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TOPOLOGICAL PRESSURE AND THE VARIATIONAL PRINCIPLE FOR NONCOMPACT SETS

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UDC 519

INTRODUCTION

The notion of topological pressure was introduced by Ruelle in [8] in the case of compact metric spaces (for homeomorphisms that separate points). In the same paper he formulated a variational principle for the topological pressure. According to this principle, for every continuous mapping f of the compact space X and every continuous function φ on X

$$P(\varphi)=\sup_{\mu}\Big(h_{\mu}(f)+\int_{X}\varphi\, d\mu\Big),$$

where $P(\varphi)$ is the topological pressure, μ are f-invariant measures, and $h_{\mu}(f)$ is the metric entropy of mapping f. In the particular case $\varphi=0$, we recover the variational principle for the topological entropy h(f) = P(0) (see [4, 5]).

A complete proof of the variational principle in the general case was given by Walters [9]. A discussion of these topics can be found in [1]. For noncompact subsets of compact metric spaces Bowen introduced the notion of topological entropy and proved the corresponding variational principle. Here we give a definition of topological pressure for noncompact subsets of compact metric spaces and prove the variational principle. Our results may be regarded as a generalization of the results of Walters and Bowen. Let us make some preliminary remarks.

I. We deal with the following situation: X is a compact metric space, Y is a (generally noncompact) subset of X, and $f:Y \to Y$ is a continuous mapping. Generally speaking, it is not assumed that f can be extended to a continuous mapping of X. In this aspect our setting differs from that analyzed by Bowen in [2] and permits us to cover the case of discontinuous mappings of X [where the role of Y is played by the set $X\setminus\bigcup_{n} f^{-n}(Z)$, where Z is the set of discontinuity points of f].

In particular, we prove the variational principle for one-dimensional discontinuous mappings (see Sec. 3). Our results may be used to prove the variational principle for Lorenz-type attractors (see [10]).

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II. The fact that the set Y considered here is a subset of a compact metric space means essentially that we use only the presence of the metric structure on Y induced by the metric of X. An equivalent description: for every $\varepsilon > 0$ there is a finite covering of Y by balls of radius $\leqslant\varepsilon$. The notion of topological pressure can be also defined for the general case of a noncompact space Y (with an arbitrary metric; see Sec. 4). However, in the general case the results that we prove are generally speaking no longer valid.

III. The notion of topological pressure on Y can be defined in analogy with the notion of Hausdorff dimension; one introduces a special outer measure on Y, $m_\lambda(Z)\ (Z \subset Y, \lambda \in \mathbf{R})$, which is not an increasing function of λ and assumes either the value 0 or ∞ at all points (except, possibly, for one). The critical value of λ is exactly the (topological) pressure. When formally applied to noncompact sets, the recipe for defining the pressure for compact sets Y given in [1] leads, generally speaking, to a finitely-semiadditive outer measure $m_\lambda{}^\times(Z)$. For this reason, in this paper we give a different definition of topological pressure for which the corresponding outer measure is countably semiadditive (see Sec. 1). In the case where Y is compact our definition agrees with that given in [1] (see Sec. 1). An important consequence of our approach is the following: the pressure on the union of the sets $Z_n \subset Y, n \in Z$, equals the supremum of the pressure on the sets Z_n.

IV. For either definition, the pressure corresponding to function $\varphi = 0$ must coincide with the topological entropy. In this way we obtain a new definition of the topological entropy of a continuous mapping of a noncompact set, and we show (see Sec. 4) that it agrees with Bowen's definition of topological entropy [2] (and hence, in the case where Y is compact, with the classical definition). Moreover, Bowen's definition, unlike ours, cannot be generalized to functions $\varphi \neq 0$.

V. We show that for a noncompact set Y the inequality

$$\sup_\mu \Big(h_\mu(f) + \int_Y \varphi\, d\mu\Big) \leqslant P_Y(\varphi)$$

holds (see Theorem 1), in which the supremum is taken over all f-invariant measures μ satisfying $\mu(Y) = 1$, and $P_Y(\varphi)$ denotes the pressure on Y corresponding to function φ (we assume that φ is continuous on $\bar{Y}$).

In the compact case at least two approaches to the proof of this inequality are known. One of them goes back to Goodwyn [5] (who proved it for $\varphi = 0$; a proof in the general case, based on the same considerations, is given in [1]) and the other – to Dinaburg [4] (under the assumption that the topological dimension of X is finite). The attempts to generalize Goodwyn's proof to the noncompact case encounter considerable obstacles of a topological character. Dinaburg's idea of proof works in the noncompact case too (again under the assumption that X has finite topological dimension). In the present paper we propose another idea of proof (apparently mastered by Margulis) which allows us to establish the indicated inequality in the general case. A yet another variant of the proof, based on consideration made by Denker [3], is given in papers [6, 7]; the idea of this proof is, in certain respects, close to ours.

In the general case strict inequality holds; we give some rather severe and, generally speaking, difficult to verify, supplementary conditions ensuring equality (see Theorem 2). Then we show that these conditions are fulfilled for one-dimensional discontinuous mappings (see Theorem 6); in [10] these conditions were verified for Lorenz's type attractors. Another variant of the variational principle is given by Theorem 3 and asserts that $h_\mu(f) + \int \varphi\, d\mu = P_{G_\mu}(\varphi)$ for any f-invariant measure μ, where G_μ is the set of typical-forward points for measure μ (see Sec. 2).

In Sec. 3 we give sufficient conditions for the existence of equilibrium states, i.e., of measures μ_φ with the property that

$$h_{\mu_\varphi}(f) + \int \varphi\, d\mu_\varphi = \sup_\mu \Big(h_\mu(f) + \int \varphi\, d\mu\Big) = P_Y(\varphi).$$

The authors became aware of the importance of the construction using the outer measure thanks to discussions with D. V. Anosov. He also examined carefully the manuscript and made many other useful remarks which contributed significantly to the effort of improving the paper. We express him our most sincere gratitude. We are also indebted to Ya. G. Sinai for the encouragement and attention given to our work.

1. Definition of Topological Pressure

1. Let X be a compact metric space, $Y \subset X$, and $f: Y \to Y$ a continuous mapping. Let $\mathcal{U}$ be a finite open cover of X. We denote by $\mathcal{W}_m(\mathcal{U})$ the set of collections of length m of elements of cover $\mathcal{U}$: $\underline{U} = U_{i_0} U_{i_1} \ldots U_{i_{m-1}}$. For a continuous function φ on X we set

$$Z(\underline{U}) = \{x \in Z: \quad f^k(x) \in U_{ik}, \; k = 0, \ldots, m-1\},$$

$$S_m\varphi(\underline{U}) = \sup \left\{ \sum_{k=0}^{m-1} \varphi(f^k(x)) : \; x \in Z(\underline{U}) \right\}.$$

If $Z(\underline{U}) = \emptyset$, we shall consider that $S_m\varphi(U) = -\infty$. Set $\mathcal{W}(\mathcal{U}) = \bigcup_{m \geqslant 0} \mathcal{W}_m(\mathcal{U})$. We will say that $\Gamma \subset \mathcal{W}(\mathcal{U})$ covers Z if $Z \subset \bigcup_{\underline{U} \in \Gamma} Z(\underline{U})$. The number of element of collection $\underline{U}$ will be denoted by $m(\underline{U})$. Set

$$M(\mathcal{U}, \lambda, Z, \varphi, N) = \inf_{\Gamma \subset \mathcal{W}(\mathcal{U})} \left\{ \sum_{\underline{U} \in \Gamma} \exp(-\lambda m(\underline{U}) + S_{m(\underline{U})} \varphi(\underline{U})) : \right.$$

Γ covers Z and for every $\underline{U} \in \Gamma, m(\underline{U}) \geqslant N\}$. It is readily verified that function $M(\mathcal{U}, \lambda, Z, \varphi, N)$ increases monotonically with the growth of N. This guarantees the existence of the limit

$$m(\mathcal{U}, \lambda, Z, \varphi) = \lim_{N \to \infty} M(\mathcal{U}, \lambda, Z, \varphi, N). \tag{1}$$

For any λ and given $\mathcal{U}$ and φ, function $m(\mathcal{U}, \lambda, Z, \varphi)$ is a regular outer Borel measure on the family of all subsets of Y.

For fixed Z, function $m(\mathcal{U}, \lambda, Z, \varphi)$ has the following property: there is a λ_0 such that $m(\mathcal{U}, \lambda, Z, \varphi) = 0$ for $\lambda > \lambda_0$, and $m(\mathcal{U}, \lambda, Z, \varphi) = \infty$ for $\lambda < \lambda_0$. Let $P_Z(\mathcal{U}, \varphi) = \inf\{\lambda: m(\mathcal{U}, \lambda, Z, \varphi) = 0\}$. The quantity $P_Z(\mathcal{U}, \varphi)$ enjoys the following properties:

1) $P_\emptyset(\mathcal{U}, \varphi) = 0$;

2) if $Z_1 \subset Z_2 \subset Y$, then $P_{Z_1}(\mathcal{U}, \varphi) \leqslant P_{Z_2}(\mathcal{U}, \varphi)$;

3) if $Z = \bigcup_i Z_i \subset Y$, then $P_Z(\mathcal{U}, \varphi) = \sup P_{Z_i}(\mathcal{U}, \varphi)$.

Proposition 1. The following limit exists:

$$P_Z(\varphi) = \lim_{\operatorname{diam} \mathcal{U} \to 0} P_Z(\mathcal{U}, \varphi). \tag{2}$$

The *proof* is a slightly modified version of the proof of Lemma 2.8 of [1].

Proposition 2. 1) Let X, X' be compact spaces, $Y \subset X$, $Y' \subset X'$ Borel subsets, $f: Y \to Y$, $f': Y' \to Y'$ continuous mappings. Suppose that $\chi: X \to X'$ is a continuous mapping such that $\chi(Y) = Y'$, $\chi \circ f = f' \circ \chi$. Then $P_Z(\varphi) \leqslant P_{\chi^{-1}(Z)}(\tilde{\varphi})$ for all continuous functions φ on X' and $Z \subset Y'$, where $\tilde{\varphi}(x) = \varphi(\chi(x))$.

2) If under the conditions of assertion 1), mapping χ is a homeomorphism, then $P_Z(\varphi) = P_{\chi^{-1}(Z)}(\tilde{\varphi})$.

Proof. Assertion 1) is an immediate consequence of the definition of topological pressure and Proposition 1. Assertion 2) is a straightforward consequence of 1). It says that topological pressure is a topological invariant.

Now set

$$\gamma = \gamma(\mathcal{U}) = \sup_{U_i} \{|\varphi(x) - \varphi(y)|: \; x, y \in U_i\}. \tag{3}$$

2. We denote by $\bar{P}(\varphi)$ the pressure for function φ on space X defined in [1]. Let us recall the definition. If $\mathcal{U}$ is a finite covering of X, we set (f is defined everywhere on X)

$$\bar{P}(\mathcal{U}, \varphi) = \lim_{m \to \infty} \frac{1}{m} \log Z_m(\mathcal{U}, \varphi),$$

where

$$Z_m(\mathcal{U}, \varphi) = \inf_\Gamma \sum_{\underline{U} \in \Gamma} \exp(S_m\varphi(\mathbf{U})),$$

and Γ runs through all possible subsets of $\mathcal{W}_m(\mathcal{U})$ covering X. Now $\bar{P}(\varphi)$ is defined as

$$\bar{P}(\varphi) = \lim_{\operatorname{diam}\mathcal{U}\to 0} \bar{P}(\mathcal{U}, \varphi).$$

Proposition 3. $\bar{P}(\varphi) = P_X(\varphi)$.

Proof. Let us show that $P_X(\mathcal{U}, \varphi) = \bar{P}(\mathcal{U}, \varphi)$. Fix some $\lambda > \bar{P}(\mathcal{U}, \varphi)$ and an arbitrary $\varepsilon > 0$. Pick N so large that

$$m(\mathcal{U}, \lambda, X, \varphi) \leqslant M(\mathcal{U}, \lambda, X, \varphi, N) + \varepsilon. \tag{4}$$

From the above discussion it follows that there is an $m > N$ such that

$$\left|\bar{P}(\mathcal{U}, \varphi) - \frac{1}{m}\log Z_m(\mathcal{U}, \varphi)\right| \leqslant \varepsilon.$$

Hence

$$Z_m(\mathcal{U}, \varphi) \leqslant \exp\left[(\bar{P}(\mathcal{U}, \varphi) + \varepsilon)m\right]. \tag{5}$$

Inequality (5) implies

$$M(\mathcal{U}, \lambda, X, \varphi, N) = \inf_{\Gamma\subset\mathcal{W}(\mathcal{U})} \left\{\sum_{\underline{U}\in\Gamma} \exp(-\lambda m(\underline{U}) + S_{m(\underline{U})}\varphi(\underline{U}))\right\} \leqslant$$

$$\leqslant \inf_{\Gamma\subset\mathcal{W}_m(\mathcal{U})} \left\{\sum_{\underline{U}\in\Gamma} \exp(-\lambda m + S_m\varphi(\underline{U}))\right\} = \exp(-\lambda m) Z_m(\underline{U}, \varphi) \leqslant \exp\left[(-\lambda + \bar{P}(\mathcal{U}, \varphi) + \varepsilon)m\right].$$

For sufficiently small ε and sufficiently large m (4) yields $m(\mathcal{U}, \lambda, X, \varphi) \leqslant 2\varepsilon$. This means that $\lambda \geqslant P_X(\mathcal{U}, \varphi)$, whence $\bar{P}(\mathcal{U}, \varphi) \geqslant P_X(\mathcal{U}, \varphi)$. Now let us verify the converse inequality. Fix $\lambda > P_X(\mathcal{U}, \varphi)$ and $\varepsilon > 0$. There are an N and a $\Gamma \subset \mathcal{W}(\mathcal{U})$ covering X, such that

$$\sum_{\underline{U}\in\Gamma} \exp(-\lambda m(\underline{U}) + S_{m(\underline{U})}\varphi(\underline{U})) \leqslant \varepsilon.$$

From Lemma 2.14 of [1] it follows that $\exp(-\lambda) \leqslant \exp(-\bar{P}(\mathcal{U}, \varphi))$. Therefore, $\lambda > \bar{P}(\mathcal{U}, \varphi)$, and hence $P_X(\mathcal{U}, \varphi) \geqslant \bar{P}(\mathcal{U}, \varphi)$.

2. The Variational Principle

1. Let X be a compact metric space, Y a Borel subset of X, and $f: Y \to Y$ a continuous mapping. Let M(X), $M_f(X)$, and $M_f(Y)$ denote, respectively, the set of normalized Borel measures on X, the set of f-invariant† measures $\mu \in M(X)$, the set of measures $\mu \in M_f(X)$, such that $\mu(Y) = 1$, and the set of ergodic measures $\mu \in M_f(Y)$.

THEOREM 1. If $\mu \in M_f(Y)$, then $h_\mu(f \mid Y) + \int_Y \varphi\, d\mu \leqslant P_Y(\varphi)$.

Proof. One can readily verify the following statement.

LEMMA 1. For every $\varepsilon > 0$ there are a $\delta \in (0, \varepsilon)$, a finite Borel partition $\xi = \{C_1, \ldots, C_m\}$, and a finite open covering $\mathcal{U} = \{U_1, \ldots, U_k\}$, $k \geqslant m$, of X, such that

1. $\operatorname{diam} U_i \leqslant \varepsilon$, $\operatorname{diam} C_j \leqslant \varepsilon$, $i = 1, \ldots, k$, $j = 1, \ldots, m$.
2. $\bar{U}_i \subset C_i$, $i = 1, \ldots, m$.
3. $\mu(C_i \setminus U_i) < \delta$, $i = 1, \ldots, m$.
4. $\mu(\bigcup_{i=m+1}^{k} U_i) < \delta$.
5. $2\delta \ln m \leqslant \varepsilon$.

Now fix an $\varepsilon > 0$ and take the number δ, covering $\mathcal{U}$, and partition ξ provided by Lemma 1. Let $\tilde{\xi}$ and $\tilde{\mathcal{U}}$ be the partition and the cover of Y induced by ξ and $\mathcal{U}$, respectively.

We may assume that measure μ is ergodic. In fact, consider the partition η of Y into ergodic components Y_s, $s \in S$, of measure μ. Denote by μ_s the measure on Y_s (then $f^*\mu_s = \mu_s$) and by ν the measure on the quotient space Y/η. Then

†Measure μ is called f-invariant if $\mu(f^{-1}(A)) = \mu(A)$ for every measurable subset $A \subset Y$.

$$h_\mu = \int_{Y/\eta} h_{\mu_s}(f \mid Y)\, d\nu(s), \qquad \int_Y \varphi\, d\mu = \int_{Y/\eta} \left(\int_{Y_s} \varphi\, d\mu_s\right) d\nu(s).$$

There is a component Y_s such that $\mu_s(Y_s) = 1$, $h_{\mu_s} + \int_{Y_s} \varphi d\mu_s \geqslant h_\mu + \int_Y \varphi\, d\mu$. Thus we shall assume that $\mu \in \mathfrak{M}_f(Y)$.

For $y \in Y$ we denote by $t_n(y)$ the number of integers l, $0 \leqslant l < n$, such that $f^l(y) \in U_i$, where $i = m + 1, \ldots, k$. From Lemma 1 and Birkhoff's theorem it follows that there are an $N_1 > 0$ and a set $A_1 \subset Y$ such that $\mu(A_1) \geqslant 1 - \delta$, and for all $y \in A_1$ and $n > N_1$

$$n^{-1} t_n(y) < 2\delta. \tag{6}$$

Let $\tilde{\xi}_n = \xi \vee f^{-1}\xi \vee \ldots \vee f^{-n}\xi$. From the Shannon–McMillan theorem it follows that there are an $N_2 > 0$ and a set $A_2 \subset Y$ such that $\mu(A_2) \geqslant 1 - \delta$ and for all $y \in A_2$ and $n > N_2$

$$\mu(C_{\tilde{\xi}_n}(y)) \leqslant \exp[-(h_\mu(f \mid Y, \tilde{\xi}) - \delta)\, n]. \tag{7}$$

Finally, Birkhoff's theorem guarantees that there are an $N_3 > 0$ and a set $A_3 \subset Y$ such that $\mu(A_3) > 1 - \delta$ and for all $y \in A_3$ and $n > N_3$

$$\left| n^{-1} \sum_{i=0}^{n-1} \varphi(f^i(y)) - \int_Y \varphi\, d\mu \right| < \delta. \tag{8}$$

Set $N = \max\{N_1, N_2, N_3\}$, $A = A_1 \cap A_2 \cap A_3$. We have

$$\mu(A) \geqslant 1 - 3\delta. \tag{9}$$

Pick an arbitrary $\lambda < h_\mu(f \mid Y, \tilde{\xi}) + \int_Y \varphi\, d\mu - \gamma(\mathfrak{U})$ and an arbitrary $n > N$. There is a $\Gamma \subset \mathcal{W}(\mathfrak{U})$, covering Y such that $m(\underline{U}) \geqslant n$ and

$$\left| \sum_{\underline{U} \in \Gamma} \exp(-\lambda m(\underline{U}) + S_{m(\underline{U})} \varphi(\underline{U})) - M(\mathfrak{U}, \lambda, Y, \varphi, n) \right| < \delta. \tag{10}$$

Let $\Gamma_l \subset \Gamma$ denote the set of collections $\underline{U}$ with the properties $m(\underline{U}) = l$ and $Y(\underline{U}) \cap A \neq \emptyset$. Let $P_l = \operatorname{card} \Gamma_l$, $Y_l = \bigcup_{\underline{U} \in \Gamma_l} Y(\underline{U})$.

<u>LEMMA 2.</u>

$$P_l \geqslant \mu(Y_l \cap A) \exp[(h_\mu(f \mid Y, \tilde{\xi}) - \delta - 2\delta \ln m)\, l].$$

<u>Proof.</u> We denote by L_l the number of those elements of partition $\tilde{\xi}_l$ such that

$$C_{\tilde{\xi}_l} \cap Y_l \cap A \neq \emptyset. \tag{11}$$

It is readily checked that

$$\sum \mu(C_{\tilde{\xi}_l}) \geqslant \mu(Y_l \cap A), \tag{12}$$

where the sum is taken over all elements $\tilde{\xi}_l$ which satisfy (11). On the other hand, since $C_{\tilde{\xi}_l} \cap A_2 \neq \emptyset$, inequalities (7) and (12) imply

$$L_l \geqslant \mu(Y_l \cap A) \exp[(h_\mu(f \mid Y, \tilde{\xi}) - \delta)\, l]. \tag{13}$$

Fix a collection $\underline{U} \in \Gamma_l$. Since $Y(\underline{U}) \cap A_1 \neq \emptyset$, (6) yields the following estimate of the number $S(\underline{U})$ of those elements $C_{\tilde{\xi}_l}$ of partition $\tilde{\xi}_l$ for which $Y(\underline{U}) \cap C_{\tilde{\xi}_l} \cap A \neq \emptyset$:

$$S(\underline{U}) \leqslant m^{2\delta l} = \exp(2\delta l \ln m). \tag{14}$$

Now (13) and (14) yield the desired estimate for P_l.

From Lemma 2 and inequalities (8) and (9) it follows that

$$\sum_{\underline{U} \in \Gamma} \exp(-\lambda m(\underline{U}) + S_{m(\underline{U})} \varphi(\underline{U})) \geqslant \sum_{l=N}^{\infty} \sum_{\underline{U} \in \Gamma_l} \exp(-\lambda l + S_l \varphi(\underline{U})) \geqslant \sum_{l=N}^{\infty} P_l \exp\left[\left(-\lambda + \int_Y \varphi\, d\mu - \delta - \gamma(\mathfrak{U})\right) l\right] \geqslant$$

$$\geqslant \sum_{l=N}^{\infty} \mu(Y_l \cap A) \exp\left[\left(h_\mu(f \mid Y, \tilde{\xi}) + \int_Y \varphi\, d\mu - 2\delta - 2\delta \ln m - \gamma(\mathfrak{U}) - \lambda\right) l\right] \geqslant \sum_{l=N}^{\infty} \mu(Y_l \cap A) = \mu(A) \geqslant 1 - 3\delta.$$

Here we used the fact that for sufficiently small δ

$$h_\mu(f|Y,\tilde{\xi}) + \int_Y \varphi\, d\mu - \gamma(\mathcal{U}) - 2\delta - 2\delta \ln m - \lambda > 0.$$

From this and inequality (10) it follows that $M(\mathcal{U}, \lambda, Y, \varphi, n) \geqslant 1 - 4\delta \geqslant 1/2$ for sufficiently small δ. Therefore, from the definition of pressure it follows that $P_Y(\mathcal{U}, \varphi) > \lambda$. Hence $P_Y(\mathcal{U}, \varphi) \geqslant h_\mu(f|Y,\tilde{\xi}) + \int_Y \varphi\, d\mu - \gamma(\mathcal{U})$. Since ε is arbitrary and in view of assertions 1 and 2 of Lemma 1, the foregoing discussion implies the desired result.

2. Let $x \in Y$. Consider the sequence of normalized measures

$$\mu_{x,n} = n^{-1} \sum_{k=0}^{n-1} \mu_{f^k(x)}, \tag{15}$$

where μ_y is the normalized measure (unit mass) placed at the point y. Let V(x) denote the set of limit measures (in the weak topology in X) of the sequences of measures $\mu_{x,n}$. It is readily checked that $V(x) \subset M_f(X)$.

THEOREM 2. Suppose that for each $x \in Y$ the intersection $V(x) \cap M_f(Y) \neq \varnothing$. Then

$$P_Y(\varphi) = \sup_{\mu \in M_f(Y)} \left(h_\mu(f|Y) + \int_X \varphi\, d\mu \right). \tag{16}$$

Proof. For $A \subset Y$ we denote by $(\bar{A})_Y$ and $(\operatorname{int} A)_Y$ the closure and, respectively, the interior of the set A in the topology of Y (induced by the topology of X). It is not hard to verify the following statement.

LEMMA 2. Let $\mathcal{V} = \{V_1, \ldots, V_t\}$ be a finite open covering of Y, and let $\xi = \{D_1, \ldots, D_t\}$ be a Borel partition of Y with the property that $(\bar{D}_i)_Y \subset V_i$, for $i = 1, \ldots, t$. Then for every $\beta > 0$ there are a Borel partition $\eta = \{V_1^*, \ldots, V_t^*\}$ of the set Y and compact subsets K_i of X such that

$$K_i \subset D_i \cap (\operatorname{int} V_i^*)_Y,\ \mu(D_i \setminus K_i) \leqslant \beta,\ (\bar{V}_i^*)_Y \subset V_i.$$

Let E be a finite set, and let $\underline{a} = (a_0, \ldots, a_{k-1}) \in E^k$. Define a measure $\mu_{\underline{a}}$ on E by the formula $\mu_{\underline{a}}(e) = k^{-1} \times$ (the number of indices j such that $a_j = e$).

Set $H(\underline{a}) = -\sum_{e \in E} \mu_{\underline{a}}(e) \ln \mu_{\underline{a}}(e)$.

Let $\mathcal{U}$ be a finite open covering of X and pick $\varepsilon > 0$.

LEMMA 3. Let $x \in Y$, and $\mu \in V(x) \cap M_f(Y)$. Then there are a number m and a sufficiently large number N such that one can find a collection $\underline{U} \in \mathcal{W}_N(\mathcal{U})$, which satisfies the following conditions:

(a) $x \in Y(\underline{U})$; (b) $S_N\varphi(\underline{U}) \leqslant N\left(\int_X \varphi\, d\mu + \gamma(\mathcal{U}) + \varepsilon\right)$; (c) $\underline{U}$ contains a subcollection of length $km \geqslant N - m$ which, on representing it as $\underline{a} = (a_0, \ldots, a_{k-1}) \in (\mathcal{U}^m)^k$, satisfies the inequality

$$m^{-1} H(\underline{a}) \leqslant h_\mu(f|Y) + \varepsilon. \tag{17}$$

Proof. Suppose that $\mathcal{U} = \{U_1, \ldots, U_r\}$ is an open cover of X. There is a Borel partition ζ of the set X into subsets $C_1, \ldots, C_r$ with $\bar{C}_i \subset U_i$. Let $\tilde{\zeta}$ denote the partition of Y with elements $\tilde{C}_i = C_i \cap Y$ and let $\tilde{\mathcal{U}}$ denote the covering of Y with the elements $\tilde{U}_i = U_i \cap Y$. There is a number m such that

$$m^{-1} H_\mu(\tilde{\zeta} \vee \ldots \vee f^{-(m+1)}\tilde{\zeta}) \leqslant h_\mu(f, \tilde{\zeta}) + \frac{\varepsilon}{2} \leqslant h_\mu(f|Y) + \frac{\varepsilon}{2}.$$

Let $D_1, \ldots, D_t$ be the nonempty elements of partition $\xi = \tilde{\zeta} \vee \ldots \vee f^{-(m+1)}\tilde{\zeta}$. Fix $\beta > 0$ and apply Lemma 2 to the covering $\mathcal{V} = \tilde{\mathcal{U}} \vee \ldots \vee f^{-(m+1)}\tilde{\mathcal{U}}$ and the partition ξ of Y to produce the partition $\eta = \{V_1^*, \ldots, V_t^*\}$ of Y. Now using the fact that $\mu_{x,n_j} \to \mu$ for some sequence $n_j \to \infty$ and repeating the arguments given in the proof of Lemma 2.15 of [1] we verify our claim.

For each $m > 0$ we denote by Y_m the set of these points $y \in Y$ for which the assertion of Lemma 3 is valid with the given m and some measure $\mu \in V(y) \cap M_f(Y)$. The assumptions of the theorem imply that $Y = \bigcup_{m>0} Y_m$. Let $Y_{m,u}$ be the set of those points $x \in Y_m$ for which the assertion of Lemma 3 is valid for some measure $\mu \in V(x) \cap M_f(Y)$ that satisfies the condition $\int \varphi \, d\mu \in [u - \varepsilon, u + \varepsilon]$. Set

$$c = \sup_{\mu \in M_f(Y)} \left(h_\mu(f \mid Y) + \int_X \varphi \, d\mu \right).$$

For $x \in Y_{m,u}$ the corresponding measure μ satisfies the inequality $h_\mu(f \mid Y) \leqslant c - u + \varepsilon$. Let $\Gamma_{m,u}$ denote the set of all collections $\underline{U}$ introduced by Lemma 3, taken for all points $x \in Y_{m,u}$ and all numbers N larger than N_0. Set $\overline{R}(k, h, E) = \{\underline{a} \in E^k: H(\underline{a}) \leqslant h\}$. From (17) it follows that for each $x \in Y$ the subcollection constructed in Lemma 3 (see assertion 3) is contained in $R(k, m(h + \varepsilon), E^m)$, where $h = c - u + \varepsilon$. Therefore, the number of all possible collections $\underline{U}$ constructed in Lemma 3 does not exceed $b(N) = |E|^m |R(k, m(h + \varepsilon), E^m)|$. By Lemma 2.16 of [1],

$$\overline{\lim_{N \to \infty}} N^{-1} \ln b(N) \leqslant h + \varepsilon. \tag{18}$$

From the above discussion it follows that $\Gamma_{m,u}$ covers $Y_{m,u}$. Hence, by Lemma 3 and (18)

$$M(\mathcal{U}, \lambda, Y_{m,u}, \varphi, N_0) \leqslant \sum_{N=N_0}^{\infty} b(N) \exp(-\lambda N + S_N\varphi(\underline{U})) \leqslant \sum_{N=N_0}^{\infty} b(N) \exp\left(-\lambda N + N\left(\int_X \varphi \, d\mu + \gamma(\mathcal{U}) + \varepsilon\right)\right).$$

If N_0 is sufficiently large, then $b(N) \leqslant \exp(N(h + 2\varepsilon))$. Therefore,

$$M(\mathcal{U}, \lambda, Y_{m,u}, \varphi, N_0) \leqslant \beta^{N_0}/(1 - \beta), \tag{19}$$

where $\beta = \exp(-\lambda + h + \int_X \varphi \, d\mu + \gamma(\mathcal{U}) + 3\varepsilon)$. For every $\lambda > c + \gamma(\mathcal{U}) + 4\varepsilon$, the last inequality shows that $m(\mathcal{U}, \lambda, Y_{m,u}, \varphi) = 0$. Consequently, $\lambda \geqslant P_{Y_{m,u}}(\mathcal{U}, \varphi)$. Next, suppose that the points $u_1, \ldots, u_r$ constitute an ε-net in $[-\|\varphi\|, \|\varphi\|]$. Then $Y = \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{r} Y_{m,u_i}$. By the foregoing argument, $\lambda \geqslant P_{Y_{m,u_i}}(\mathcal{U}, \varphi)$ for some m and i. Hence $\lambda \geqslant \sup_{m,i} P_{Y_{m,u_i}}(\mathcal{U}, \varphi) = P_Y(\mathcal{U}, \varphi)$. This implies that $c + \gamma(\mathcal{U}) + 4\varepsilon \geqslant P_Y(\mathcal{U}, \varphi)$. Since ε is arbitrary, $c + \gamma(\mathcal{U}) \geqslant P_Y(\varphi)$. Letting $\operatorname{diam} \mathcal{U}$ tend to zero, we conclude that $c \geqslant P_Y(\varphi)$. The inverse inequality is a corollary of Theorem 1.

3. Let $\mu \in \tilde{M}_f(Y)$. Denote by G_μ the set of typical-forward points for measure μ: these are defined as the points $x \in Y$ such that the measures $\mu_{x,n}$ converge weakly to measure μ. The next statement is an immediate consequence of Theorem 2.

THEOREM 3. For every measure $\mu \in \tilde{M}_f(Y)$ and every function $\varphi \in C(X)$

$$h_\mu(f \mid Y) + \int_X \varphi \, d\mu = P_{G_\mu}(\varphi).$$

Theorem 2 admits the following generalization.

THEOREM 4. Let $Z \subset Y$ be an f-invariant subset and $Z_1 = \{x \in Z: V(x) \cap M_f(Z) \neq \varnothing\}$. Then for every function $\varphi \in C(X)$

$$\sup_{\mu \in M_f(Z)} \left(h_\mu(f \mid Z) + \int_Z \varphi \, d\mu \right) = P_{Z_1}(\varphi).$$

Proof. Repeating the proof of Theorem 2 it is readily verified that

$$A = \sup_{\mu \in M_f(Z)} \left(h_\mu(f \mid Z) + \int_Z \varphi \, d\mu \right) \geqslant P_{Z_1}(\varphi).$$

Now take measures $\mu_n \in M_f(Z)$ such that

$$\sup_{\mu_n} \left(h_{\mu_n}(f \mid Z) + \int_Z \varphi \, d\mu_n \right) = A.$$

On decomposing the measures μ_n into ergodic components and repeating the arguments given in the proof of Theorem 1 it is checked easily that there is a sequence of ergodic measures $\tilde{\mu}_n$ with the same property.

Since for $x \in G_{\tilde{\mu}_n}$ we have $V(x) = \tilde{\mu}_n \subset M_f(Z)$, $G_{\tilde{\mu}_n} \subset Z_1$. Now Theorem 3 implies that

$$A = \sup_n P_{G_{\tilde{\mu}_n}}(\varphi) \leqslant P_{Z_1}(\varphi).$$

4. Next, we give an example of a set Y which does not satisfy the condition of Theorem 2, and for which equality (16) is not valid (for $\varphi = 0$).

Let (X, f) be a topological Bernoulli shift with two states, 0 and 1 (f-shift). Set $A = \bigcup_{\mu \in M_f(X)} G_\mu$ and $Y = X \setminus A$. Obviously, $\sup_{\mu \in M_f(X)} h_\mu(f|Y) = 0$. Consider a Bernoulli measure μ such that $\mu(\omega_0 = 1) = p$ and $\mu(\omega_0 = 0) = 1 - p = q$, $p \neq q$, and $|h_\mu(f) - \log 2| \leqslant \delta$ with $\delta \ll 1$. Consider the following partition of the integers into two subsets Q_1 and Q_2: $k \in Q_1$ if $(2n)! \leqslant |k| \leqslant (2n+1)!$ for some $n \geqslant 1$; and Q_2 is of course the complement of Q_2. Now consider the homeomorphism $\Psi: X \to X$ defined by the rule

$$(\Psi\omega)_n = \begin{cases} \omega_n, & n \in Q_1, \\ \omega_n + 1 \pmod 2, & n \in Q_2. \end{cases}$$

Set $Z = \Psi G_\mu$.

LEMMA 4. $Z \subset Y$.

Proof. Let χ denote the indicator of the set $\{\omega: \omega_0 = 1\} \subset X$. Pick some point $x \in Z$. Then, by Birkhoff's theorem and the definitions of measure and homeomorphism Ψ,

$$\lim_{n \to \infty} \frac{1}{(2n+1)!} \sum_{i=0}^{(2n+1)!} \chi(f^i(x)) = p,$$

$$\lim_{n \to \infty} \frac{1}{(2n)!} \sum_{i=0}^{(2n)!} \chi(f^i(x)) = q.$$

Since $p \neq q$, the sequence $a_n = n^{-1} \sum_{i=0}^{n-1} \chi(f^i(x))$ has no limit for $n \to \infty$, which shows that $x \in Y$ and proves the lemma.

We denote by ξ the partition of X with the two elements $A_1 = \{\omega: \omega_0 = 0\}$ and $A_2 = \{\omega: \omega_0 = 1\}$. Fix an $m > 0$ and let $\eta_m = \bigvee_{j=-m}^{m} f^j\xi$, $\xi_n = \bigvee_{j=0}^{n-1} f^j\eta_m$.

LEMMA 5. For μ-almost every $x \in G_\mu$

$$\lim_{n \to \infty} \left(-\frac{1}{n} \log \mu(C_{\xi_n}(\Psi(x)))\right) = h_\mu(f). \tag{20}$$

Proof. We have

$$I \stackrel{\text{def}}{=} \lim_{n \to \infty} \left(-\frac{1}{n} \log \mu(C_{\xi_n}(\Psi(x)))\right) = \lim_{n \to \infty} \left(-\frac{1}{n} \log \prod_{j=-m}^{m+n-1} \mu(C_{f^j\xi}(\Psi(x)))\right) = I_1 + I_2,$$

where $Q(i, m, n) = Q_i \cap [-m, m + n - 1]$, $i = 1, 2$,

$$I_i = \lim_{n \to \infty} \left(-\frac{|Q(i, m, n)|}{n} \cdot \frac{1}{|Q(i, m, n)|} \sum_{j \in Q(i, m, n)} \log \mu(C_{f^j\xi}(\Psi(x)))\right),$$

and $|A|$ denotes the number of elements of the set A. By the strong law of large numbers for the Bernoulli shift,

$$\lim_{n \to \infty} \left(-\frac{1}{|Q(i, m, n)|} \sum_{j \in Q(i, m, n)} \log \mu(C_{f^j\xi}(\Psi(x)))\right) = h_\mu(f).$$

Since the involution $0 \to 1$, $1 \to 0$, takes the measure μ into a Bernoulli measure with the same entropy

$$\lim_{n \to \infty} \left(-\frac{1}{|Q(i, m, n)|} \sum_{j \in Q(i, m, n)} \log \mu(C_{f^j\xi}(\Psi(x)))\right) = h_\mu(f).$$

This yields $I = h_\mu(f)$. The lemma is proved.

LEMMA 6. $P_Z(0) = \log 2$.

Proof. Since X is endowed with the open–closed topology, partition η_m is also a finite open covering of X. Fix an arbitrary $\gamma > 0$. Lemma 5 guarantees the existence of a set D and a number $N > 0$ such that $\mu(D) > 1 - \gamma$ and for every $x \in D$ and $n \geqslant N$

$$\mu(C_{(\eta_m)_n}(\Psi(x))) \leqslant \exp(-n(h_\mu(f) - \gamma)). \tag{21}$$

Fix $n \geqslant N$, and choose $\Gamma_n \subset W(\eta_m)$ such that

$$\left| M(\eta_m, \lambda, Z, 0, n) - \sum_{\underline{U} \in \Gamma_n} \exp(-\lambda n) \right| \leqslant \gamma,$$

and Γ_n covers Z. We write $\Gamma_{n,l} = \{\underline{U} \in \Gamma_n : m(\underline{U}) = l\}$, K_l is the number of elements in $\Gamma_{n,l}$ and $E_l = \bigcup_{U \in \Gamma_{n,l}} Z(\underline{U})$. Since $Z(\underline{U}') \cap Z(\underline{U}'') = \varnothing$ for every choice of $\underline{U}', \underline{U}'' \in \Gamma_{n,l}$, with $\underline{U}' \neq \underline{U}''$, inequality (21) implies that

$$K_l \geqslant \frac{\mu(E_l \cap D)}{\exp(-l(h_\mu(f) - \gamma))}.$$

Therefore, for every $\lambda < h_\mu(f) - \gamma$

$$\sum_{\underline{U} \in \Gamma_n} \exp(-\lambda n) = \sum_{l=n}^{\infty} \exp(-\lambda l) \cdot K_l \geqslant \sum_{l=n}^{\infty} \mu(E_l \cap D) \exp[(-\lambda + h_\mu(f) - \gamma) l] \geqslant (1 - \gamma) \exp[(-\lambda + h_\mu(f) - \gamma) n]$$

Passing to the limit $n \to \infty$, we get $m(\eta_m, \lambda, Z, 0) = \infty$, and hence $P_Z(\eta_m, 0) \geqslant h_\mu(f) - \gamma \geqslant \log 2 - \delta - \gamma$. Since $\operatorname{diam} \eta_m \to 0$ as $m \to \infty$, the last inequality shows that $P_Z(0) \geqslant \log 2 - \delta - \gamma$. Taking into account that δ and γ are arbitrary numbers and $P_Z(0) \leqslant P_X(0) = \log 2$ we obtain the needed statement. Now Lemma 6 follows from the following chain of equalities:

$$\log 2 = P_X(0) \geqslant P_Y(0) \geqslant P_Z(0) = \log 2.$$

3. Equilibrium States

1. Measure $\mu = \mu_\varphi$ is called an equilibrium state for function $\varphi \in C(X)$ on Y if $\mu_\varphi \in M_f(Y)$ and

$$h_{\mu_\varphi}(f \mid Y) + \int_X \varphi \, d\mu_\varphi = \sup_{\mu \in M_f(Y)} \left(h_\mu(f \mid Y) + \int_X \varphi \, d\mu \right).$$

THEOREM 5. Suppose that mapping f satisfies the following conditions:

1) f is a homeomorphism of Y;

2) f separates points [i.e., there is an $\varepsilon > 0$ such that for every $x, y \in Y$ the inequality $\rho(f^k(x), f^k(y)) \leq \varepsilon$ for all k implies that $x = y$];

3) the set $M_f(Y)$ is closed in M(X) (in the weak topology).

Then for every function $\varphi \in C(X)$ there is an equilibrium state.

Proof. Let ε be the separating constant for f. Then by repeating the proof of Proposition 2.5 in [1] and taking into account conditions 1) and 2) it is readily checked that $h_\mu \times (f \mid Y) = h_\mu(f \mid Y, \xi)$ for any measure $\mu \in M_f(Y)$ and any Borel partition ξ of Y with $\operatorname{diam} \xi \leqslant \varepsilon$. Using this fact and Lemma 4 and repeating the arguments given in the proof of Proposition 2.19 of [1] one can prove that $\mu \to h_\mu(f \mid Y)$ is upper semicontinuous. This in turn implies that function $\mu \to h_\mu(f \mid Y) + \int_X \varphi \, d\mu$. By condition 3), this function must attain its supremum on the set $M_f(Y)$.

The next result is a straightforward consequence of Theorems 2 and 5.

THEOREM 6. Suppose that mapping f verifies the conditions of Theorems 2 and 5 and let μ_φ be an equilibrium state for function φ. Then:

$$h_{\mu_\varphi}(f \mid Y) + \int_X \varphi \, d\mu_\varphi = P_Y(\varphi). \tag{22}$$

2. We apply the results obtained above to one-dimensional discontinuous mappings. Let $X = [0, 1]$, and let $A = \{a_l\}_{l=0}^{q}$ define a partition of segment X by points $0 = a_0 < a_1 < \ldots < a_q = 1$. Let $I_l = (a_{l-1}, a_l)$. Suppose that $T : X \setminus A \to X$ is a mapping such that:

a) T is continuous and monotonic on each interval I_l, and hence extends to a continuous mapping of I_l into X;

b) $T(a_l - 0) \neq T(a_l + 0)$, $l = 1, \ldots, q-1$.

Set $R = \{x \in X: T^n(x) \in A$ for some $n \geq 0\}$. It is readily checked that T is continuous on the noncompact set $Y = X \setminus R$. We call the point $T_+(x)$ the right image of the point x if there is a sequence $x_n \in X \setminus A$, whose terms lie at the right of x, such that $x_n \to x$ and $T \times (x_n) \to T_+(x)$. The left image $T_-(x)$ of x is defined in the same manner. Clearly, $T_+(x) = T_-(x) = T(x)$ whenever $x \in X \setminus A$. We call the sequence of points x_n, $n = 1,\ldots,p$ a generalized periodic trajectory of period p if $T_\delta(x_n) = x_{n+1}$ for $n = 1,\ldots,p-1$ and $T_\delta(x_p) = x_1$, where δ equals plus or minus.

THEOREM 7. Suppose that mapping T satisfies conditions a), b), and

c) the set R is dense in X;

d) T has no generalized periodic trajectories $\{x_n\}$ with $x_1 \in R$.

Then for every continuous function φ on X equality (16) holds and there exists an equilibrium state μ_φ satisfying equality (22).

Proof. The following argument was suggested by M. Lyubich. Let (Σ, σ) be a one-sided Bernoulli shift with q states, and let $\Psi: Y \to \Sigma$ be a mapping such that $\Psi(x) = (\omega_n)$, where $T^n(x) \subset I_{\omega_n}$, $n \geq 0$. Using conditions a), b), and c) one can show that mapping Ψ enjoys the following properties: 1) it maps Y homeomorphically onto its image $Q = \Psi(Y)$; 2) $\overline{Q} \setminus Q$ is countable; 3) Ψ^{-1} extends to a continuous mapping of $\overline{Q}$ onto X; 4) $\{\omega_n\} \in \overline{Q} \setminus Q$ if and only if $\Psi^{-1}(\omega) \in \bigcup_n T_\pm^n(R)$. Let φ be a continuous function on X. From Proposition 2 and properties 1)-3) of Ψ it follows that $P_Y(\varphi) \leq P_Q(\tilde{\varphi}) \leq P_{\overline{Q}}(\tilde{\varphi})$, where function $\tilde{\varphi}(\omega) = \varphi(\Psi^{-1}(\omega))$ is continuous on $\overline{Q}$. Since the set $\overline{Q}$ is compact, the variational principle holds for the mapping $\sigma|\overline{Q}$, i.e.,

$$\sup_{\mu \in M_\sigma(\overline{Q})} \left(h_\mu(\sigma) + \int_{\overline{Q}} \tilde{\varphi}\, d\mu\right) = P_{\overline{Q}}(\tilde{\varphi}). \tag{23}$$

From property 2) of mapping Ψ we deduce the measure $\mu \in M_\sigma(\overline{Q} \setminus Q)$ must have a component supported on a periodic trajectory of mapping σ, which is impossible in view of property 4) of Ψ and condition d) of Theorem 7. Consequently,

$$\sup_{\mu \in M_\sigma(\overline{Q})} \left(h_\mu(\sigma) + \int_{\overline{Q}} \tilde{\varphi}\, d\mu\right) = \sup_{\mu \in M_\sigma(Q)} \left(h_\mu(\sigma) + \int_Q \tilde{\varphi}\, d\mu\right). \tag{24}$$

Property 1) of mapping Ψ implies that

$$\sup_{\mu \in M_\sigma(Q)} \left(h_\mu(\sigma) + \int_Q \tilde{\varphi}\, d\mu\right) = \sup_{\mu \in M_f(Y)} \left(h_\mu(T|Y) + \int_Y \varphi\, d\mu\right). \tag{25}$$

From the foregoing discussion, equalities (23)-(25), and Theorem 1 we obtain the variational principle for $P_Y(\varphi)$. The existence of an equilibrium state μ_φ is a straightforward consequence of the foregoing discussion and the existence of an equilibrium state $\mu_{\tilde{\varphi}}$ for function $\tilde{\varphi}$ and mapping $\sigma|\overline{Q}$ (moreover, $\mu_\varphi = \Psi^{-1} * \mu_{\tilde{\varphi}}$).

We note a particular case in which condition d) of Theorem 7 is superfluous.

THEOREM 8. Suppose that under conditions a), b), and c) function φ is such that:

$$\sup \varphi - \inf \varphi \leq P_Y(0). \tag{26}$$

Then, for function φ equality (16) holds and there is a measure $\mu_\varphi \in M_f(X)$, satisfying (22).

Proof. If $P_Y(0) = 0$, then $\varphi = 0$. Since the set $\overline{Q} \setminus Q$ is countable, given any measure $\mu \in M_\sigma(\overline{Q})$ supported on $\overline{Q} \setminus Q$ we have $h_\mu(\sigma) = 0$. Hence [cf. (24)]

$$\sup_{\mu \in M_\sigma(\overline{Q})} h_\mu(\sigma) = \sup_{\mu \in M_\sigma(Q)} h_\mu(\sigma).$$

From now on we repeat the proof of Theorem 7. Suppose now that $P_Y(0) > 0$ and $\varphi \in C(X)$ satisfies condition (26). As in the proof of Theorem 7, we have that $P_{\overline{Q}}(0) \geq P_Y(0) > 0$. Since $\overline{Q}$ is compact, there is a measure μ such that $h_\mu(\sigma) = P_{\overline{Q}}(0)$. Pick any measure ν supported on $\overline{Q} \setminus Q$. Then, by (26),

$$\left(h_\mu(\sigma) + \int_{\bar Q} \varphi\, d\mu\right) - \left(h_\nu(\sigma) + \int_{\bar Q} \varphi\, d\nu\right) = P_{\bar Q}(0) + \left(\int_{\bar Q} \varphi\, d\mu - \int_{\bar Q} \varphi\, d\nu\right) \geqslant P_{\bar Q}(0) - (\sup\varphi - \inf\varphi) \geqslant 0.$$

Consequently, equality (24) holds for function φ, and to complete the proof one repeats the proof of Theorem 7.

Theorem 8 permits us to give a different and yet equivalent definition of topological pressure in the case of one-dimensional discontinuous mappings considered here. Specifically, if mapping T satisfies conditions a)-c) we set $P_Y(\varphi) = P_{\bar Q}(\bar\varphi)$. This is exactly the definition that was used in [12].† If mapping T satisfies condition d) or if function φ satisfies (26), then this definition is equivalent to ours. In the same manner, the topological entropy $h_Y(T)$ may be defined as $h_{\bar Q}(\sigma)$. This is the usual definition of topological entropy for one-dimensional mappings [11].

4. Topological Entropy

1. Let Y be an arbitrary (generally speaking) noncompact metric space, $f: Y \to Y$ a continuous mapping, and φ a continuous function on Y. Consider an arbitrary finite open covering $\mathcal{U}$ of Y. Define on Y an outer measure $m(\mathcal{U}, \lambda, Z, \varphi), Z \subset Y$, by formula (1), and then take the corresponding pressure $P_Z(\mathcal{U}, \varphi)$ as in Sec. 1, No. 1. Now define the topological pressure on the set Z corresponding to function φ, by the equality $P_Z^*(\varphi) = \sup P_Z(\mathcal{U}, \varphi)$, where the supremum is taken over all finite open coverings $\mathcal{U}$ of Y. If Y is a subset of the compact metric space X and the metric of Y is induced by the metric of X one can readily show that $P_Z^*(\varphi) = P_Z(\varphi)$ for all $\varphi \in C(X)$. If Y is arbitrary the last equality does not hold in general and Theorems 1 and 2 are not valid for pressure $P_Z^*(\varphi)$. In what follows we confine our discussion to the case $\varphi = 0$.

2. We call topological entropy of the mapping f of the (noncompact) set Y the quantity $h^*(Y, f) = P_Y^*(0)$. In [2] Bowen gave a different definition of topological entropy. We recall it here. Let $\mathcal{U}$ be a finite open covering of Y, and let $\{E_i\}$ be a family of set covering Y. We write $\{E_i\} \prec \mathcal{U}$ whenever each set E_i is included in some element of $\mathcal{U}$. Set

$$n_f(E_i) = \min\{n: f^k(E_i) \prec \mathcal{U} \text{ for } k = 0, 1, \ldots, n-1 \text{ and } f^n(E_i) \not\prec \mathcal{U}\}.$$

When $f^k(E_i) \prec \mathcal{U}$ for all k, we set $n_f(E_i) = \infty$. Next, let

$$D(\{E_i\}, \lambda) = \sum_i \exp(-\lambda n_f(E_i)),$$
$$\tilde M(\mathcal{U}, \lambda, Y, N) = \inf_{\{E_i\}} \{D(\{E_i\}, \lambda): \bigcup_i E_i \supset Y,\ n_f(E_i) \geqslant N\}.$$

It is readily verified that function $\tilde M(\mathcal{U}, \lambda, Y, N)$ does not decrease with the growth of N, which guarantees the existence of the limit

$$\tilde m(\mathcal{U}, \lambda, Y) = \lim_{N\to\infty} \tilde M(\mathcal{U}, \lambda, Y, N).$$

Further, it is readily verified that $\tilde m(\mathcal{U}, \lambda, Y)$, as a function of λ, enjoys the following properties: there is a λ_0 such that $m(\mathcal{U}, \lambda, Y) = 0$ for $\lambda > \lambda_0$ and $\tilde m(\mathcal{U}, \lambda, Y) = \infty$ for $\lambda < \lambda_0$. Now set

$$\tilde h(\mathcal{U}, Y, f) = \inf\{\lambda: \quad \tilde m(\mathcal{U}, \lambda, Y) = 0\},$$
$$\tilde h(Y, f) = \sup \tilde h(\mathcal{U}, Y, f),$$

where the supremum is taken over all finite open coverings $\mathcal{U}$ of Y.

Proposition 4. $h^*(Y, f) = \tilde h(Y, f)$.

Proof. Let us show that $h^*(\mathcal{U}, Y, f) = \tilde h(\mathcal{U}, Y, f)$ for every finite covering $\mathcal{U}$ of Y.

1) Suppose that $\lambda > \tilde h(\mathcal{U}, Y, f)$. Then there exists a collection of sets $\{E_i\}$ covering Y such that $D(\{E_i\}, \lambda) < 1$. Now attach to each E_i with $n_f(E_i) < \infty$ the set $Y(\underline{U}^i)$, where $\underline{U}^i = \{U_1^i, \ldots, U_{m(\underline{U}^i)}^i\}$ is a collection with the properties $m(\underline{U}^i) = n_f(E_i)$ and $f^k(E_i) \subset U_k^i$ for $k = 0, 1, \ldots, m(\underline{U}^i)$. Also, attach to each E_i with $n_f(E_i) = \infty$ the set $Y(\underline{U}^i)$, where $\underline{U}^i = \{U_1^i, \ldots, U_{m(\underline{U}^i)}^i\}$ is a collection with the properties $E_i \subset Y(\underline{U}^i)$ and $\exp(-\lambda m(\underline{U}^i)) \leqslant \exp(-i)$.

†Note, however, that for arbitrary functions φ all results in [12] were established under condition (26), i.e., when the two definitions are equivalent. In this case, in [12] are described the ergodic properties of measure μ_φ.

Then it is readily checked that $m(\mathcal{U}, \lambda, Y, 0) \leqslant D(\{E_i\}, \lambda) + \sum_i \exp(-i) < \infty$. Therefore, $h^*(\mathcal{U}, Y, f) = P_Y^*(\mathcal{U}, 0) \leqslant \lambda$, and consequently $h^*(\mathcal{U}, Y, f) \leqslant \tilde{h}(\mathcal{U}, Y, f)$.

2) Suppose that $\lambda > h^*(Y, f)$. Then for every $N > 0$ there is a collection $\Gamma_N \in \mathcal{W}(\mathcal{U})$, such that 1) $m(\underline{U}) \geqslant N$ for all $\underline{U} \in \Gamma_N$; 2) $Y \subset \bigcup_{\underline{U} \in \Gamma_N} Y(\underline{U})$; 3) $\sum_{U \in \Gamma_N} \exp(-\lambda m(\underline{U})) < \infty$.

Set $E(\underline{U}) = Y(\underline{U})$. Then clearly $n_f(E\underline{U})) \geqslant m(\underline{U}) \geqslant N$. It is readily checked that the family of sets $E(\underline{U})$ covers Y and

$$D(\{E(\underline{U})\}, \lambda) \leqslant \sum_{\underline{U} \in \Gamma_N} \exp(-\lambda n_f(E(\underline{U}))) \leqslant \sum_{\underline{U} \in \Gamma_N} \exp(-\lambda m(\underline{U})) < \infty.$$

Therefore, $\tilde{m}(\mathcal{U}, Y, \lambda) < \infty$, whence $\tilde{h}(\mathcal{U}, Y, f) \leqslant \lambda$, which means that $\tilde{h}(\mathcal{U}, Y, f) \leqslant h^*(\mathcal{U}, Y, f)$.

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