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ON THE DIFFICULTY OF MAKING SOCIAL CHOICES

ABSTRACT. The difficulty of making social choices seems to take on two forms: one that is related to both preferences and the method used in aggregating them and one which is related to the preferences only. In the former type the difficulty has to do with the discrepancies of outcomes resulting from various preference aggregation methods and the computation of winners in elections. Some approaches and results which take their motivation from the computability theory are discussed. The latter 'institution-free' type of difficulty pertains to solution theory of the voting games. We discuss the relationships between various solution concepts, e.g. uncovered set, Banks set, Copeland winners. Finally rough sets are utilized in an effort to measure the difficulty of making social choices.

Keywords: Social choice, computer simulation, computational complexity, Banks set, Copeland winner, uncovered set, NP-completeness, rough set.

1. INTRODUCTION

That different voting systems may produce different winning alternatives for fixed voter opinions and voting strategies is not a new observation (see, e.g. Riker, 1982). Each voting system has a background intuition of what constitutes a good social choice. For example, the plurality system is based on the idea that no alternative should be regarded as best or socially most acceptable unless it gets more votes than any other alternative when every voter can vote for one and only one alternative. The fact that, given fixed voter opinions, two systems end up with different winning alternatives can thus be regarded as an indication that the underlying intuitions are different.

It would seem natural that the more often the winning alternatives of voting systems differ, the more different are their underlying intuitions. The frequency of different choice sets depends, however, on the distribution of voter opinions, i.e. preference profiles. Generating the (connected and transitive) preference ordering of each voter over the set of alternatives randomly and independently of the preference

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orders of other voters, one obtains an impartial culture (IC, for brevity) preference profile. Given the number of alternatives and the size of the electorate the relative frequency of the occurrences in which two voting systems produce different winning alternatives gives an estimate of how different the intuitions underlying those systems are (see Nurmi, 1988a).

One of the difficulties of making social choices is that all procedures in use have some plausible and some implausible properties (see e.g. Richelson, 1979, and Riker, 1982). So the distinct choice sets of procedures are not due to the fact that some procedures are bad and others good, *simpliciter*. One could argue, however, that some of the difficulties of social choices are procedure-related, i.e. can only be discussed when both the choice situation and the preference aggregation procedure are given. I shall call this type I difficulty.

On the other hand, there are circumstances in which all reasonable systems agree, i.e. produce identical choice sets. In those circumstances, thus, the choice of the procedure is of no consequence. An example of such a circumstance would, of course, be a profile where each voter has an identical preference ordering over the alternatives. A contrast to this profile would be one in which no pair of voters would agree on the rank of any alternative, i.e. each alternative would get a different rank from each voter (e.g. the Condorcet voting paradox profile). The contrast between these two types of profiles would seem to suggest that an aspect of the difficulty of making social choices is related to profiles only. I shall call this type II difficulty.

In this article I shall discuss both types of difficulty of making social choices. Type I difficulty is first focused upon. I review some computer simulations pertaining to the discrepancies of the choice sets of various procedures in various cultures. I shall also discuss results on the computational complexity of some procedures. Thereafter, I shall look at type II difficulty and start with a description of the relationships between various solution concepts in tournaments. I shall then try to outline an approach to type II difficulty, viz. one based on the theory of rough sets (see Pawlak, 1982). Finally, some concluding remarks are presented.

2. TYPE I DIFFICULTY

2.1. Computer Simulations

The probabilistic and simulation studies of voting procedures can be divided into two groups: (1) those producing probability estimates of the criterion violations of procedures; and (2) those looking at the probability that various procedures end up with different choice sets. Of the former group the best-known are results on the Condorcet efficiency of various voting procedures (see e.g. Merrill, 1988) and on the probability of cyclic majorities (Niemi and Weisberg, 1968; Gehrlein, 1981; Gehrlein, 1983; Berg, 1985).

The computer simulations of ICs shed some light on what factors make social choices difficult in the sense of increasing the probability of different choices being made by different procedures. Obviously, if procedures end up with identical choices for a fixed preference profile, then the social choice setting is not difficult, whereas if the choice depends on the method chosen, then the setting is difficult. Now, in ICs various factors seem to be accompanied with the probability of discrepancies in the choice sets (see Nurmi, 1988b, 1990, 1992).

First of all one would expect that Condorcet extension methods would have more discrepancies with non-Condorcet extension methods than with each other. It will be recalled that a method is a *Condorcet extension* if it always chooses the Condorcet winning alternative whenever one exists. A *Condorcet winning alternative* or *Condorcet winner* is an alternative that defeats all the other alternatives with a simple majority of votes, provided that each voter votes according to his/her (hereinafter his) preferences. An example of a Condorcet extension method is Copeland's rule. To determine the winner among k alternatives, one performs for each alternative x all $(k-1)$ pairwise comparisons and computes, on the basis of the preference profile, how many alternatives x defeats by a simple majority. Thus, one obtains the Copeland score $C(x)$ of x. The alternative(s) with the highest score is (are) Copeland winner(s). Another Condorcet extension is Nanson's method. It is an elimination procedure based on Borda scores (see Nanson, 1883; Niou, 1987; Nurmi, 1989, for various interpretations of the method). The version we shall discuss here eliminates at each stage the alternative that has the smallest Borda score. After the elimination, new scores are computed for the remaining alternatives and the procedure is continued until only one alternative remains.

Plurality voting, Bordal count, plurality runoff method and Hare's method, on the other hand, are not Condorcet extensions, i.e. they may exclude a Condorcet winner from the choice set. Hare's method is the single transferable vote (STV) method when only one candidate is elected.

Tables I and II report the discrepancies expressed in percentages between Hare's system, on the one hand, and plurality runoff and Nanson's method, respectively, on the other (Nurmi, 1992). Table I thus gives us an idea of what is the percentage of different choices between two non-Condorcet extension methods, whereas Table II

TABLE I

CANDIDATES	The Number of Voters: 5	7	11	15	25	51	101	151	201	301	999
3	0	4	1	0	2	0	0	0	0	0	0
4	6	13	10	11	11	11	12	12	12	12	13
5	15	18	17	18	19	20	21	22	21	21	22
6	22	22	23	25	26	27	29	29	28	29	30
7	29	26	28	29	31	32	33	34	35	35	36
13	48	45	45	46	50	53	54	56	55	57	57
15	52	50	46	52	54	56	59	59	59	61	63

TABLE II

CANDIDATES	The Number of Voters: 5	7	11	15	25	51	101	151	201	301	999
3	5	8	6	7	7	7	7	7	7	7	7
4	11	14	13	14	14	14	14	14	14	13	14
5	19	21	19	19	20	21	20	19	20	19	20
6	24	26	25	25	25	25	25	25	25	24	25
7	29	30	28	29	29	29	29	30	29	30	29
13	63	66	65	66	65	51	63	64	52	53	53
15	69	72	72	72	72	54	70	70	55	56	56

indicates the discrepancies between a non-Condorcet extension (Hare's system) and a Condorcet extension method. Both tables are based on the IC assumption. Moreover, ties are broken randomly in all simulations. All estimates are based on 10 000 IC elections.

The first impression one gets from these tables is that the discrepancies are roughly of the same order of magnitude. However, the discrepancies in Table I are generally somewhat smaller than in Table II. A more important observation is that the discrepancies tend to increase with the number of alternatives. Increasing the size of the electorate does not have the same effect on the discrepancies over the range of electorates focused upon in the above tables.

The IC assumption is, of course, very restrictive if one wishes to come up with 'realistic' probability estimates. This, however, is not our intention. IC serves as a benchmark in assessing the impact that various groupings have on discrepancies. In bipolar cultures (BC, for short) it is assumed that there are two groups of voters, each comprising 1/3 of the electorate and having diametrically opposing preferences. Within each group the preferences of individuals are identical. The rest of the electorate is an IC.

Some of the discrepancy estimates remain largely unaffected when the IC assumption is replaced by the BC one (see Nurmi, 1992). This is, for example, the case with Hare's and Nanson's systems. However, the discrepancies between the plurality runoff and Hare's systems almost vanish when BCs are considered.

BC is, of course, a special type of culture. More reliable estimates concerning the robustness of the IC results can be obtained by making minor modifications to the IC assumption, such as assuming that there is a group of voters with identical preferences that consists of 10% of the electorate while the remaining 90% form an IC. Let us call these cultures UPC cultures.

The effects of replacing ICs with UPCs on the discrepancies of various voting procedures are considerable. Roughly speaking, in large electorates the discrepancies between voting systems disappear. In particular, this happens to Hare's system *vis-à-vis* plurality runoff and Nanson's methods (Nurmi, 1992). Thus, a fairly small perturbation of the IC assumption brings about a qualitatively different choice behaviour in preference aggregation systems (see Tables III and IV).

TABLE III

The Number of Voters		5	7	11	15	25	51	101	151	201	301	999
C												
A												
N	3					1	0	0	0	0	0	0
D	4					11	10	6	4	3	1	0
I	5					18	16	11	7	5	2	0
D	6					24	22	16	11	8	4	0
A	7					28	27	19	14	10	5	0
T												
E	13					46	45	34	25	18	10	0
S	15					50	46	35	26	20	11	0

TABLE IV

The Number of Voters		5	7	11	15	25	51	101	151	201	301	999
C												
A												
N	3					7	6	5	4	3	2	0
D	4					13	13	10	8	6	3	0
I	5					19	17	15	11	8	4	0
D	6					26	26	22	17	13	7	0
A	7					32	32	28	22	17	9	0
T												
E	13					50	50	44	37	30	19	0
S	15					55	54	47	40	32	20	0

The computer simulations referred to above thus suggest that the main sources of the difficulty in making social choices are large candidate or alternative sets and – to a lesser extent – large electorates. However, these factors lose most of their importance in cultures where even small homogeneous groupings exist, surrounded by ICs. One could thus argue that the difficulty encountered in simulation studies has something to do with the underlying restrictive culture assumptions. We now turn to another approach to studying the type I difficulty of making social choices, viz. by looking at the computational difficulty of voting games.

2.2. *Computational Difficulty of Procedures*

2.2.1. Difficulty of finding winners. The theory of computational complexity offers a precise way to evaluate the difficulty of various preference aggregation procedures. The complexity pertains to algorithms used for computing the values of functions. As preference aggregation methods can be viewed as functions, it is natural to look at their computability properties (see, e.g. Hopcroft and Ullman, 1979; Rogers, 1967; Salomaa, 1985, for basic concepts and results of the theory of computation).

A function is (recursively) computable if there is a Turing machine that, given the argument of the function, computes its values. The basic and coarsest division of functions separates computable from noncomputable functions. The latter have the distinction that no algorithm exists for the determination of their values. A classic example of noncomputable functions is the halting problem. All distinct Turing machines can be characterized or indexed by positive integers. Consider an integer i, given in binary form, and function $f(i)$ from positive integers to $\{0, 1\}$. The problem of whether the ith Turing machine computes the value of $f(i)$, i.e. whether the ith machine halts after processing the input i, is undecidable. Thus, noncomputable functions exist.

Of course, all voting procedures are computable in the sense that, given the preference profiles, the procedures determine the winners in finite time. An interesting problem in this context is, however, whether the conditions characterizing various voting procedures are sufficient to guarantee computability. Kelly (1988a, 1988b) discusses this problem, focusing on two axiomatized voting procedures, viz. the simple majority rule for two alternatives and the Borda count. The former procedure has been axiomatized by May (1952) and the latter by Young (1974).

Kelly (1988a) shows that of the three individually necessary and jointly sufficient conditions for simple majority rule – anonymity, neutrality and positive responsiveness – all are needed to guarantee computability. The demonstration proceeds by showing that, should any one of the conditions not be satisfied, there would exist noncomputable functions satisfying the rest of the conditions. On the other hand, if all three are satisfied, the function – the simple majority rule – is obviously computable.

In a similar way Kelly (1988b) discusses the consequences of dropping some of Young's axioms for the Borda count: faithfulness, consistency, neutrality and cancellation. He shows that leaving consistency aside while retaining the other conditions would allow for noncomputable social choice rules. The same observation holds for the case in which cancellation is dropped while retaining the other three conditions. Whether dropping the faithfulness or neutrality would have a similar effect on the computability is an open question. Kelly conjectures that if a social choice function satisfies consistency, neutrality and cancellation, then it will necessarily be computable. Similarly, he conjectures that if a social choice function is faithful, consistent and has the cancellation property, then it must be computable.

Now, the above results are useful mainly in the contexts of axiomatized rules. They deal with properties of functions that are compatible or incompatible with the requirement that no noncomputable choice functions be allowed. They tell us which requirements exclude the noncomputable functions. Of more practical nature are results that examine the difficulty of determining winners of elections from the computational complexity point of view.

Suppose that the winner in an election is determined by the scores of alternatives so that the alternative(s) with the largest score is (are) elected. The scores may be determined by the number of first ranks given to each alternative, as in the plurality voting, or by the number of other alternatives defeated by each alternative by a simple majority in pairwise comparisons, as in Copeland's rule or by some other method. Given a preference profile, an alternative set, a fixed number L and the scores of each alternative, one may ask if the score of a given alternative is less than or equal to L.

For a Dodgson system, answering this question is an NP-complete problem. This has been shown by Bartholdi *et al.* (1989a). Dodgson's system is a Condorcet extension method. The Dodgsons scores are determined by computing for each alternative the number of preference changes that would be needed to make it the Condorcet winner. Obviously, when a Condorcet winner exists, it is the Dodgson winner as well. The fact that discovering whether the Dodgson score of an alternative is less than or equal to a fixed number L is an NP-complete problem means: (i) that no polynomial time algorithm for its solution is known; and (ii) that each instance of some known NP-complete problem is an instance of the problem related to Dodgson scores. (For a discussion and an extensive list of NP-complete problems, see Garey and Johnson, 1979.) In fact, the requirement (i) is redundant since (ii) already guarantees that (i) holds.

In demonstrating that the computation of the Dodgson score is an NP-complete problem, Bartholdi *et al.* (1989a) proceed by pointing out first that, once a tentative answer to the question of whether the Dodgson score of alternative is less than or equal to L has been given, its correctness can be checked in polynomial time. They then demonstrate that each instance of a problem known to be NP-complete can be translated in polynomial time into an instance of the problem of computing the Dodgson scores. In this case the NP-complete problem is that of exact cover by 3-sets which Garey and Johnson (1979) have shown to be NP-complete. An instance of this problem is the following: given a set X of objects with cardinality of $3p$ (where p is an integer) and a collection Y consisting of 3-element subsets of X, is there a subcollection Y' of Y that would cover X in the sense that each element of X appears in one and only one element of subcollection Y'? For each instance of this problem there exists a preference profile such that, given a solution to the exact cover by the 3-sets problem, the Dodgson score problem can be computed in polynomial time.

The computation of Kemeny scores is NP-complete as well. This has also been shown by Bartholdi *et al.* (1989a). Given a preference profile, Kemeny's rule produces a collective preference relation that can be obtained from the individual voter's preference relations with minimum number of pairwise preference changes. It is thus rather similar in spirit to Dodgson's rule, but while the latter looks at the number of preference changes needed to make an alternative the Condorcet winner, Kemeny's rule focuses on the entire preference relation.

The NP-complete problem, which can be translated into the problem of computing Kemeny scores, is known as the feedback arc set problem (Garey and Johnson, 1979): given a directed graph with R nodes and a positive integer L, is there a set of at most L edges that includes at least one edge from each cycle in the graph?

Prima facie, it is somewhat strange that computing the Dodgson and Kemeny scores, and thus determining the winners, is computationally intractable. After all, one could envision elections where these methods are used. For fixed numbers V and A of voters and alternatives, respectively, one can impose a constant upper bound on the number of computations needed to determine the scores (see Bartholdi *et al.*, 1989a). Given a preference profile, a fixed alternative x can be in one of A positions in a fixed voter's preference order. Thus, x can be in A^V positions in the preference profile as a whole. For each preference profile, one can form an $n \times n$ matrix of pairwise comparisons with $(i, j) = n_{ij}$, the number of individuals preferring the alternative represented by row i to the alternative represented by column j. In determining x's Dodgson score one identifies those columns y whose entries on x's row are no larger than $V/2$. The necessary changes have to be made in the preferences of those individuals who prefer those y's to x. Let us denote this subset of V by V_{yx}. One then determines those individuals in V_{yx} whose preferences can be changed from yPx to xPy with minimum number of secondary changes. Continuing in this way, one eventually ends up with a modified preference profile where x defeats y with a minimal simple majority. Clearly, there is a constant upper limit in the number of computations needed to determine the Dodgson scores.

The same is true of the Kemeny scores. Given a fixed preference profile, one can enumerate the $A!$ potential collective preference relations, and for each one of them one may determine how many individuals have to make binary changes in their preferences to render the preference order unanimous.

Now, even though an upper limit can be imposed on the number of computations needed to determine Dodgsons and Kemeny scores for fixed preference profiles, it is easy to see that the number of computations can be huge. Moreover, this number tends to increase more rapidly with the increase of the number of alternatives than with the increase of the number of voters. The results on computational complexity are, however, typically 'worst case' ones. In other words, they pertain to situations of maximal computational effort. They are thus not representative samples of the amount of computational effort one has to expend in typical circumstances. For example, in preference profiles where more than 50% of the voters position the same alternative to the first rank, it is *eo ipso* the Condorcet winner and can thus immediately be spotted.

2.2.2. Difficulty of resorting to strategic behaviour. The computational complexity considerations pertain to the difficulty of making social choices in other respects as well. In particular, issues of strategic behaviour of agenda-setters and voters have been discussed in the light of the amount of computation one needs in order to control the election (see Bartholdi *et al.*, 1989b, 1989c; Bartholdi and Orlin, 1991). In this context the computational complexity is rather a virtue than a vice of procedures. The more difficult it is for the voters to benefit from strategic misrepresentation of their preferences, the more likely it is that they vote according to their true preferences. And this is often the very rationale of resorting to voting. Similarly, the more difficult it is for the agenda-setter to benefit from strategic behaviour, the less likely he is to resort to such maneuverings.

The difficulty of strategically misrepresenting one's preferences has been studied on the basis of empirical data by Chamberlin (1985). Hare's system, i.e. STV for a one-member constituency, turns out to be very resistant to manipulation. The amount of information about the preference profile as a whole one needs to benefit from preference misrepresentation is typically considerably larger than in positional systems, e.g. plurality voting (see Nurmi, 1987, pp. 118–124). Bartholdi and Orlin (1991) give an exact account of the intuitive difficulty of preference misrepresentation in Hare's system: the problem of discovering whether there is an effective preference order to get one's favourite elected is NP-complete. The known NP-complete problem used in the demonstration is 3-cover (Garey and Johnson, 1979). In this problem one is given a set A with cardinality a and a family of 3-element subsets of A: $A_1, \ldots, A_k$. The problem is to determine if there exists a subfamily of sets A_i of $a/3$ members such that their union coincides with A.

An equally important, if not more important, problem than the strategic misrepresentation of preferences for democratic institutions is the possibility agenda manipulation. Although the theoretical results of social choice theory suggest that the possibilities are ubiquitous (see e.g. Riker, 1982) and also some laboratory evidence seems to lend support to this contention (see Plott and Levine, 1978), the computational requirements for successful agenda manipulation may be prohibitive. Bartholdi *et al.* (1989c) give an interesting discussion of two types of voting procedures from the point of view of how much computational effort is needed to successfully control agendas using some particular techniques of manipulation. The procedures are: (i) Condorcet extensions in profiles where a Condorcet winner is assumed to exist; and (ii) the plurality voting method.

The specific techniques discussed are: (1) those used in modifying the alternative set; and (2) those used in modifying the set of voters. It turns out that Condorcet extension methods are computationally resistant to manipulations through adding and deleting voters or through partitioning of the voter set, whereas the plurality method is computationally resistant to adding and deleting alternatives or partitioning of the alternative set (Bartholdi *et al.*, 1989c). Moreover, Condorcet extensions are immune to manipulation through adding alternatives in profiles where a Condorcet winner exists as obviously a new alternative may possibly defeat the Condorcet winner, but it cannot make any other 'old' alternative the Condorcet winner. Deleting alternatives, on the other hand, may succeed without excessive computations in rendering the desired alternative the Condorcet winner.

Manipulating the plurality procedure through adding suitable voters is, of course, computationally easy, as is the deletion of undesired voters. In contrast, adding voters in Condorcet extension methods may have surprising overall results, even though in some pairwise contests the added voters might make the desired change in majority preference relation. Thus, the Condorcet extension methods are computationally difficult to manipulate through adding or deleting voters.

Computational complexity considerations would thus seem to affect the choice theoretic properties that voting procedures have. If the determination of winners in elections is computationally difficult, then it is not advisable to utilize such methods in large scale elections. In interpreting those results one should, however, bear in mind that they pertain to worst case situations. Thus, even in elections with large numbers of candidates and voters, the determination of winners is not necessarily or even typically exceedingly difficult. As was already pointed out above, the determination of winners of any Condorcet extension method – be it Kemeny's of Dodgson's or some other method – is very easy in elections where one candidate is ranked first by more than 50% of the voters. With a fixed number of voters and candidates, the amount of computations can always be bounded by a constant which, however, may be huge.

It is also worth reiterating that computational complexity need not be an especially bad thing, if it pertains to operations needed to manipulate either by preference misrepresentation or by agenda manipulation. In these circumstances it would, *ceteris paribus*, be advisable to utilize procedures where these types of strategic behaviour are computationally difficult.

3. TYPE II DIFFICULTY

The above considerations on the difficulty of making social choices stem from the use of various preference aggregation procedures. In this section we focus on what have been called 'institution-free' properties of social choice (McKelvey, 1986). In other words, we shall focus on the relationships of various solution concepts of voting games, regardless of the procedures that could be used.

3.1. Definitions and Solution Configurations

The most obvious requirement to be imposed on a social choice is that it results in a Pareto-undominated outcome. In other words, if x is chosen from the set A, then it should not be the case that there exists a y in A that would be regarded as at least as good as x by all voters and strictly better than x by some voters. This requirement is in a way minimal and yet there are procedures that do not rule out the selection of Pareto-dominated outcomes. One example of such a procedure is the amendment method, widely used in contemporary legislatures (see e.g. McKelvey, 1976; Kramer, 1977). However, the conditions under which the Pareto requirement is violated include the assumption that all voters vote myopically in all phases of the process.

That the Condorcet winner should be elected whenever it exists, is often regarded as almost equally obvious requirement. It has the obvious virtue of being unique. In situations where majority undefeatable alternatives exist, even though no alternative is the Condorcet winner, the subset consisting of them – called the core – is almost equally natural solution set. In contradistinction to the set of Pareto-undominated outcomes, the core may be empty.

One generalization of the Condorcet winner concept leads to the top cycle set (Miller, 1977). It consists of the smallest set T of alternatives that has the property that each alternative in T defeats each alternative outside T. Obviously this set reduces to the Condorcet winner or core when these solutions are nonempty. However, it is possible that there exists a majority cycle through several alternatives that form the set T with the above defining properties. None of the alternatives in T is then undefeated. Thus, the core is empty and yet the top cycle set exists.

Although the Condorcet winner is necessarily included in the Pareto undominated set, its generalization – the top cycle set – may contain alternatives that are not included in the Pareto set (see Banks, 1985; Miller *et al.*, 1986). The following 3-person preference profile over $\{a, b, c, d\}$ illustrates this possibility (Nurmi, 1988b):

person 1	*person 2*	*person 3*
a	d	b
b	c	d
d	a	c
c	b	a

Here the top cycle consists of a, b, c and d and yet c is Pareto dominated by d.

Both the top cycle and Pareto sets are often far too large to be of much help in specifying which alternatives should be chosen. Nor do we have much by way of institutional behaviour theory that would tell us which kinds of mechanisms would lead to outcomes in those two sets. The myopic voting assumption – as was pointed out above – in fact 'predicts' that the social outcomes are by no means restricted to those sets. In this respect a much more useful solution concept is that of uncovered set defined by Miller (1980). Alternative x covers alternative y – in symbols xCy – if x defeats by a simple majority everything that y defeats by a simple majority. If all preferences are strict, then xCy implies that x defeats y.

The set UC of uncovered alternatives, i.e. those alternatives that are covered by no other alternative, contains all outcomes ensuing from sophisticated voting (Shepsle and Weingast, 1984). However, UC is in general too large for a complete characterization of those outcomes. A subset of UC currently called the Banks set contains all the outcomes that can ensue from sophisticated voting and only those outcomes (Banks, 1985). Thus we have a solution set and the corresponding institutional behaviour theory.

The Banks set is defined by the following algorithm. Given a preference profile, choose any alternative, say x, and find out whether there is another alternative, say y, that defeats x by a simple majority. If the answer is no, then the Banks chain starting from x also ends at x. If such a y exists, then one determines if there is an alternative, say z, that would defeat all the previous alternatives in the chain starting from x – i.e. x and y. Supposing that no such z exists, we conclude that the Banks chain starting from x has the endpoint y. Otherwise, one continues looking for other alternatives that defeat all the preceding ones until no such alternative exists. The last one found is then the end point of the Banks chain starting from x.

A Banks chain is constructed from each alternative. The Banks set B consists of the end points of all Banks chains. It is always a subset of UC. For alternative sets with at most six elements $UC = B$ (Miller *et al.*, 1986). For larger sets B may be a proper subset of UC (see Moulin, 1986).

Another subset of UC, viz. the set of Copeland winners CW, is also of some interest as a solution concept. It turns out that with small alternative sets the B and CW sets coincide. However, when the number of alternatives is at least 13, B and CW do not necessarily have common elements (Moulin, 1986). However, they both remain within UC. Thus the configuration of the above solution concepts can

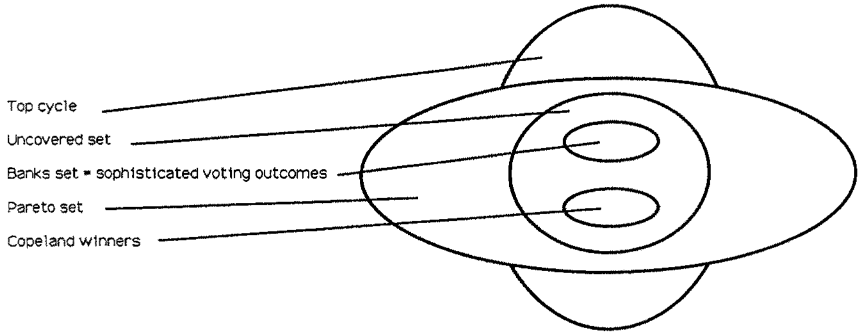

Fig. 1.

be depicted as in Figure 1, assuming that the discrepancies between them are maximal (Nurmi, 1988b).

3.2. Rough Sets and Approximate Solutions

Suppose that – given any preference profile – one has an intuitive idea of which alternatives should necessarily belong to the social choice set. Suppose, moreover, that one has a similar intuition concerning the set of alternatives that could possibly be included in the social choice set. The former set could for example consist of alternatives that most voting systems would choose whenever they exist. The latter set, in turn, could contain alternatives that one would not outright exclude from further consideration, even though one would admit that the set may often include implausible alternatives. These two intuitions would enable us to approach the difficulty of making social choices from the perspective provided by the theory of rough sets introduced and elaborated by Pawlak (1982, 1984; see also Slowinski and Stefanowski, 1989). More specifically, it would be possible to construct an approximate classification of outcomes (alternatives) and define a measure of the difficulty one is confronted with when starting from a fixed preference profile and alternative set.

Consider a set A of alternatives together with a binary relation S defined over A. The pair $(A, S) = X$ is called the approximation space. The relation S is called the indiscernibility relation. Thus, $a_i S a_j$ means that a_i cannot be discerned from a_j. S is assumed to be an equivalence relation and the classes of the relation are called elementary sets.

A definable set in X is any finite union of elementary sets. Consider now a subset Y of X. Y's upper approximation is the smallest definable set in X that contains Y and is denoted by Y^U. Y's lower approximation, Y^L, in turn, is the largest definable set in X which is contained in Y. Finally, the boundary of Y in X is defined as $Y^U - Y^L$.

Obviously, the smaller the boundary of Y in X, the more accurate is the approximation. Pawlak (1984) defines the following measure of accuracy for the set Y in X:

$$u_X(Y) = \text{card}(Y^L)/\text{card}(Y^U) .$$

Clearly, the values of $u_X(Y)$ are in the interval [0, 1]. When $u_X(Y) = 1$, the accuracy is perfect, while $u_X(Y) = 0$ is the case when no element of A necessarily belongs to Y.

Let us apply these ideas to the social choice problem. Suppose that we have several plausible solution concepts, e.g. uncovered set, Banks set and Copeland winners. We could then view the union of these sets as the upper approximation Y^U of a good social choice. In other words, we could maintain that no alternative outside this union will be eligible. We could rest assured that we will not be left empty handed: these sets are never empty. On the other hand, our intuition could dictate that the intersection of these sets be the lower approximation Y^L of a good choice: the set of alternatives satisfying the requirements of all these solutions would necessarily qualify as the social choice.

One could then use $v_X(Y) = 1 - u_X(Y)$ as the measure of the difficulty of making social choices. This measure reflects the discrepancy between the various intuitions concerning plausible choices: the more different the solution concepts, the smaller the value of $u_X(Y)$ and, consequently, the larger the value of $v_X(Y)$.

The measure is, of course, no better than the solution concepts used in its construction. As the main dividing line between various solutions seems to distinguish positional from binary solutions, a plausible measure of the difficulty would take solutions from both classes, e.g. Borda winners and Copeland winners.

4. CONCLUDING REMARKS

In the preceding we have reviewed approaches to difficulty of making social choices. As the numerous negative results of the social choice theory seem to cast doubt on an effort to find optimal social choice procedures, it is worthwhile trying to characterize the situations in which one could expect the existing procedures to work without anomalous results. The first approach discussed above looks into the contexts where all 'reasonable' procedures almost always end up with identical outcomes. It turns out that even a relatively small degree of consensus among the voters is sufficient to make the discrepancies between methods vanish. This is perhaps what one would intuitively expect. The simulation results referred to above are intended to show that the various culture assumptions underlying various probability estimates should be taken seriously and the assumptions subjected to sensitivity analysis.

Looking at the difficulty of making social choices from a more exact angle, viz. from the computational complexity perspective, reveals that some procedures may require an astonishing amount of computational effort for the winners to be determined. Although in practice these types of results just show how quickly the computational work increases as a function of adding candidates or voters, they also show that in some procedures the strategic manipulation – either in the sense of preference misrepresentation by the voters or of agenda control by the agenda setters – is much more difficult than in others. These results are of obvious value in institutional design, even when their worst case nature is appreciated.

Viewing the difficulty from the institution-free angle leads one to discuss the discrepancy between different solution concepts of voting games. The more the solutions differ, the more difficult the decision situation becomes. A simple measure of the difficulty was designed in the preceding section. There are certainly other ways of measuring the difficulty, but rather than insisting on a particular measure, the main point is to stress that the contextual difficulty plays an important role in the intuitive 'success' of various methods of making social choices. Although voting or some other preference aggregation method is often resorted to just because the group members have different opinions about the matters to be decided, it is often recognized that the preferences are so different that no matter which decision will be made, it is bound to be viewed unreasonable by some individuals. The importance of measuring the decision context difficulty is precisely in showing us when one should not expect the social choice methods to work. Every institution works within some contextual limits. When these are exceeded, the results are more or less arbitrary. To quote Ordeshook (1991):

> Put simply, there are circumstances in which constitutions ought to 'fail' and in which such 'failures' ought to be construed as successes. If a constitution is constructed on a morally corrupt foundation, then success is the eventual destruction of that foundation and it merely remains for posterity to decide whether the method of destruction provided the most efficient feasible route to that end.

The study of the difficulty in choice making situations aims at outlining the range of reasonable behaviour of the decision methods.

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