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COMPLETE INTEGRABILITY OF ORDINARY DIFFERENTIAL EQUATIONS ON SUPERMANIFOLDS

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We say that an ordinary differential equation is completely integrable [2] if its solution can actually be constructed, i.e., it can be obtained by a finite number of algebraic operations, changes of coordinates, integrations, and computations of partial derivatives.

1. Let $\mathscr{M}$ be a supermanifold, $\mathscr{T}$ – time, I – the ideal generated by all the uneven functions in $C^\infty(\mathscr{M})$, and write $\mathrm{Vect}_{\mathscr{T}}\,\mathscr{M} = \{D \in \mathrm{Vect}\ \ \mathscr{M}\times\mathscr{T} \mid D\mid_{C^\infty(\mathscr{T})} = 0\}$.

THEOREM 1. Suppose that $D_1, D_2 \in \mathrm{Vect}_{\mathscr{T}}\,\mathscr{M}$ satisfy $D_1 \equiv D_2 \bmod I^2\,\mathrm{Vect}_{\mathscr{T}}\mathscr{M}$, where $p(D_i) = 0\,(1)$ if $\dim \mathscr{T} = (1, 0)$ (respectively, if $\dim \mathscr{T} = (1,1)$), and let φ_i be the solution of the differential equation corresponding to D_i. Then, from a given φ_1, one can construct φ_2, and conversely.

COROLLARY. The integration of an even field D on $\mathscr{M}^{r,s}$ reduces to the integration of the corresponding (see [2]) field πD on the underlying manifold M and to that of a system of linear, nonautonomous equations on $\mathbf{R}^s$. In particular, the differential equations on $\mathscr{R}^{0,s}$ and $\mathscr{R}^{1,s}$ are completely integrable. The integration of the Hamiltonian system defined by the Hamiltonian H with respect to the even form $\Sigma dp_i dq_i + \Sigma \varepsilon_j d\xi_j^2$, where $\varepsilon_j = \pm 1$, reduces to the integration of the system defined by the Hamiltonian πH with respect to the form $\Sigma\, dp_i dq_i$ and to that of a system of linear nonautonomous equations having the matrix (a_{ij}), where $D \equiv \pi(D) + \Sigma a_{ij}\xi_i \frac{\partial}{\partial \xi_j} \bmod I^2$, and ξ are the uneven coordinates on $\mathscr{M}$.

Recall that the integration of an uneven equation having (1, 1)-dimensional time reduces to the integration of an even equation [2].

As examples of systems to which the theorem and its corollary apply, we may take dynamical systems on the orbits of the coadjoint representations of supergroups for which all invariant polynomials on their Lie superalgebras are even (such are the simple Lie superalgebras forming the series sl, osp, SH(0|2n), and the corresponding Kac–Moody superalgebras; the corresponding dynamical systems are the super-Liouville system, the (p, q)-dimensional top, the heavy superbody, and their generalizations [3, 4].

Let us note that the above mentioned linear system of nonautonomous equations can be solved by applying the T-exponent, and in this way one can extend the range of applicability of our theorem to all systems having an integrable underlying system.

2. If $\Sigma du_i d\xi_i$ is an uneven form, then the field πD on the underlying manifold corresponding to an uneven Hamiltonian $H \equiv \Sigma \xi_i f_i \bmod I^3$, has the absolutely general form $\Sigma f_i(u)\,\partial/\partial u_i$, and the integration of the field D requires special methods.

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Let $\mathcal{M}$ be a supermanifold with canonical form ω, and let $\{,\}$ be the corresponding bracket. A function f is said to be a first integral of the system with Hamiltonian H if $\{f, H\} = 0$. The functions $f_1,\ldots,f_n$ are said to be in involution if $\{f_i, f_j\} = 0$ for all i, j.

THEOREM 2 (Analogue of Liouville's Theorem [1]). On the supermanifold $\mathcal{M}^{r,s}$, $r+s=2n$, assume that a Hamiltonian system has n first integrals $f_1,\ldots,f_n$ which are in involution and such that $df_1,\ldots,df_n$ are linearly independent at the point $m \in \mathcal{M}$. Then there exists a neighborhood $\mathcal{U} \ni m$ with coordinates $f_1,\ldots,f_n$, $g_1,\ldots,g_n$, and, in addition $\omega = \Sigma df_i dg_i$.

Proof of Theorem 1. The equalities $D_{\mathcal{T}}(\varphi_1^* - \varphi_2^*) \equiv 0 \bmod I^2 \operatorname{Vect}_{\mathcal{T}}\mathcal{M}$ and $\varphi_1^* - \varphi_2^*|_{\mathcal{T}} = 0$ (see [2] for the definition of the field $D_{\mathcal{T}}$) imply that $\varphi_1^* \equiv \varphi_2^* \bmod I^2$. Set $\varphi_2^* = (1+A)\,\varphi_1^*$, where $A: C^\infty(\mathcal{M}\times\mathcal{T}) \to C^\infty(\mathcal{M}\times\mathcal{T})$ is an operator. Then the equation satisfied by A has the form $(1+A)\,\varphi_1^*(D_1 - D_2)(\varphi_1^*)^{-1} = [D_{\mathcal{T}}, A]$.

For our purpose it is enough to define the action of the operator A on $C^\infty(\mathcal{M})$ inductively, and construct $A_i \equiv A \bmod I^i$, starting with $A_1 = 0$. Set $\triangle D = \varphi_1^*(D_1 - D_2)(\varphi_1^*)^{-1}$. Then $\Delta D\,(C^\infty(\mathcal{M})) \subset I$ and $(1+A)\,\Delta D = D_{\mathcal{T}}A$ on $C^\infty(\mathcal{M})$, whence $D_{\mathcal{T}}A_{i+1} \equiv (1 + A_i + A_{i+1} - A_i) \times \triangle D \equiv (1 + A_i)\triangle D \bmod I^{i+1}$ and A_{i+1} is found by integration; $A = A_{s+1}$.

Proof of Theorem 2. Let $\mathcal{U} = \mathcal{W} \times \mathcal{V}$ be a neighborhood of the point m, such that the given first integrals $f_1,\ldots,f_n$, form a system of coordinates on $\mathcal{W}$, and let $h_1,\ldots,h_n$ be coordinates on $\mathcal{V}$. Since ω is nondegenerate, the operator $A_\omega: \operatorname{Vect}\mathcal{U} \to \Omega^1\mathcal{U}, A_\omega D = i(D)\,\omega$ is an isomorphism. The collections $\{\partial/\partial f_1,\ldots,\partial/\partial f_n\}$ and $\{D_i = A_\omega^{-1}(df_i),\ 1 \le i \le n\}$ consists of weakly nondegenerate vector fields (the first by hypothesis, and the second due to the invertibility of the operator A_ω). These vector fields are actually all linearly independent (because the equality $\Sigma k_i \partial/\partial f_i = \Sigma k_i' D_i = D$ implies $D(f_j) = \Sigma k_j'\{f_i, f_j\} = 0$ and $D(h_i) = \Sigma k_i \partial h_i/\partial f_i$) and so they form a basis in $\operatorname{Vect}\mathcal{U}$. By the Poincaré Lemma for supermanifolds, $\omega = dl$, where $l = \Sigma(a_i df_i + b_i dh_i)$. Define d' as the $C^\infty(\mathcal{W})$-linear extension of the differential d from $\Omega(\mathcal{V})$ to $C^\infty(\mathcal{U})\,\Omega(\mathcal{V}) \subset \Omega(\mathcal{U})$; and the set $l' = \Sigma b_i dh_i$ and $\omega' = d'l'$. Since $i(D_j) = \omega = df_j$ and $i(\partial/\partial f_j)\omega' = 0$, we have $\omega' = 0$; thus $l' = d'\Phi = \Sigma \partial\Phi/\partial h_i dh_i$ and $l = d\Phi + \Sigma(a_i - \partial\Phi/\partial f_i)df_i$. We conclude that $\omega = \Sigma df_i dg_i$, where $g_i = a_i - \partial\Phi/\partial f_i$, and the fact that ω is nondegenerate shows that $f_1,\ldots,f_n$, $g_1,\ldots,g_n$ is a system of coordinates.

The Corollary follows by observing that $D = \pi D + \Sigma a_{ij}(u)\xi_i \partial/\partial\xi_j \bmod I^2$ whenever $p(D) = 0$.

Theorem 2 applies, for example, to dynamical systems on the orbits of the coadjoint representation of the supergroup $\mathfrak{Q}(n)$, that represent the functor $C \to Q(n; C) = \{X \in \operatorname{Mat}(n|n; C) \mid [X, \pi_n] = 0\}$, where $\pi_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. The invariant polynomials on $\mathfrak{Q}(n)^*$ are $H_k = \operatorname{otr} X^k$, where $\operatorname{otr}\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \operatorname{tr} B$. The choice H_2 yields an exotic analogue of the Toda lattice. Other examples are connected to the reduction $\mathfrak{Q}(n) \to \operatorname{Po}(0 \mid 2n+1)$, [5] and to the Kac–Moody superalgebras corresponding to the series $\mathfrak{Q}$ and Po.

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