(x, k) = \int Q_\gamma^{(\pm)}(k, l)\, u^{(\pm)}(x, l)\, dl. \tag{3.1}$$

In this section we will describe the set of such operators $\hat{Q}_\gamma^{(\pm)}$, which are in some sense an analog of the matrices $M_i^{(\pm)}(k)$ described in Sec. 3 Chap. 1. An appropriate expression for the Fredholm determinant $\Delta_\gamma(k)$ will be found in the next section.

Let us compare Green's functions $G^{(\pm)}(x, |k|)$ and $G_\gamma(x, k)$ occurring in the integral equations (1.3) and (2.4). Equations (1.4) and (2.5) imply that

$$\begin{aligned} G_\gamma(x, k) &= G^{(+)}(x, |k|) + \frac{2\pi i}{(2\pi)^3} \int e^{i(l, x)} \delta(l^2 - k^2)\, \theta[(l - k, \gamma)]\, dl = \\ &= G^{(-)}(x, |k|) - \frac{2\pi i}{(2\pi)^3} \int e^{i(l, x)} \delta(l^2 - k^2)\, \theta[(k - l, \gamma)]\, dl. \end{aligned} \tag{3.2}$$

Here $\theta(t)$ is the Heaviside function. Using the first of these equations we can rewrite Eq. (2.4) for $u_\gamma(x, k)$ in the form

$$u_\gamma(x, k) = e^{i(k, x)} + \frac{2\pi i}{(2\pi)^3} \int e^{i(l, x - y)} \delta(l^2 - k^2)\, \theta[(l - k, \gamma)] \times$$
$$\times v(y)\, u_\gamma(y, k)\, dy\, dl + \int G^{(+)}(x - y, |k|)\, v(y)\, u_\gamma(y, k)\, dy.$$

We consider the first two terms in the right side as a new free term. Setting

$$Q_\gamma^{(+)}(k, l) = \delta(k - l) + 2\pi i \delta(k^2 - l^2)\, \theta[(l - k, \gamma)]\, h_\gamma(k, l), \tag{3.3}$$

where

$$h_\gamma(k, l) = \left(\frac{1}{2\pi}\right)^3 \int e^{-i(l, x)} v(x)\, u_\gamma(x, k)\, dx, \tag{3.4}$$

we can rewrite them in the form

$$\int Q_\gamma^{(+)}(k, l)\, e^{i(l, x)}\, dl,$$

i.e., as a linear combination of free terms in Eq. (1.3) for $u^{(+)}(x, k)$. The integral operator in the resulting equation also coincides with the operator of Eq. (1.3). We may assert based on the uniqueness theorem for this equation that Eq. (3.1) holds if the kernel $Q_\gamma^{(+)}(k, l)$ is given by Eq. (3.3). Analogously we can find that

$$Q_\gamma^{(-)}(k, l) = \delta(k - l) - 2\pi i \delta(k^2 - l^2)\, \theta[(k - l, \gamma)]\, h_\gamma(k, l). \tag{3.5}$$

Equations (3.3) and (3.5) also constitute the desired equations determining the operators $\hat{Q}_\gamma^{(\pm)}$, which operate in $\mathfrak{H}_0$ according to the formula

$$\hat{Q}_\gamma^{(\pm)} \varphi(k) = \varphi(k) \pm 2\pi i \int h_\gamma(k, l)\, \theta[\pm(l - k, \gamma)]\, \delta(k^2 - l^2)\, \varphi(l)\, dl.$$

Concepts by now quite standard here demonstrate that the operators $\hat{Q}_\gamma^{(\pm)}$ define a factorization of the scattering operator. Comparing Eqs. (1.9), (1.11), and (3.1) we find that

$$\hat{S} = \hat{Q}_{\gamma}^{(+)-1}\hat{Q}_{\gamma}^{(-)}. \tag{3.6}$$

We now note that the kernel $h_{\gamma}(k, l)$ that occurs is Eqs. (3.3) and (3.4) is the same in these equations. We thus use Eq. (3.6) to uniquely determine $h_{\gamma}(k, l)$ in terms of the scattering amplitude. In fact, we rewrite it in the form

$$\hat{Q}_{\gamma}^{(+)}\hat{S} = \hat{Q}_{\gamma}^{(-)}$$

and then substitute Eq. (1.11), (3.3), and (3.5) for $\hat{S}$ and $\hat{Q}_{\gamma}^{(\pm)}$. We obtain the equation

$$h_{\gamma}(k, l) = f(k, l) + 2\pi i \int h_{\gamma}(k, m)\,\theta\,[(m-k, \gamma)]\,\delta(m^2-k^2)\,f(m, l)\,dm, \tag{3.7}$$

which can be considered as a linear integral equation for determining $h_{\gamma}(k, l)$ in terms of given $f(k, l)$. This equation involves only the angular variables of the kernels occurring in it. The length of all equal vectors occur in it only as parameters. In the next section we will verify that condition C is equivalent to a unique solvability condition on this equation.

We now note an important property of the functions $h_{\gamma}(k, l)$ which is implied by analyticity properties on the solutions $u_{\gamma}(x, k)$. We consider the integral representation of Eq. (3.4) for $h_{\gamma}(k, l)$ and set $(k, \gamma) = (l, \gamma) = s$ in it. As a consequence of a bound of the type of Eq. (2.2), the integrand is absolutely integrable for all nonsingular s in the upper half-plane. It therefore follows that the function $h_{\gamma}(k, l)$ when $(k, \gamma) = (l, \gamma) = s$ and for fixed $k_{\perp}$, $l_{\perp}$, and γ has an analytic continuation into the upper half-plane of the variable s and poles of finite order at singular s.

We emphasize that the locality of v(x) is highly important for deriving this analyticity property. In fact the growth of the solution $u_{\gamma}(x, k)$ with respect to x at Im s > 0 is compensated by decreasing $e^{-i(l, x)}$ only because these functions are multiplied within the integral in Eq. (3.4). For nonlocal V, the independent variables x and y on which these functions will depend in an equation of the type of Eq. (3.4) will differ and no such compensation will occur. Henceforth we will verify in studying the inverse problem that the necessary analyticity condition we have obtained is essentially also a sufficient condition on the scattering amplitude corresponding to a local potential. We should now state that this condition is a generalization of the analyticity of the foward scattering amplitude noted in Sec. 2. In fact it is evident from Eq. (3.7) that the amplitudes $f(k, l)$ and $h_{\gamma}(k, l)$ coincide when $k \parallel \gamma$. Under this condition $k = l$ also if $(k, \gamma) = (l, \gamma)$. Thus,

$$h_{\gamma}(k, l)|_{k \parallel \gamma;\, (k,\gamma)=(l,\gamma)} = f(k, k)$$

and the analyticity we have indicated for the function on the left side implies the analyticity of the right side, which was noted above.

We present one more useful equation relating the kernels $h_{\gamma}(h, l)$ for different γ. For this purpose we use the factorization of Eq. (3.6) and a unitarity condition on $\hat{S}$. Rewriting the equation

$$\hat{S}^* = \hat{S}^{-1}$$

in terms of Eq. (3.6), using different γ in the left and right sides, we find

$$\hat{Q}_{\gamma}^{(-)}\hat{Q}_{\gamma'}^{(-)*} = \hat{Q}_{\gamma}^{(+)}\hat{Q}_{\gamma'}^{(+)*}. \tag{3.8}$$

We now set $\gamma' = -\gamma$ and rewrite the resulting equation in the form

$$\hat{Q}_{\gamma}^{(-)}\hat{Q}_{-\gamma}^{(-)*} = \hat{Q}_{\gamma}^{(+)}\hat{Q}_{-\gamma}^{(+)*}.$$

The operators $\hat{Q}_{\gamma}^{(\pm)}$ are Volterra, which is explicitly evident from the presence of the Heaviside function in their definition. The Volterra property of the operators in the right and left sides of the last equation are in opposite directions. It is thus consistent only if each side is separately a unit operator. We have arrived at the important equation

$$\hat{Q}_{\gamma}^{(\pm)*} = \hat{Q}_{-\gamma}^{(\pm)-1}. \tag{3.9}$$

We also rewrite the general equation (3.8) in more detail in terms of the kernel of the operators occurring in it,

$$h_\gamma(k, l) - h_{\gamma'}(-l, -k) = \\ = 2\pi i \int h_\gamma(k, m)\, h_{\gamma'}(-l, -m)(\theta[(m-k, \gamma)] - \\ - \theta[(l-m, \gamma')])\,\delta(k^2 - m^2)\, dm. \tag{3.10}$$

Here we have used also the equation

$$\overline{h_\gamma(-k, -l)} = h_\gamma(k, l),$$

which follows from the property of Eq. (2.8) and from the integral representation of Eq. (3.4). Equation (3.10) constitutes a generalization of Eq. (3.7).

Because of Eq. (3.9) we must solve integral equations to determine operators inverse to the normalizing factors $\hat{Q}_\gamma^{(\pm)}$, which occur in the construction of the weight operator W_γ.

4. Differential Equations With Respect to the Parameter γ

Let us now turn to the transformation operator U_γ. The normalizing factors corresponding to it we nearly already calculated; they are constructed by means of the operators $\hat{Q}_\gamma^{(\pm)}$ and the Fredholm determinant $\Delta_\gamma(k)$. Explicit equations can be written in the form

$$\hat{N}_\gamma^{(\pm)} \varphi(k) = \int \overline{Q_\gamma^{(\mp)}(l, k)}\, \overline{\Delta_\gamma(l)}\, \varphi(l)\, dl, \tag{4.1}$$

and the operators $N_\gamma^{(\pm)}$ acting in $\mathfrak{H}$ are described by the equation

$$N_\gamma^{(\pm)} = T_0^* \hat{N}_\gamma^{(\pm)} T_0; \quad U_\gamma = U^{(\mp)} N_\gamma^{(\pm)}. \tag{4.2}$$

In this section we will prove that the determinant $\Delta_\gamma(k)$ can be explicitly expressed in terms of the kernel $h_\gamma(k, l)$ and thereby in terms of the scattering amplitude.

In order to state the Gel'fand–Levitan-type equation we must obtain, in addition to normalizing factors, an expression for the generalized element χ_γ, which is the pre-image of the eigenfunction of the discontinuous spectrum u(x) under the map U_γ. This can easily be carried out on the basis of the already noted coinciding of the function $u_\gamma(x, k)$ and $u^{(+)}(x, k)$ for $k \parallel \gamma$ and Eq. (2.10).

We will proceed on the basis of the equation

$$f_\gamma(x, k) = e^{i(k,x)} + \int\limits_{(y-x,\gamma)>0} A_\gamma(x, y)\, e^{i(k,y)} dy, \tag{4.3}$$

which constitutes a concrete variant of the more abstract definition of Eqs. (2.1) and (2.3). We note its analogy to Eqs. (I.1.8) and (I.1.9). Setting here $k = s\gamma$ and expressing $f_\gamma(x, k)$ in terms of $u^{(+)}(x, k)$ and $\Delta_\gamma(k)$, we find

$$u^{(+)}(x, s\gamma)\, \Delta^{(+)}(s) = e^{is(x,\gamma)} + \int\limits_{(y-x,\gamma)>0} A_\gamma(x, y)\, e^{is(y,\gamma)} dy.$$

Here we will use the already noted coinciding of $\Delta_\gamma(s\gamma)$ and $\Delta^{(+)}(s)$. Under our assumption on the simplicity of the discontinuous eigenvalue, the determinant $\Delta^{(+)}(s)$ has a simple zero at $s = i\varkappa$, so that setting $s = i\varkappa$ in the last equation and using Eq. (1.12), we find

$$u(x) = m(\gamma)\left[e^{-\varkappa(x,\gamma)} + \int\limits_{(y-x,\gamma)>0} A_\gamma(x, y)\, e^{-\varkappa(y,\gamma)} dy\right],$$

where

$$m(\gamma) = \left[\frac{d}{ds}\Delta^{(+)}(s)\Big|_{s=i\varkappa} c(\gamma)\right]^{-1}$$

and $c(\gamma)$ was introduced in Eq. (1.13). This equation is also the result we require; we will see that

$$u = U_\gamma \chi_\gamma; \; \chi_\gamma(x) = m(\gamma)\, e^{-\varkappa(x,\gamma)}. \tag{4.4}$$

Thus to express all the variables occurring in the Gel'fand–Levitan equation in terms of the scattering amplitude, it remains for us to find an expression for $\Delta_\gamma(k)$ and $c(\gamma)$. For this purpose differential equations for the functions $u_\gamma(x, k)$, $h_\gamma(h, l)$, $\Delta_\gamma(k)$, and $c(\gamma)$ with respect to the parameter γ will turn out to be convenient. This variable runs through the unit sphere and it is therefore convenient for differentiation to use the Lie operator corresponding to the operation of a group of rotations. We will not have to write explicit equations for these operators, the following single equation being sufficient:

$$M_\xi f[(\gamma, a)] = f'[(\gamma, a)](a, \gamma \times \xi).$$

Here (γ, a) is the scalar product of γ and an arbitrary vector a, f(t) is an arbitrary function, M_ξ is the Lie operator corresponding to differentiation in the direction of ξ, and $\gamma \times \xi$ is the vector product of γ and ξ.

Differentiating Eq. (3.2) we find

$$M_\xi G_\gamma(x, k) = \left(\frac{1}{2\pi}\right)^3 \int \omega_{\gamma,\xi}(k, l)\, e^{i(l,x)} dl, \tag{4.5}$$

where

$$\omega_{\gamma,\xi}(k, l) = 2\pi i \delta(l^2 - k^2)\, \delta[(l-k, \gamma)]\, (l-k, \gamma \times \xi).$$

This equation leads to a differential equation for all the variables mentioned at the beginning of the section. We begin with the function $u_\gamma(x, k)$. Differentiating Eq. (2.4) we find the equation

$$M_\xi u_\gamma(x, k) = \int M_\xi G_\gamma(x-y, k)\, v(y)\, u_\gamma(y, k)\, dy + \\ + \int G_\gamma(x-y, k)\, v(y)\, M_\xi u_\gamma(y, k)\, dy.$$

This equation differs from Eq. (2.4) only in a free term, which can be written in the form of a linear combination of plane waves of the form

$$\int R_{\gamma,\xi}(k, l)\, e^{i(l,x)} dl, \tag{4.6}$$

where

$$R_{\gamma,\xi}(k, l) = \omega_{\gamma,\xi}(k, l)\, h_\gamma(k, l). \tag{4.7}$$

The expression for $\omega_{\gamma,\xi}(k, l)$ includes δ-functions, so that $l^2 = k^2$ and $(l, \gamma) = (k, \gamma)$ in the integral of Eq. (4.6). We recall that Green's function $G_\gamma(x, k)$ depends on k only in terms of k^2 and (k, γ). This together with Eq. (2.4) implies that

$$M_\xi u_\gamma(x, k) = \int R_{\gamma,\xi}(k, l)\, u_\gamma(x, l)\, dl.$$

Now using the definition of Eq. (3.4) for the kernel $h_\gamma(k, l)$ we have

$$M_\xi h_\gamma(k, l) = \int h_\gamma(k, m)\, \omega_{\gamma,\xi}(k, m)\, h_\gamma(m, l)\, dm \tag{4.8}$$

which is an integrodifferential equation for $h_\gamma(k, l)$.

We now pass to the differential equations for the Fredholm determinant $\Delta_\gamma(k)$. A determination of this expression is given by Eq. (2.9). Differentiating it with respect to γ and using Eq. (4.5), we find

$$M_\xi \ln \Delta_\gamma(k) = -\int [h_\gamma(l, l) - q]\, \omega_{\gamma,\xi}(k, l)\, dl,$$

where the constant

$$q=\left(\frac{1}{2\pi}\right)^{3}\int v(x)\,dx$$

is the asymptotic of the kernel $h_\gamma(l, l)$ for large $|l|$. The resulting differential equation can be explicitly solved. We consider the function

$$g_\gamma(k)=-2\pi i\int[h_\gamma(l,l)-q]\,\theta[(l-k,\gamma)]\,\delta(k^2-l^2)\,dl.$$

Differentiating it with respect to γ, we find

$$M_\xi g_\gamma(k)=-2\pi i\int h_\gamma(l,m)\,h_\gamma(m,l)\,\omega_{\gamma,\xi}(l,m)\,\theta[(l-k,\gamma)]\times$$
$$\times\delta(k^2-l^2)\,dl-\int[h_\gamma(l,l)-q]\,\omega_{\gamma,\xi}(k,l)\,dl.$$

The first term here vanishes. In fact the function $\omega_{\gamma,\xi}(l, m)$ is antisymmetric with respect to l and m, while the remaining part of the integrand in this term is symmetric, so that we may set $(m, \gamma) = (l, \gamma)$. Thus, the differential equations for $\ln \Delta_\gamma(k)$ and $g_\gamma(k)$ coincide so that these functions coincide to within a term independent of γ. We however know that when $k \parallel \gamma$ the determinant $\Delta_\gamma(k)$ coincides with the Fredholm determinant $\Delta^{(+)}(|k|)$ of the integral equation of scattering theory [Eq. (1.3)]. At the same time $g_\gamma(k) = 0$ when $\gamma \parallel k$. We arrive at the equation

$$\ln\Delta_\gamma(k)=\ln\Delta^{(+)}(|k|)-2\pi i\int h_\gamma(l,l)\,\theta[(l-k,\gamma)]\,\delta(k^2-l^2)\,dl.$$

We may analogously study the Fredholm determinant $\tilde{\Delta}_\gamma(k)$ of the integral equation (3.7) in which an integral operator $P_\gamma(k)$ with kernel

$$P_\gamma(k;l,m)=2\pi i\theta[(l-k,\gamma)]\,\delta(l^2-m^2)\,f(l,m)$$

occurs. In this case we may prove that

$$\ln\tilde{\Delta}_\gamma(k)=\mathrm{Tr}\ln(I-P(k))=g_\gamma(k),$$

so that

$$\Delta_\gamma(k)=\Delta^{(+)}(|k|)\,\tilde{\Delta}_\gamma(k). \tag{4.9}$$

On the basis of the Kato theorem we know that $\Delta^{(+)}(|k|)\neq 0$ for real $|k|$. Then Eq. (4.9) also demonstrates the equivalence between the unique solvability problem for Eqs. (3.7) and (2.4).

Let us now express the determinant $h_\gamma(k,l)$ in terms of $\Delta^{(+)}(|k|)$. For this purpose we recall that $\Delta^{(+)}(|k|)$ has an analytic continuation in the upper half-plane of the variable $|k| = s$, there has a unique zero at $s = i\varkappa$, and when $\mathrm{Im}\, s = 0$, Eq. (2.11) holds. Suppose $Q_\gamma^{(+)}(|k|)$ and $Q_\gamma^{(-)}(|k|)$ are operators in $L_2(S^2)$ defined in terms of $\hat{Q}_\gamma^{(+)}$ and $\hat{Q}_\gamma^{(-)}$ in the same way as $S(|k|)$ was defined in terms of $\hat{S}$.

The factorization of Eq. (3.6) leads to the equation

$$S(|k|)=Q_\gamma^{(+)^{-1}}(|k|)\,Q_\gamma^{(-)}(|k|),$$

which leads to the equation

$$\ln\det S(|k|)=\mathrm{Tr}\ln S(|k|)=\mathrm{Tr}\,[\ln Q_\gamma^{(-)}(|k|)-\ln Q_\gamma^{(+)}(|k|)],$$

which implies that

$$\ln\det S(|k|)=-\pi\int h_\gamma(l,l)\,\delta(l^2-k^2)\,dl=2i\arg\Delta^{(+)}(|k|), \tag{4.10}$$

since all the degrees of the operators $Q_\gamma^{(\pm)} - I$ in a logarithmic decomposition, other than the first degree, yield a null contribution to the trace as a consequence of the Volterra property of $\hat{Q}_\gamma^{(+)}$ and $\hat{Q}_\gamma^{(-)}$. If we know $\arg\Delta^{(+)}(|k|)$, we can reconstruct this function using the equation

$$\Delta^{(+)}(s) = \exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta(t)}{t-s}\right\}\frac{s-i\varkappa}{s+i\varkappa}, \qquad \operatorname{Im} s > 0, \tag{4.11}$$

where

$$\eta(t) = \arg\Delta^{(+)}(t) + 2\operatorname{arctg}\frac{\varkappa}{t}. \tag{4.12}$$

Combining Eqs. (4.10), (4.11), and (4.12), we find a explicit expression for $\Delta^{(+)}(|k|)$ in terms of $h_\gamma(k, l)$.

We will now demonstrate how to express the function $c(\gamma)$ occurring in the construction of the vector χ_γ in terms of the scattering amplitude $f(k, l)$ [cf. Eq. (4.4)]. We consider the scattering amplitude $f(k, l)$ as a function of the variable $\gamma = l/|l|$ and apply to it the operator M_ξ in terms of this variable. Using the integral representation of Eq. (1.7) and the analyticity properties of the solution $u^{(+)}(x, k)$, we find that

$$M_\xi f(k, |k|\gamma)\,|_{k\parallel\gamma} = g(|k|, \gamma)$$

like $f(k, k)$ has an analytic continuation into the upper half-plane of the variable $s = |k|$ with pole at $s = i\varkappa$. It can be easily verified using Eq. (1.12) that the corresponding residue has the form

$$\operatorname{Res} g(s, \gamma)|_{s=i\varkappa} = \frac{2i\varkappa}{(2\pi)^3}\, c(\gamma)\, M_\xi c(-\gamma).$$

Comparing this equation to Eq. (1.15) we find that

$$M_\xi \ln c(\gamma) = \frac{g(s, -\gamma)}{f(s, -\gamma)}\bigg|_{s=i\varkappa}. \tag{4.13}$$

Solving this equation, assuming that the right side is known, we are able to obtain the equation

$$c(\gamma) = b(\gamma)\, c(-\gamma),$$

where $b(\gamma)$ is expressed in terms of integral of this right side. Here multiplying both side by $c(\gamma)$ we find finally

$$c^2(\gamma) = \operatorname{Res} f(s, \gamma)|_{s=i\varkappa}\,\frac{(2\pi)^3}{2i\varkappa}\, b(\gamma),$$

so that the square of $c(\gamma)$ is explicitly expressed in terms of the scattering amplitude. It is precisely this square that occurs in the Gel'fand–Levitan equation. We can now proceed to a direct formulation of this equation.

5. Investigation of Inverse Problem

By assuming that the potential $v(x)$ satisfies the conditions

a) $v(x)$ is a continuous, rapidly decreasing function, and

b) Eq. (2.7) does not have nontrivial restrictive solutions for all real k we have proved that the scattering amplitude $f(k, l)$ possesses the properties:

1. The integral equation (3.7) is uniquely solvable for any $\gamma \in S^2$ defining a family of kernels $h_\gamma(k, l)$.

2. The function $\Delta_\gamma(k)$, constructed in terms of $h_\gamma(k, l)$ by Eqs. (4.9), (4.10), and (4.11), has a restricted analytic continuation with respect to the variable $s = (k, \gamma)$ into the upper half-plane.

3. The function $h_\gamma(k, l)\Delta_\gamma(k)$ when $(k, \gamma) = (l, \gamma)$ also has such a continuation with respect to $s = (k, \gamma)$ for arbitrary fixed $l_\perp$ and $k_\perp$.

The last property presents a rich collection of necessary conditions that somewhat explicitly decrease the parameters in the scattering amplitude. It turns out that it substantially exhausts the necessary conditions on the scattering amplitude, about which we spoke in the introduction. That is, we will prove that if it holds, there exists a local potential v(x), such that f(k, l) is the scattering amplitude. This will be carried out using the formalism of the inverse problem.

We begin by writing an integral equation for determining the transformation operator U_γ. The idea of the proof of this equation was already set forth in the introduction. The corresponding weight operator W_γ has the kernel

$$W_\gamma(x, y)=\left(\frac{1}{2\pi}\right)^3\int e^{-i(k,x)}\tilde{W}_\gamma(k, l)\, e^{i(l,y)}\, dk\, dl + m^2(\gamma)\, e^{-\varkappa(x+y,\gamma)}, \tag{5.1}$$

where

$$\tilde{W}_\gamma(k, l)=\frac{1}{\Delta_\gamma(-k)}\int Q_{-\gamma}(k, m)\overline{Q_{-\gamma}(l, m)}\, dm\,\frac{1}{\Delta_\gamma(l)}.$$

In writing these equations we will use an abstract definition of the weight operator given in the introduction, a concrete form of the normalizing factors $N_\gamma^{(\pm)}$ from Eq. (4.1), and Eq. (3.9).

The Gel'fand–Levitan equation has the form

$$A_\gamma(x, y)+\Omega_\gamma(x, y)+\int\limits_{(z-x,\gamma)>0} A_\gamma(x, z)\,\Omega_\gamma(z, y)\, dz=0,$$
$$(x, \gamma)<(y, \gamma), \tag{5.2}$$

where

$$\Omega_\gamma(x, y)=W_\gamma(x, y)-\delta(x-y).$$

The kernel $\Omega_\gamma(x, y)$ is completely reconstructed in terms of the scattering amplitude f(k, l), as was demonstrated in the preceding section.

Equation (5.2) is an equation for $A_\gamma(x, y)$ as a function of the variable y, while x and γ play the role of parameters. The kernel of this equation is evidently positive, which ensures its unique solvability. Thus, we will find it possible to reconstruct the transformation operator U_γ in terms of the scattering amplitude.

We construct using the transformation operators we have found a family of operators

$$H_\gamma=U_\gamma H_0 U_\gamma^{-1}. \tag{5.3}$$

We intend to prove that if the properties formulated at the beginning of this section hold, the operator H_γ is independent of γ and the corresponding operator $V_\gamma = H_\gamma - H_0$ is an operator for multiplication by the real function v(x), where the initial function f(k, l) is the scattering amplitude for this potential.

The formal proof scheme is significantly simplified if we assume that no discontinuous spectrum exists. We will limit ourselves here to a presentation of only this case, so that we will assume that the second term in Eq. (5.1) is absent.

The investigation begins according to a scheme entirely analogous to the one-dimensional case of Sec. 5 Chap. 1. It is possible to prove that the Volterra operator

$$U_\gamma=I+A_\gamma,$$

constructed in terms of the solution $A_\gamma(x, y)$ of Eq. (5.2) satisfies

$$U_\gamma W_\gamma U_\gamma^*=I, \tag{5.4}$$

which implies that H_γ is self-adjoint.

We now prove that the operator $V_\gamma = H_\gamma - H_0$ is local in the γ direction, i.e., that its kernel, a generalized function, is expressed in the form

$$V_\gamma(x, y) = \delta((x, \gamma) - (y, \gamma))\widetilde{V}(x, y).$$

We note for this purpose that in the identity

$$H_0 A_\gamma - A_\gamma H_0 = -V_\gamma(I + A_\gamma), \tag{5.5}$$

which follows from Eq. (5.3), there exist terms with singularities of various orders on the plane $(x, \gamma) = (y, \gamma)$. The kernel $A_\gamma(x, y)$ itself has a discontinuity-type singularity,

$$A_\gamma(x, y) = \theta[(y - x, \gamma)]A_\gamma(x, y).$$

If it commutes with H_0 there arises the more singular term of the form

$$2\delta((x, \gamma) - (y, \gamma))\left(\frac{\partial}{\partial(x, \gamma)} + \frac{\partial}{\partial(y, \gamma)}\right)A_\gamma(x, y),$$

which can be compensated for in the right side of Eq. (5.5) only by the potential $V_\gamma(x, y)$, so that we will find an explicit expression for the potential $V_\gamma(x, y)$ in terms of the kernel $A_\gamma(x, y)$,

$$V_\gamma(x, y) = -2\delta((x, \gamma) - (y, \gamma))\frac{\partial}{\partial(x, \gamma)}(A_\gamma(x, y)|_{(x,\gamma)=(y,\gamma)}).$$

Until now we have not used the analyticity conditions on the kernel $h_\gamma(k, l)$. This property makes it possible for us to assert that the operator H_γ is independent of γ, so that the potential V_γ is local in every direction and is therefore a function-multiplication operator. For this purpose we naturally use differential equations with respect to the parameter γ.

We in turn note that the differential equation (4.8) can be derived by proceeding on the basis of the definition of $h_\gamma(k, l)$ by means of the integral equation (3.7). In fact differentiating Eq. (3.7) with respect to γ, we find

$$M_\xi h_\gamma(k, l) = 2\pi i\int h_\gamma(k, m)\delta[(m - k, \gamma)]\delta(k^2 - m^2)(m - k, \gamma \times \xi) \times$$
$$\times f(m, l)\,dm + 2\pi i\int M_\xi h_\gamma(k, m)\theta[(m - k, \gamma)]\delta(k^2 - m^2)f(m, l)\,dm.$$

The free term here can be written in the form

$$\int R_{\gamma,\xi}(k, m)f(m, l)\,dm$$

[cf. definition of Eq. (4.7) of the kernel of $R_{\gamma,\xi}$]. On the other hand, multiplying Eq. (3.7) by $R_{\gamma,\xi}(q, k)$ and integrating over k, we obtain for

$$r_\gamma(q, l) = \int R_{\gamma,\xi}(q, k)h_\gamma(k, l)\,dk$$

an equation with integral term of the form

$$2\pi i\int R_{\gamma,\xi}(q, k)h_\gamma(k, m)\theta[(m - k, \gamma)]\delta(m^2 - k^2)f(m, l)\,dk\,dm.$$

Because of the presence of $\delta[(q - k, \gamma)]$ in the kernel $R_{\gamma,\xi}(q, k)$ we can here replace $\theta[(m - k, \gamma)]$ by $\theta[(m - q, \gamma)]$. The equations for $M_\xi h_\gamma(q, l)$ and $r_\gamma(q, \gamma)$ then coincide, so that Eq. (4.8) can now be said to hold for the kernel $h_\gamma(k, l)$ reconstructed with respect to $f(k, l)$.

We now recall that the very procedure for constructing $\Delta_\gamma(k)$ in terms of $h_\gamma(k, l)$ is based on Eq. (4.8) for this function. The kernel $L_\gamma(k, l)$ of the operator $\hat{L}_\gamma = \hat{N}_\gamma^{(+)-1}$ can then be said [cf. Eqs. (4.1) and (4.2)] to satisfy the differential equation

$$M_\xi L_\gamma(k, l) = \int \Pi_{\gamma,\xi}(k, m)L_\gamma(m, l)\,dm,$$

where

$$\Pi_{\gamma,\xi}(k,l)=R_{-\gamma,\xi}(k,l)-\delta(k-l)\int[h_{-\gamma}(m,m)-q]\,\omega_{-\gamma,\xi}(m,l)\,dm.$$

We introduce with this equation the operator $\hat{\Pi}_{\gamma,\xi}$ in the space $\mathfrak{H}_0$. The analyticity of $h_\gamma(k, l)$ when $(k, \gamma) = (l, \gamma)$ reduces to the operator $\Pi_{\gamma,\xi}=T_0^*\Pi_{\gamma,\xi}T_0$ being triangular,

$$\Pi_{\gamma,\xi}(x,y)=\left(\frac{1}{2\pi}\right)^3\int e^{-i(k,x)}\Pi_{\gamma,\xi}(k,l)\,e^{i(l,y)}dk\,dl=0,\quad (x,\gamma)>(y,\gamma).$$

We also note that

$$\Pi^*_{\gamma,\xi}=-\Pi_{-\gamma,\xi},$$

which is a simple corollary of Eq. (3.9).

Recalling the definition of W_γ we find that

$$M_\xi W_\gamma=\Pi_{\gamma,\xi}W_\gamma+W_\gamma\Pi_{-\gamma,\xi}. \tag{5.6}$$

We now prove, proceeding on the basis of Eq. (5.4), that

$$M_\xi U_\gamma=-U_\gamma\Pi_{\gamma,\xi}. \tag{5.7}$$

For this purpose we note that the operator $M_\xi U_\gamma$ is triangular and therefore uniquely determined by the equation

$$(M_\xi U_\gamma)\,W_\gamma+U_\gamma(M_\xi W_\gamma)=-U_\gamma^{*-1}(M_\xi U_\gamma^*)\,U_\gamma^{*-1},$$

which is obtained by differentiating Eq. (5.4). In fact if we explicitly write out this equation in terms of the kernels of the operators occurring in it, the right side will vanish when $(x,\gamma)<(y,\gamma)$ and we will obtain a linear integral equation for the kernel of $M_\xi U_\gamma$, which differs from the Gel'fand–Levitan equation only in the free term. Using Eqs. (5.4) and (5.6) we easily find that $-U_\gamma\Pi_{\gamma,\xi}$ satisfies this equation. The Volterra property of U_γ and the triangularity of $\Pi_{\gamma,\xi}$ implies that $U_\gamma\Pi_{\gamma,\xi}$ is also triangular. Equation (5.7) may therefore be said, because of the uniqueness noted above, to be proved.

Let us now turn to the operator H_γ introduced in Eq. (5.3). We have

$$M_\xi H_\gamma=(M_\xi U_\gamma)\,H_0U_\gamma^{-1}-U_\gamma H_0U_\gamma^{-1}(M_\xi U_\gamma)\,U_\gamma^{-1}=$$
$$=U_\gamma H_0\Pi_{\gamma,\xi}U_\gamma^{-1}-U_\gamma\Pi_{\gamma,\xi}H_0U_\gamma^{-1}=0,$$

since the operator $\Pi_{\gamma,\xi}$ commutes with H_0. We have found the promised constancy factor on H_γ as a function γ and together with it the locality of the potential V.

In transferring this scheme to the case when a discontinuous spectrum is present, it is necessary to observe care in differentiating the contribution to Eq. (5.2) from the improper vector χ_γ. It is convenient to first rotate all the variables so that a variation in γ will not vary the Volterra direction of the desired operators. Differentiation with respect to γ then no longer causes complications. In proving an equation of the type of Eq. (5.6) it will be necessary to use a differential equation of the form of Eq. (4.13).

It remains for us to prove that the initial kernel f(k, l) is the scattering amplitude for the given Schrödinger operator H. We need only prove here that the solutions $u_\gamma(x, k)$ constructed with respect to the transformation operator U_γ and the determinant $\Delta_\gamma(k)$ using Eqs. (4.3) and (2.10) have for large $|x|$ the asymptotic

$$u_\gamma(|x|\gamma,k)\big|_{|x|\to\infty}=e^{i|x|(k,\gamma)}+o(1). \tag{5.8}$$

In fact once this equation has been proved, we easily verify that the set of solutions

$$u^{(+)}(x,k)=\int\overline{Q^{(+)}_{-\gamma}(l,k)}\,u_\gamma(x,l)\,dl,$$

which is independent of γ because of Eq. (5.7), has the asymptotic of Eq. (1.2) in which the initial kernel f(k, l) occurs. We will not present the arguments proving Eq. (5.8), since they require a significantly more detailed study of the kernel $A_\gamma(x, y)$ than we have so far limited ourselves to.

This investigation can be carried out until the formal discussions of this chapter have been made rigorous. The variants of it available to us are too cumbersome to fit within the present survey. We hope that the formal scheme for solving the multidimensional inverse scattering problem presented here will be a stimulus for some readers to develop better founded analytic models for its justification.

With this we conclude the description of the state of the inverse problem of quantum scattering theory through the beginning of 1973.

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