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METHOD OF ORBITS IN THE REPRESENTATION THEORY OF COMPLEX LIE GROUPS

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1. Introduction

The method of orbits (see [2, 3, and 13]) originated as a method of constructing a large class of unitary representations of an arbitrary Lie group. Each representation is defined by an orbit of the group in the space dual to its Lie algebra, and it seems that one can express in terms of an orbit the properties of the corresponding representation: to calculate its character, the spectrum of the restriction to a subgroup, etc.

In the work of the French School, the algebraic version of the method of orbits is used to study enveloping algebras, their centers, and primitive ideals (i.e., the kernels of irreducible representations). In particular, for solvable Lie algebras, they have succeeded in describing the structure of the field of partial envelopings of the algebra, and finding all its primitive ideals (see [1, 4]). Similar results are proved by induction on the dimension of a Lie algebra. This method runs into difficulties when the Lie algebra is unsolvable. Sometimes these difficulties can be overcome, but the complexity of the relevant proofs increases considerably. The direct methods put forward in the present article allow us both to simplify the proofs of certain standard theorems, and to obtain new results. Our approach is intermediate between the analytic and algebraic ones. It is close to the theory of quantization, whose connection with the method of orbits was discovered by Kostant [3].

We briefly state our main results. We consider the set of generalized functions concentrated on the identity of a Lie group G. They form an algebra under convolution that is none other than the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ corresponding to the group G. If G is $\mathbf{R}^n$, then the Fourier transformation establishes an isomorphism between $U(\mathfrak{g})$ and algebras of polynomials. It turns out that if G is arbitrary, then there is a mapping (we denote it by J) of the algebra $U(\mathfrak{g})$ into the set $C[\mathfrak{g}^*]$ of polynomials on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ that plays the same role as the Fourier transformation in the above example. By means of J we carry over the multiplication from $U(\mathfrak{g})$ into $C[\mathfrak{g}^*]$ [i.e., we define it by the formula $\Phi \circ \Psi = J(J^{-1}\Phi * J^{-1}\Psi)$]. When $G = \mathbf{R}^n$, this operation is the usual product. In the general case it sends the space of polynomials into an algebra H isomorphic to $U(\mathfrak{g})$. By going over from $U(\mathfrak{g})$ to H we can construct a fairly large commutative subalgebra in $U(\mathfrak{g})$: the corresponding subalgebra in H consists of all polynomials constant on specified submanifolds in $\mathfrak{g}^*$.

We illustrate the construction of these submanifolds by the example of the group $G = SL(2, \mathbf{R})$. It can be verified that the orbits of the action of $G = SL(2, \mathbf{R})$ in $\mathfrak{g}^*$ that is dual to the associated action in $\mathfrak{g}$ are hyperboloids (or their components) in the three-dimensional space $\mathfrak{g}^*$. In a suitable Cartesian coordinate system they are defined by the equations $x^2 + y^2 = z^2 + c$. We consider the domain of the hyperboloids of one sheet ($c > 0$). It is a standard fact that such a hyperboloid has a system of linear generators which can be chosen in

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two ways. For each hyperboloid we fix one of these ways so that the systems chosen on near hyperboloids are compatible. We define A as the subspace of those polynomials that are constant on each linear generator of each hyperboloid. It turns out that A is a commutative subalgebra in H, and the operation in H coincides in A with the usual multiplication of polynomials. In other words, the restriction of the mapping J^{-1}: $C[\mathfrak{g}^*] \to U(\mathfrak{g})$ to A is a homomorphism of algebras.

We can choose a subalgebra A with these properties in the case of an arbitrary complex Lie group G. As a corollary we obtain the following result of Duflo (see [1, 5, and 8]): J is an isomorphism of the center of $U(\mathfrak{g})$ onto a ring of polynomials on $\mathfrak{g}^*$ invariant under G. For semisimple Lie algebras this fact was discovered by Harish-Chandra.

We consider the representation π_Ω of a Lie algebra $\mathfrak{g}$ that corresponds to an orbit Ω in general position. We extend it to a representation of the algebra $U(\mathfrak{g})$ with H (by using J); we can regard π_Ω as a representation of H. Then it turns out that the elements of the commutative subalgebra A go over into diagonal operators. More precisely, π_Ω can be realized in the space of cross sections of the bundle over Ω, and it sends a function $\Psi \in A$ into the operator of multiplication by Ψ. In particular, the center of $U(\mathfrak{g})$ goes into scalar operators: $\pi_\Omega(z) = J(z)|_\Omega$.

Next we consider π_Ω as a unitary representation of a group G. The character of π_Ω is a generalized function on G, which is closely connected with the δ-function of the orbit Ω. For orbits in general position we shall prove that $\mathrm{tr}\, \pi_\Omega = J^{-1}(\delta_\Omega)$ (supporting a conjecture of Kirillov).

The material in this article is arranged as follows: the basic definitions, constructions, and theorems (with outlines of a part of the proofs) are gathered together in Sec. 2. The principal results are proved in Sec. 3 by "deforming" Lie algebras and their representations. The proofs of assertions about polarization are presented in Sec. 4.

The main results of the present article were announced in [9]. The author is glad to have this opportunity to thank A. A. Kirillov for stimulating discussions on the theory of representations.

2. Definitions and Basic Results

Let G be a connected complex Lie group, and $\mathfrak{g}$ be its Lie algebra. The space $\mathfrak{g}^*$ dual to $\mathfrak{g}$ splits into orbits under the action of G dual to the associated action. Every G-orbit is a symplectic manifold. We recall the construction of a 2-form on an orbit. In accordance with the action of G in $\mathfrak{g}^*$, to an element $x \in \mathfrak{g}$ there corresponds a vector field ξ_x (of an infinitesimal translation) touching the orbit. The value of a 2-form on the vector fields ξ_x and ξ_y at a point f is f([x, y]).

A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ that is also a maximal isotropic subspace of the form f([x, y]) (defined on $\mathfrak{g}$) is called a polarization of the functional f. It is a fact that if an orbit through f has maximal dimension (in which case f is called a regular point), then polarizations of f exist. If, e.g., $\mathfrak{g}$ is semisimple and f is a functional dual to a vector in general position in a Cartan subalgebra, then $\mathfrak{p}$ can be taken to be a Borel subalgebra.

On an orbit containing f each polarization defines a Lagrange distribution in a neighborhood of f. To determine it we must consider the image of $\mathfrak{p}$ under the mapping $x \to \xi_x(f)$ of the algebra $\mathfrak{g}$ onto the tangent space to the orbit at f, and extend the resulting subspace to other points of the orbit. It can be verified that close to f the result does not depend on the method of extension and defines a G-invariant integrable Lagrangian fibration in a neighborhood of f. We obtain (locally) a partition of an orbit into fibers. Another method of obtaining the fiber through f is to consider the orbits of f under the subgroup P corresponding to the algebra $\mathfrak{p}$.

It turns out [13] that every fiber is "plane." More precisely, let $\mathfrak{p}^\perp$ be the subspace of $\mathfrak{g}^*$ consisting of functionals that annihilate $\mathfrak{p}$. Then the following proposition holds.

Proposition 2.1. The fiber Pf is an open dense set in the linear manifold $f + \mathfrak{p}^\perp$ whose complement is an algebraic submanifold in $f + \mathfrak{p}^\perp$

Proof. First of all, $f + \mathfrak{p}^\perp$ is stable under P; therefore $Pf \subset f + \mathfrak{p}^\perp$. We prove that Pf and $f + \mathfrak{p}^\perp$ have the same dimension. For $l \in \mathfrak{g}^*$ we consider the form $B_l(x, y) = l([x, y]), x, y \in \mathfrak{g}$. We set $n = {}^1/_2 \operatorname{rank} B_f$; then the Lagrange manifold of Pf is n-dimensional. If the dimension of the kernel B_f is k, then $\dim \mathfrak{g} = 2n + k$, and $\dim \mathfrak{p} = n + k$, so that $\dim \mathfrak{p}^\perp = \dim \mathfrak{g} - \dim \mathfrak{p} = n$.

In fact, the fiber Pf coincides with the set $\mathcal{F}$ of all those points $l \in f + \mathfrak{p}^\perp$ for which $\operatorname{rank} B_l \geq \operatorname{rank} B_f$. For the dimension of a polarization decreases as rank B_f increases; hence $\mathfrak{p}$ is a polarization of any point of $\mathcal{F}$. Therefore, the P-orbits partition $\mathcal{F}$ into open sets. But $\mathcal{F}$ is connected since its complement is a complex submanifold in $f+\mathfrak{p}^{\perp}$ defined by the equation $B_l \wedge \ldots \wedge B_l = 0$ (n factors). Consequently, $\mathcal{F}$ consists of a single orbit Pf. QED.

As a corollary we shall see that $Pf = f + \mathfrak{p}^{\perp}$ almost always (Theorem 2.2).

Now we discuss the question as to when a Lagrange distribution defined near to f can be extended to the whole orbit. For this it is necessary and sufficient that the Lagrange subspace at f is invariant under the action of its stabilizer G(f). This means that the subalgebra $\mathfrak{p}$ must be stable under the associated action of the group G(f). We call such polarizations $\mathfrak{p}$ global. [Let us note that $\mathfrak{p}$ is trivially invariant under a connected component of the group G(f) since its Lie algebra is contained in $\mathfrak{p}$. Therefore, the problem of the global property of polarizations only arises when G(f) is not connected.]

We consider the closed subgroup $P^{\#}$ of those elements of G that send $f+\mathfrak{p}^{\perp}$ into itself. The Lie algebra of $P^{\#}$ is $\mathfrak{p}$, and so the group P coincides with a connected component of $P^{\#}$. In particular, P is closed. In the proof of Proposition 2.1 we saw that the fiber $\mathcal{F}$ consists of all points l for which the rank of B_l is a maximum. Consequently, $P^{\#}$ coincides with a subgroup of elements of G that preserve $\mathcal{F}$. If $\mathfrak{p}$ is a global polarization, then $P^{\#} = G(f) \cdot P$, since P acts trivially on a fiber.

In Sec. 4 we shall prove

THEOREM 2.2. Let f be a point in general position in $\mathfrak{g}^*$.

a) There is a polarization $\mathfrak{p}$ satisfying Pukanszky's condition: $Pf = f + \mathfrak{p}^{\perp}$.

b) If stabilizer G(f) of f is commutative, then there is a global polarization $\mathfrak{p}$ invariant under G(f) and satisfying Pukanszky's condition.

It can be shown that if G is an algebraic group and f is a point in general position in $\mathfrak{g}^*$, then for G(f) to be commutative it is sufficient that the fundamental group of the orbit containing f is commutative.

From now on until the end of Sec. 3 we consider the complex Lie algebra $\mathfrak{g}$ as a real one, and denote by $\mathfrak{g}^*$ the space of real functionals on $\mathfrak{g}$.

We turn to the determination of the mapping J that sends functions on G into functions on $\mathfrak{g}^*$, and consider the open set

$$\{x \in \mathfrak{g} : |\text{ the imaginary parts of the eigenvalues of } \operatorname{ad} x \,| < \pi\}$$

in $\mathfrak{g}$. Under the mapping $\exp\colon \mathfrak{g} \to G$ this set is mapped diffeomorphically onto its image W. For a function φ with support in W, $J(\varphi)$ is defined as follows: we need to map φ onto $\mathfrak{g}$, by using exp, and then multiply by $j(x) = \det S(\operatorname{ad} x)$, where $S(t) = \left(\frac{\exp(t/2) - \exp(-t/2)}{t}\right)^{1/2}$. By taking the Fourier transformation of the resulting function, we obtain a rapidly decreasing function $J(\varphi)$ on $\mathfrak{g}^*$. In a similar way the mapping J is defined for generalized functions on W, and J sends generalized functions concentrated on the identity of the group into polynomials.

The choice of f is explained as follows. Let $f \in \mathfrak{g}^*$, $\mathfrak{p}$ be a polarization of f, $\mathcal{F}$ be a fiber through f, and δ_P and $\delta_{\mathcal{F}}$ be δ-functions of the manifolds P and $\mathcal{F}$ defined by the Haar measure on P and by the Lebesgue measure on $f+\mathfrak{p}^{\perp}$, respectively. We consider the character χ_f of the group P (correctly defined on $P \cap W$), whose differential is $2\pi i f(x) - {}^1/_2 \operatorname{tr}_{\mathfrak{g}/\mathfrak{p}} \operatorname{ad} x$. Let $\bar{\chi}_f$ be the complex conjugate character.

Proposition 2.3. Let the point $f \in \mathfrak{g}^*$ be regular. We can choose a solvable polarization $\mathfrak{p}$, so that $J(\bar{\chi}_f \cdot \delta_P) = \delta_{\mathcal{F}}$, and $\langle \chi_f \cdot \delta_P, \varphi \rangle = \langle \delta_{\mathcal{F}}, J(\varphi) \rangle$ for every φ with support in W. If the algebra $\mathfrak{g}$ is algebraic, then for $\mathfrak{p}$ we can take any polarization satisfying Pukanszky's condition.

We are going to prove that $J(\bar{\chi}_f \cdot \delta_P) = \delta_{\mathcal{F}}$ (the second assertion is proved similarly). On going over to the Lie algebra, $\bar{\chi}_f \cdot \delta_P$ goes into

$$\exp(-2\pi i f(x) - {}^1/_2 \operatorname{tr}_{\mathfrak{g}/\mathfrak{p}} \operatorname{ad} x) \cdot \mu_P(x)/\mu_G(x) \cdot \delta_{\mathfrak{p}},$$

where μ_P and μ_G are the Jacobians of the transformation from P to $\mathfrak{p}$ and from G to $\mathfrak{g}$, respectively. We must multiply this function by f(x) and compare it with $F^{-1}(\delta_{\mathcal{F}})$ (F stands for the Fourier transformation). Since $\mathcal{F}$ is a subset of complete measure in $f+\mathfrak{p}^{\perp}$, $F^{-1}(\delta_{\mathcal{F}})$ coincides with $F^{-1}(\delta_{f+\mathfrak{p}^{\perp}})$, which is equal to $\exp(-2\pi i f(x)) \cdot \delta_{\mathfrak{p}}$. Now we only need to verify that

$$\mu_G(x) = j(x) \cdot \mu_P(x) \cdot \exp(-{}^1/_2 \operatorname{tr}_{\mathfrak{g}/\mathfrak{p}} \operatorname{ad} x) \quad \text{for} \quad x \in \mathfrak{p}.$$

We use a lemma which is proved in Sec. 4.

LEMMA 2.4. If f is regular, then there is a solvable polarization $\mathfrak{p}$ such that for all $x \in \mathfrak{p}$, the nonzero eigenvalues of the operator adx in the spaces $\mathfrak{p}$ and $\mathfrak{g}/\mathfrak{p}$ are opposite in sign (with due regard for multiplicity).

We call these polarizations admissible.

Remark. The sense of the lemma is obvious when $\mathfrak{p}$ is the Borel subalgebra of a semisimple algebra $\mathfrak{g}$, and x is an element of the Cartan subalgebra.

By using the lemma and standard expressions for μ_G and μ_P (see [2]), we find that

$$\mu_P(x) = \prod_\lambda \frac{1-e^{-\lambda}}{\lambda}, \quad \mu_G(x) = \prod_\lambda \frac{(1-e^{-\lambda})}{\lambda} \cdot \frac{(e^\lambda - 1)}{\lambda}, \quad j(x) = \prod_\lambda \frac{e^{\lambda/2} - e^{-\lambda/2}}{\lambda},$$

where λ ranges over the nonzero eigenvalues of adx in $\mathfrak{p}$. The assertion is now obvious. QED.

Now we turn to unitary representations of the group G. Let $P^\#$ be a global polarization of $f \in \mathfrak{g}^*$. We assume that the character χ_f is extended (from $P \cap W$) to the whole group $P^\#$. In this case we can define the representation of G induced from the character χ_f of the subgroup $P^\#$, i.e., the representation in the space of functions on G that satisfy the condition $s(pg) = \chi_f(p) \cdot s(g)$, $p \in P^\#$, $g \in G$. The group G acts in this space by right translations. We denote this representation by π_f. If instead of f we take another point of the same orbit and choose the polarization of it and character obtained from $P^\#$ and χ_f by conjugation, then the resulting representation is equivalent to π_f. The representation π_f is unitary. Its geometrical meaning will be accounted for below, but first we calculate the character of π_f.

Let G be a complex algebraic group, f be a point of an orbit Ω of maximal dimension, and G(f) be its stabilizer. We assume that the functional $2\pi i f$ on the Lie algebra of the group G(f) is extended to a character of G(f) [since $G(f) \subset P^\#$, this condition is necessary for the existence of a character χ_f of $P^\#$]. Under these conditions the following theorem holds (cf. [14]).

THEOREM 2.5. Let $P^\#$ be a global polarization of f satisfying Pukanszky's condition; then the corresponding representation π_f is defined. If φ is a positive definite finite function on G with support in W, and if the integral of the restriction of $J(\varphi)$ to Ω with respect to the measure induced by the symplectic structure exists, then the operator $\pi_f(\varphi)$ has the trace:

$$\operatorname{tr} \pi_f(\varphi) = \int_\Omega J(\varphi). \tag{2.5}$$

Proof. This is based on a standard formula (see [2]) for the characters of induced representations

$$\operatorname{tr} \pi_f(\varphi) = \int_{P^\# \setminus G} \left(\int_{P^\#} \chi_f(p) \cdot \varphi(u^{-1}pu)\, dp \right) du.$$

It is not difficult to show that $P^\# \cap W = P \cap W$; therefore, the inner integral reduces to an integral over P. By making the substitution $w = u^{-1}pu$ it reduces to $\langle \chi_{uf} \cdot \delta_{u^{-1}pu}, \varphi \rangle$, which by Proposition 2.3 is equal to $\langle \delta_{u\mathcal{F}}, J(\varphi) \rangle$. Thus we obtain the expression $\int_{P^\# \setminus G} \left(\int_{u\mathcal{F}} J(\varphi) \right)$, in which the inner integral is over fibers and the outer over a set of fibers. This repeated integral defines a G-invariant measure on an orbit that consequently only differs from the symplectic measure by a multiplier. To evaluate this multiplier we compare both measures at f. We identify the tangent space to $P^\# \setminus G$ at f with $\mathfrak{g}/\mathfrak{p}$; then to the measure defined by the repeated integral there corresponds a volume form on an orbit, which is equal to $\mu_{\mathfrak{g}/\mathfrak{p}} \wedge \mu_{f+\mathfrak{p}^\perp}$ at f. Here $\mu_{\mathfrak{g}/\mathfrak{p}}$ and $\mu_{f+\mathfrak{p}^\perp}$ are Lebesgue volume forms, and the form on $f + \mathfrak{p}^\perp$ is obtained as follows: we identify $\mathfrak{p}^\perp$ with the space dual to $\mathfrak{g}/\mathfrak{p}$ and take on it the form $\mu_{\widehat{\mathfrak{g}/\mathfrak{p}}}$ dual to $\mu_{\mathfrak{g}/\mathfrak{p}}$ (this is a property of the Fourier transformation). Thus, we must compare the form $\mu_{\mathfrak{g}/\mathfrak{p}} \wedge \mu_{\widehat{\mathfrak{g}/\mathfrak{p}}}$ with the symplectic volume form on the tangent space to the orbit. We choose $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{g}$ so that the functionals $f([x_1, \cdot]), \ldots, f([y_n, \cdot])$ form a symplectic basis for the tangent space. Clearly, the last n of the functionals form a basis in $\widehat{\mathfrak{g}/\mathfrak{p}}$ dual to the basis $x_1, \ldots, x_n$ in $\mathfrak{g}/\mathfrak{p}$. Therefore, in the coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ corresponding to our basis on the tangent space, we have $\mu_{\mathfrak{g}/\mathfrak{p}} = dq_1 \wedge \ldots \wedge dq_n$ and $\mu_{\widehat{\mathfrak{g}/\mathfrak{p}}} = dp_1 \wedge \ldots \wedge dp_n$; their product is a symplectic volume form.

Remark. In fact we have used not the algebraic nature of $\mathfrak{g}$ but the admissibility of a polarization $\mathfrak{p}$.

We consider a small domain consisting of points in general position in $\mathfrak{g}^*$. It splits into orbits, each of which is stratified into fibers. We choose the fibrations on orbits to be analytically dependent on an orbit (in the domain in question). For this it is sufficient to construct an analytic mapping $s: f \mapsto \mathfrak{p}^f$ associating with a point f its polarization, and compatible with the action of G; if $l = gf$, then $\mathfrak{p}^l = \mathrm{Ad}^{-1}g\,(\mathfrak{p}^f)$. We consider the manifold of pairs $(f, \mathfrak{p})$, where f is a regular point in $\mathfrak{g}^*$, and $\mathfrak{p}$ is a polarization of f (admissible in the sense of Lemma 2.4) that satisfies Proposition 2.3. The projection $(f, \mathfrak{p}) \mapsto f$ of this manifold into $\mathfrak{g}^*$ commutes with the action of G. To construct the mapping s it is sufficient to take a cross section of this projection that commutes with the action of G. Clearly, we can choose a sufficiently small domain U on which a cross section exists. We obtain a partition of U into fibers. Let A denote the algebra of all polynomials on $\mathfrak{g}^*$ which are constant on each fiber in U.

THEOREM 2.6. The restriction of the mapping $J^{-1}\colon \mathrm{C}[\mathfrak{g}^*] \to U(\mathfrak{g})$ to A is a homomorphism of algebras.

The algebra A is fairly large, it contains all invariant and semiinvariant (see [7]) polynomials.

COROLLARY (see [5, 7]). The mapping J maps the center (respectively, semicenter) of the algebra $U(\mathfrak{g})$ isomorphically onto the ring of invariant (respectively, semiinvariant) polynomials on $\mathfrak{g}^*$.

We turn the space of polynomials on $\mathfrak{g}^*$ into an algebra (denoted by H; see Sec. 1), by carrying over multiplication from $U(\mathfrak{g})$ with the help of the mapping J. Theorem 2.6 can be restated as follows.

The restriction of the operation in H to the subspace A coincides with the usual multiplication of polynomials. In particular, A is a commutative subalgebra in H.

Remark. The smooth functions on $\mathfrak{g}^*$ form a Lie algebra with respect to the Poisson bracket (see [6, 10]). The bracket of the functions Φ and Ψ is defined as the function $\{\Phi, \Psi\}\colon f \mapsto f\,([d\Phi_f, d\Psi_f])$, or, alternatively, the restriction of the function $\{\Phi, \Psi\}$ to each orbit is equal to the Poisson bracket (relative to the symplectic structure on the orbit) of the restrictions of Φ and Ψ. The set of functions that are constant on fibers is a maximal commutative subalgebra of the Lie algebra $C^\infty(\mathfrak{g}^*)$. The part of this subalgebra consisting of polynomials is A.

We return to representations. The action of the Lie algebra $\mathfrak{g}$ in the Gårding space of the representation π_f is extended to a representation of the enveloping algebra $U(\mathfrak{g})$, which we identify with H. For every function Ψ on $\mathfrak{g}^*$ we denote by Ψ_G the function $\Psi_G(g) = \Psi(gf)$ on G.

THEOREM 2.7. If $\Psi \in A$, then the operator $\pi_f(\Psi)$ is the same as a multiplication by Ψ_G.

COROLLARY (see [5]). If z is an element from the center of the enveloping algebra, then $\pi_f(z)$ is the operator of multiplication by the value of the polynomial J(z) at f.

The idea behind the proof of Theorem 2.7 is that we must check its validity not for polynomials but for the δ-functions of the separate fibers, since all functions constant on fibers are "formed" from them. By Proposition 2.3, the generalized operator corresponding to $\delta_{\mathcal{F}}$ under the representation π_f is $\int_P \pi_f(p)\overline{\chi}_f(p)\,dp$. In the next section we give a meaning to this (divergent) expression and evaluate it.

The geometrical construction of the representation π_f was pointed out by Kostant [3]. We consider a one-dimensional Hermitian bundle over the orbit Ω (containing f) with a connection ∇ whose curvature is a symplectic form on Ω multiplied by $-2\pi i$. We define a representation of $\mathfrak{g}$ in the space of smooth cross sections that are covariantly constant along the fibers of Ω. To define a scalar product in this space it is sufficient to take the measure on the set of fibers ($\approx P^\# \setminus G$). With an $x \in \mathfrak{g}$ we associate the skew-Hermitian operator

$$\pi_\Omega(x) = (\nabla_{\xi_x} - \nabla^*_{\xi_x}) + 2\pi i \cdot x. \tag{2.7}$$

The second term is the operator of multiplication by the function which is the restriction to Ω of a linear functional on $\mathfrak{g}^*$ (corresponding to x). Then ξ_x is the field on the orbit corresponding to the action of x, and $\nabla^*_{\xi_x}$ is the differential operator dual to ∇_{ξ_x}. It is easily verified that π_Ω is a representation of g, and that it is equivalent to π_f.

Let, e.g., $\mathfrak{g}$ be a Lie algebra with a basis p, q, and z, where z is a central element, and [p, q] = z (the Heisenberg algebra). To the orbit defined by the equation z = 1 there corresponds a representation in the space of functions of q (in this case the bundle is trivial):

$$\pi(p) = \partial/\partial q, \quad \pi(q) = 2\pi i q, \quad \pi(z) = 2\pi i. \tag{2.8}$$

Let $\mathfrak{g}$ be arbitrary and Ω be an orbit in $\mathfrak{g}^*$. We introduce on the intersection of Ω with the small domain U coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$ (they exist by Darboux's theorem) in which a symplectic form has the canonical form $dp \wedge dq$, and the functions $q_1, \ldots, q_n$ are constant on fibers. Under the restriction to a small neighborhood, the bundle over an orbit becomes trivial. Therefore, the representation π_Ω can be realized in functions of $q_1, \ldots, q_n$. It turns out that the operator $\pi_\Omega(x)$ is defined in fact by the same formulae (2.8) as for the Heisenberg algebra.

Proposition 2.8. a) An element $x \in \mathfrak{g}$ defines a function on Ω (the restriction of a functional on $\mathfrak{g}^*$) which in the coordinates p, q is of the first degree in p: $x(p, q) = u(q)p + v(q)$.

b) Under a suitable trivialization of the bundle, the operator $\pi_\Omega(x)$ goes into

$$\pi_\Omega(x) = {}^1/_2\,[u(q)\cdot\partial/\partial q + \partial/\partial q\cdot u(q)] + 2\pi i v(q), \tag{2.9}$$

i.e., it is obtained from x(p, q) by replacing p by $\partial/\partial q$ and by symmetrization.

This is a "physical" explanation as to why we must consider these induced representations.

Proof. a) Under the action of the group G, functions constant on fibers go into functions constant on fibers. Therefore, if $x \in \mathfrak{g}$, ξ_x is the corresponding field, and Ψ is a function of q, then $\xi_x\Psi$ does not depend on p. But $\xi_x\Psi = \{x, \Psi\} = \partial x/\partial p\cdot(\partial\Psi/\partial q) - (\partial x/\partial q)\cdot(\partial\Psi/\partial p) = (\partial x/\partial p)\cdot(\partial\Psi(q)/\partial q)$. Since $\Psi = \Psi(q)$ is arbitrary we find that $\partial x/\partial p$ does not depend on p.

b) We trivialize the bundle by using a cross section covariantly constant along fibers. On the field ξ the connection ∇ takes the form $\nabla_\xi = \xi + \alpha(\xi)$, where the differential 1-form α is equal to the curvature of the bundle. In canonical coordinates the curvature is $-2\pi i\; dp \wedge dq$, so that $\nabla_\xi = \xi - 2\pi i p\, dq(\xi) = \xi\,\theta$, where θ is some function of q. We remove θ by changing the trivializing cross section. We express $x \in \mathfrak{g}$ in terms of coordinates: $x(p, q) = u(q)p + v(q)$; then $\xi_x = u(q)\cdot\partial/\partial q - \partial x/\partial q\cdot\partial/\partial p$, and $\nabla_{\xi_x} = \xi_x - 2\pi i u(q)p$. It is easy to see that operators (2.7) and (2.9) coincide on functions of q. QED.

Formula (2.9) is very important for what is to follow. The fact that it defines a representation of the algebra $\mathfrak{g}$ can be readily verified. We remark that representation π_Ω defined by (2.9) is meaningful for any orbit Ω; we do not need to worry about the existence of the corresponding induced representation of the group G.

We rewrite (2.9) in the form $\pi_\Omega(x) = u(q)\,\partial/\partial q + 2\pi i v(q) + {}^1/_2\cdot \mathrm{tr}\, u'(q)$.†

LEMMA 2.9. If the point f has coordinates (p_0, q_0), $\mathfrak{p}$ is a polarization of f, and $x \in \mathfrak{p}$, then $\mathrm{tr}\, u'(q_0) = -\mathrm{tr}_{\mathfrak{g}/\mathfrak{p}}\, \mathrm{ad}\, x$.

For the proof we identify $\mathfrak{g}/\mathfrak{p}$ with the factor-space of the tangent space to the orbit with respect to the Lagrange subspace. The vectors $\partial/\partial q_j$ (of the p, q coordinate system) form a basis in it. We are going to calculate the trace of the operator ad x in this basis. It can be verified that the image of $\partial/\partial q_i$ under ad x coincides (modulo the Lagrange subspace) with the commutator of the field ξ_x and $\partial/\partial q_i$ at f. By expressing ξ_x in the coordinates p and q we obtain an expression for the trace $-\Sigma\partial u_i/\partial q_i$. QED.

3. Algebra $\hat{H}$

The aim of this section is to prove Theorems 2.6 and 2.7.

Alongside the algebra $\mathfrak{g}$ we consider the Lie algebra $\mathfrak{g}_t$ obtained from $\mathfrak{g}$ by multiplying the commutator in $\mathfrak{g}$ by the number t. We construct in terms of $\mathfrak{g}_t$ a Lie group G_t, a mapping J_t, and an associative algebra H_t, similar to the way this was done for the algebra $\mathfrak{g}$. The algebra H_t as a vector space is the space $\mathbb{C}[\mathfrak{g}^*]$ of polynomials. For t = 0 the Lie algebra $\mathfrak{g}_t$ is commutative, therefore J_0 coincides with the Fourier transformation, and H_0 with the algebra of polynomials on $\mathfrak{g}^*$. Thus, the family $\{H_t\}$ is a deformation of the algebra of polynomials $H_0 = \mathbb{C}[\mathfrak{g}^*]$. As we shall see, the dependence on t of the product of two polynomials in H_t is given by

$$\Phi \circ_t \Psi = \Phi\Psi + tB_1(\Phi, \Psi) + t^2B_2(\Phi, \Psi) + \ldots, \tag{3.1}$$

where the $B_i(\Phi, \Psi)$ are bidifferential operators on $\mathfrak{g}^*$ (i.e., combinations of derivatives of Φ and Ψ that are linear in Φ and in Ψ) with polynomial coefficients, and for every Φ and Ψ the sum (3.1) contains only a finite number of nonzero terms.

The formula (3.1) is verified most simply when Φ and Ψ are not polynomials but the exponentials of linear forms on $\mathfrak{g}^*$. With each $x \in \mathfrak{g}$ we associate a function on $\mathfrak{g}^*$ of the form $E_x\colon f \mapsto \exp 2\pi i\langle f, x\rangle$. The mapping J^{-1} sends E_x into $j^{-1}(x)\cdot\delta_{\exp x}$ (exp: $\mathfrak{g} \to G$ is the exponential mapping). If $x, y \in \mathfrak{g}$, then the convolution $\delta_{\exp x} * \delta_{\exp y}$ is equal to $\delta_{\exp x\cdot\exp y} = \delta_{\exp z(x,y)}$, where z(x, y) is a Campbell–Hausdorff series. Therefore, $E_x \circ E_y = j^{-1}(x)\cdot j^{-1}(y)\cdot j(z(x, y))E_{z(x,y)}$. A similar relation is true in $\mathfrak{g}_t$; by expanding the right-hand side as a power series in t, it can be realized that the coefficient of a fixed power of t is a function on $\mathfrak{g}^*$ of the form $f \to P_{x,y}(f)E_{x+y}(f)$, where $P_{x,y}$ is a polynomial whose coefficients are polynomials in x and y. In order to put

†tr u'(q) stands for $\Sigma\partial u_i/\partial q_i$.

this expression in the form (3.1) we need to remove the dependence of the coefficients of $P_{x,y}$ on x and y. For this we must replace E_{x+y} by products of suitable derivatives of E_x and E_y (e.g., the monomial $x^r y^s E_{x+y}$ by $\partial^r E_x/\partial f^r \cdot \partial^s E_y/\partial f^s$).

The above reasoning is not completely rigorous. A method of obtaining a rigorous proof (from other considerations) will be mentioned later.

We call a formal power series in t whose coefficients are analytic (respectively, smooth, generalized, etc.) functions on a set X a formal analytic (respectively, smooth, generalized, etc.) function on X. Let U be a neighborhood of a point in general position in $\mathfrak{g}^*$ and $\hat{H}$ be the space of formal analytic functions on U. The right-hand side of (3.1) is meaningful as a formal function for any formal functions $\Phi, \Psi \in \hat{H}$. Having defined a multiplication in $\hat{H}$ in this way, we obtain an associative algebra, which can be regarded as a deformation of the algebra of analytic functions on U.

We fix canonical coordinates p, q on an orbit Ω. By associating with an analytic function Ψ on U the operator of multiplication by $\Psi(0, q)$ (we are restricting Ψ to Ω), we obtain a representation of the algebra of analytic functions in the space of functions constant on fibers of Ω. Then we construct a representation of $\hat{H}$ which is a deformation of this representation.

First we define a representation of the Lie algebra $\mathfrak{g}_t$ (for every t) by adjusting the formula (2.9),

$$\pi_{t,\Omega}(x) = t\cdot u(q)\cdot\partial/\partial q + 2\pi i v(q) + t\cdot{}^1/_2\,\mathrm{tr}\,u'(q). \tag{3.2}$$

Note that $\pi_{t,\Omega}(x) = 2\pi i x(0, q)$ for $t = 0$. A representation of the Lie algebra $\mathfrak{g}_t$ defines a representation of the algebra H_t (as was explained in Sec. 2); we denote it by $\pi_{t,\Omega}$. We are going to prove that for a polynomial $\Psi \in H_t$, the operator $\pi_{t,\Omega}(\Psi)$ is a polynomial in t:

$$\pi_{t,\Omega}(\Psi) = \Psi(0, q) + t\cdot\pi_1(\Psi) + t^2\cdot\pi_2(\Psi) + \ldots, \tag{3.3}$$

where $\pi_i(\Psi)$ is a sum of expressions of the form $[P(\partial)\Psi](0, q)\cdot D$. Here $P(\partial)$ is a differential polynomial on $\mathfrak{g}^*$, and D is a differential operator in the space of functions in q that are independent of Ψ. We take this formula as the definition of the representation of the algebra H, assuming that t is a formal variable and Ψ a formal function. The passage from the commuting operators $\pi_0(\Psi) = \Psi(0, q)$ to the noncommuting operators $\pi_t(\Psi)$ can be regarded as a quantization procedure; the parameter t plays the role of Planck's constant, and $\hat{H}$ an algebra of observables.

We prove (3.3). Suppose that the polynomial $\Psi \in H_t$; then $J_t^{-1}(\Psi)$ is concentrated at the identity of the group G_t, and by definition (we omit the indices t and Ω) $\pi(\Psi) = \int \pi(g)\, J^{-1}\Psi(g)\, dg$ (integrated over G_t). Bearing in mind that $\pi(\exp x) = \exp \pi(x)$ and that $J^{-1}(\Psi) = \exp_*(j^{-1}\cdot F^{-1}\Psi)$, we obtain

$$\pi(\Psi) = \int_{\mathfrak{g}} \exp \pi_t(x)\cdot j_t^{-1}(x)\cdot\mu_t(x)\cdot F^{-1}\Psi(x)\,dx, \tag{3.4}$$

where μ_t is the Jacobian of the transformation from G_t to $\mathfrak{g}$. We expand the functions j_t^{-1} and μ_t under the integral sign as power series in t. It is clear from the definition of these functions that the coefficient of t^k is a polynomial in x with homogeneous terms of degree not less than k. We call these series decreasing.

The operator $\exp \pi_t(x)$ in (3.4) is not a decreasing series. However, we can extract from it a multiplier $\exp 2\pi i v_x$ independent of t [the function $v_x(q)$ is defined from the expansion $x = x(p, q) = u_x(q)p + v_x(q)$] so that the remainder is a decreasing series. For if $\exp \pi(x) = (\exp 2\pi i v_x)\cdot B$, where $B = \exp(-2\pi i v_x)\cdot\exp \pi(x)$, then by the Campbell–Hausdorff formula, $B = \exp(-2\pi i v_x + \pi(x) - {}^1/_2[2\pi i v_x, \pi(x)] + \ldots)$. The repeated commutators of $\pi(x)$ and $2\pi i v_x$ that contain $2\pi i v_x(q)$ more than once are zero, since $\pi(x)$ is a first-order differential operator. Commutation with $\pi(x)$ increases the degree in t and in x by 1; therefore the series under the exponential and so the exponential itself are decreasing. The expression for $\pi(\Psi)$ takes the form

$$\pi(\Psi) = \int_{\mathfrak{g}} (\exp 2\pi i v_x)\cdot B\cdot j_t^{-1}\cdot\mu_t\cdot F^{-1}\Psi(x)\,dx. \tag{3.5}$$

We group all the multipliers apart from the first and last in a single decreasing series, distinguishing its dependence on t and x by writing it in the form $\Sigma t^i P_i(x, D)$, where $P_i(x, D)$ is a differential operator in functions of q whose coefficients are polynomials in x.

The function $(\exp 2\pi i v_x)\cdot F^{-1}\Psi(x)$ on $\mathfrak{g}$ is concentrated at 0, and is therefore a combination of derivatives of a δ-function. By applying this function to a decreasing series, we obtain a terminating sum. Consequently, $\pi_{t,\Omega}(\Psi)$ is a polynomial in t.

To obtain (3.3) we now only need to replace $v_x(q)$ by $x(0, q)$ in the equality

$$\pi(\Psi) = \sum t^k \int (\exp 2\pi i v_x)\cdot P_k(x, D)\cdot F^{-1}\Psi(x)\, dx$$

and to apply the inversion formula of the Fourier transformation. QED.

Remark. Formula (3.1) for multiplication in H_t is proved in a similar way.

We consider the subspace $\hat{A}$ in $\hat{H}$ consisting of formal functions constant along fibers.

THEOREM 3.6. If $\Psi \in \hat{A}$, then $\pi_{t,\Omega}(\Psi)$ is the operator of multiplication by $\Psi|_\Omega$.

This means that when we substitute functions from A into the series (3.3) for $\pi_{t,\Omega}$, all the terms apart from the first vanish, i.e., the quantum operator $\pi_t(\Psi)$ coincides with the classical $\pi_0(\Psi)$. In particular, if this function is a polynomial, then by setting t = 1 we obtain Theorem 2.7. [In fact, the operators $\pi_f(\Psi)$ and Ψ_G are analytic, and by Theorem 3.6 they coincide on an open set, namely, the inverse image of U in G.]

To prove Theorem 3.6 we must choose canonical coordinates on different orbits compatibly. It can be shown that there are analytic coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n, z_1, \ldots, z_k$ in U such that:

a) the orbits are defined by the equations z = const;

b) the fibers are defined by the equations z = const and q = const;

c) the restrictions of p and q to each orbit define canonical coordinates on it.

In the new coordinates, functions constant on fibers are functions of q and z. It will be convenient to "paste together" the representations $\pi_{t,\Omega}$ of $\hat{H}$ corresponding to different orbits Ω into a single representation π in the space of formal functions of q and z. The addition of the z variables does not lead to significant changes; e.g., instead of $[P(\partial)\Psi](0, q)$ in (3.3) we must now write $[P(\partial)\Psi](0, q, z)$.

We consider a point $f \in U$, a fiber $\mathcal{F}$ through it, and the generalized function $\delta_{\mathcal{F}}$ on U. Let f have canonical coordinates (p_0, q_0, z_0). Since the fiber $\mathcal{F}$ is defined by the equations $z = z_0$, $q = q_0$, then it is reasonable to expect that the following lemma holds.

LEMMA 3.7. In canonical coordinates $\delta_{\mathcal{F}}$ is proportional to $\delta(q - q_0, z - z_0)$.

We introduce operator $\pi(\delta_{\mathcal{F}})$, which is given by the general formula (3.3) and sends a smooth function $s(q, z)$ into a formal generalized function on the surface p = 0. The operation of restriction to this surface, which is available in (3.3), is realized as follows: we replace $\delta_{\mathcal{F}}$ by const$\cdot\delta(q - q_0, z - z_0)$ (Lemma 3.7) and substitute into the right-hand side of (3.3), which is also expressed in the coordinates p, q, and z. Then we set p = 0.

We compare this definition of $\pi(\delta_{\mathcal{F}})$ with the heuristic formula $\pi(\delta_{\mathcal{F}}) = \int \bar{\chi}_f(p)\pi_f(p)\, dp$ (see the idea of the proof of Theorem 2.7). We consider the Jacobian ν_t of the mapping $\exp: \mathfrak{p} \to P_t$ and the character $\chi_t(x) = 2\pi i f(x) - {}^1/_2 \cdot t \cdot \mathrm{tr}_{\mathfrak{g}/\mathfrak{p}}\, \mathrm{ad}\, x$ as formal functions on $\mathfrak{p}$, and introduce the formal operator denoted symbolically by $\int \bar{\chi}(p)\, \pi(p)\, dp$. This operator sends a smooth function $s(q, z)$ into a formal generalized function on U. By definition, its value at a finite function $\varphi = \varphi(q, z)$ is

$$\int_{\mathfrak{p}} \exp \bar{\chi}_t(x) \left(\int_U \exp \pi_t(x)\, s\cdot\varphi(q, z)\, dq\, dz\right) \nu_t(x)\, dx. \qquad (3.7)$$

[If we could first carry out the integration over $\mathfrak{p}$, then we would obtain $\int \bar{\chi}(p)\, \pi(p)\, dp$.]

LEMMA 3.8. $\pi(\delta_{\mathcal{F}}) = \int \bar{\chi}(p)\, \pi(p)\, dp$; in particular, the integral (3.7) exists.

The following lemma plays a decisive role in the proof of Theorem 3.6.

LEMMA 3.9. $\pi(\delta_{\mathcal{F}}) = \mathrm{const}\cdot\delta_{\mathcal{F}}$.

Proof of Lemma 3.7. a) First we show that the Jacobian of the transformation from linear coordinates in to the canonical coordinates p, q, and z is a function independent of p. The elements of the group P act on $\mathfrak{g}$ by unimodular transformations, since for $x \in \mathfrak{p}$ the nonzero eigenvalues of the operator ad x are partitioned into pairs of numbers of opposite sign ($\mathfrak{p}$ is admissible). Consequently, the elements of P preserve Lebesgue measure on $\mathfrak{g}^*$ and (as any element of G) the measure dpdqdz. Therefore, the coefficient of proportionality λ of these two measures is constant on the orbits of P and its conjugates. In particular, λ is constant on fibers.

b) We denote the function $\delta_{\mathcal{F}}$ in the coordinates p, q, and z by $\delta_{\mathcal{F}}(p, q, z)$, and a test function by φ. Then $\langle \delta_{\mathcal{F}}(p, q, z), \varphi\rangle = \langle \delta_{\mathcal{F}}, \lambda^{-1}\varphi\rangle = \int_{\mathcal{F}} \lambda^{-1}\varphi$. Note that λ^{-1} is constant on $\mathcal{F}$, and that the integration can be with respect to the measure dp, which is proportional to Lebesgue measure since p is an affine coordinate.

Proof of Lemma 3.8. This is based on the formula $\Phi(0, q, z) = \int F^{-1}\Phi(x) \cdot \exp 2\pi i x(0, q, z)\, dx$ for the restriction of a generalized function Φ on $\mathfrak{g}^*$ to the surface p = 0 (see [12]). In greater detail, if $\varphi = \varphi(q, z)$ is a finite function, then

$$\langle \Phi(0, q, z), \varphi\rangle = \int F^{-1}\Phi(x) \left(\int (\exp 2\pi i x(0, q, z))\, \varphi(q, z)\, dq\, dz\right) dx. \tag{3.8}$$

To define the operator $\pi(\delta_{\mathcal{F}})$ by (3.3) we can use (3.8) instead of going over to the coordinates (p, q, z) to realize the restriction operation. In our case $\Phi = P(\partial)\,\delta_{\mathcal{F}}$ and $F^{-1}\Phi = P(x)\cdot \exp 2\pi i \langle f, x\rangle \cdot \delta_{\mathfrak{p}}(x)$. Here the integral (3.8) converges (this is to be expected: we know in fact that in the coordinates p, q, z the restriction exists). Actually, the inner integral in (3.8) is rapidly decreasing along $\mathfrak{p}$ since it is the Fourier transformation of $\delta(p)\varphi(q, z)$ whose wave front is transversal to $\mathfrak{p}$.

For the proof of the lemma we write $F^{-1}\Phi(x) = P(x)\, F^{-1}(\delta_{\mathcal{F}})$ and substitute (3.8) into (3.3). The resulting series is exactly the same as (3.5) if we set $\Psi = \delta_{\mathcal{F}}$. Formula (3.5) is obtained by transforming (3.4), which is equivalent to (3.7). QED.

Proof of Lemma 3.9. Since the integrand in $\int \bar{\chi}(p)\, \pi(p)\, dp$ is multiplicative, under the translation $p \to p_1 p$ the integral is multiplied by $\chi(p_1)\pi(p_1)$. But from the invariance of Haar measure it follows that the integral is unchanged under a translation. Thus, if s(q, z) is a smooth function, then its image $\Phi = \pi(\delta_{\mathcal{F}})s$ satisfies the equation $\bar{\chi}(p_1)\, \pi(p_1)\, \Phi = \Phi$. That is, for every $x \in \mathfrak{p}$ (by differentiating along x) we obtain $\bar{\chi}_t(x)\Phi + \pi_t(x)\Phi = 0$. We substitute the expressions for $\bar{\chi}_t$ and π_t and set $\Phi = \Sigma t^i \Phi_i$. By equating the coefficients of powers of t to zero, we obtain

$$2\pi i\, [f(x) - v_x(q, z)]\, \Phi_k = [u_x(q, z)\cdot \partial/\partial q + {}^1/_2 \operatorname{tr} u_x'(q, z) - {}^1/_2 \operatorname{tr}_{\mathfrak{g}/\mathfrak{p}} \operatorname{ad} x]\, \Phi_{k-1}. \tag{3.9}$$

To simplify the analysis we assume that the coordinates of f are (0, 0, 0). We prove by induction that Φ_k is proportional to $\delta(q, z)$. Assume this is true for Φ_{k-1}, then the right-hand side of (3.9) is zero. For if $x \in \mathfrak{p}$ then $\operatorname{tr}_{\mathfrak{g}/\mathfrak{p}} \operatorname{ad} x = -\operatorname{tr} u_x'(0, 0)$ (Lemma 2.9) and $u_x \cdot \delta'(q, z) = -u_x'(q, z)\cdot \delta(q, z)$ [the coefficient of $\partial\delta/\partial q$, which is $u_x(0, 0)$, vanishes]. Observing that $f(x) = x(0, 0, 0) = v_x(0, 0)$, we rewrite (3.9) in the form $[v_x(q, z) - v_x(0, 0)]\Phi_k = 0$, or extracting the linear part, in the form $[dv_x(0, 0) + o(q, z)]\Phi_k = 0$.

It follows from the definition of $\pi(\delta_{\mathcal{F}})$ [see (3.3)] that Φ_k is proportional to some derivative of $\delta(q, z)$. We need to show that this derivative has in fact order 0. Clearly, this will follow from the equality $[dv_x(0,0) + o(q, z)]\Phi_k = 0$ if the differentials $dv_x(0, 0)$, $x \in \mathfrak{p}$, generate the whole space of forms with the basis dq and dz. Let $x_1, \ldots, x_{n+k}$ be a basis in $\mathfrak{p}$. It is enough to prove that the forms $dv_{x_i}(0, 0)$ are independent. For this we note that the functional dx(f) on $\mathfrak{g}^*$ coincides as an element of $\mathfrak{g}$ with x. Therefore the forms $dx_i(f)$ are independent, but for $x \in \mathfrak{p}$

$$dx(0, 0, 0) = \partial x/\partial p\cdot dp + \partial x/\partial q\cdot dq + \partial x/\partial z\cdot dz = dv_x(0, 0),$$

since $\partial x/\partial p\,(0, 0) = u_x(0, 0)$. QED.

Proof of Theorem 3.6. We rewrite the differential polynomials $P(\partial)$ in the formula (3.3) for a representation π, in the coordinates p, q, and z. Obviously, from Lemma 3.9 it follows that each term in the resulting expression for $\pi(\Psi)$ either contains a differential of Ψ with respect to p, or does not contain derivatives of Ψ at all. Therefore, if Ψ does not depend on p, then $\pi(\Psi) = \Psi \cdot \Sigma t^i D_i$. In particular, $\pi(1) = \Sigma t^i D_i$. But 1 "arises" from the identity in the enveloping algebra $U(\mathfrak{g}_t)$, and so $\pi(1) = 1$.

THEOREM 3.10. The restriction of multiplication in $\hat{H}$ to $\hat{A}$ is the same as the usual multiplication. In particular, $\hat{A}$ is a commutative subalgebra in $\hat{H}$.

Essentially, this asserts that if $\Phi, \Psi \in \hat{A}$, then in the series (3.1) for the product $\Phi \circ \Psi$ there remains only the first term. When Φ and Ψ are polynomials, by setting t = 1 we obtain Theorem 2.6.

Proof of Theorem 3.10. Let $\Psi_1, \Psi_2 \in \hat{A}$ and $\Phi = \Psi_1 \circ \Psi_2 - \Psi_1\Psi_2$. By Theorem 3.6, $\pi(\Psi_1 \circ \Psi_2) = \pi(\Psi_1) \times \pi(\Psi_2) = \Psi_1\Psi_2 = \pi(\Psi_1\Psi_2)$; therefore, $\pi(\Phi) = 0$. If $\Phi = \Sigma t^i \Phi_i$, then the coefficient of the zeroth power of t in $\pi_t(\Phi)$ is $\pi_0(\Phi_0) = \Phi_0(0, q, z)$. Hence Φ_0 vanishes on the surface p = 0. Since this is true for an arbitrary canonical coordinate system p, q, z, we have $\Phi_0 = 0$. Now we divide Φ by t. By repeating this argument we obtain $\Phi_1 = 0$, $\Phi_2 = 0$, etc. Hence, $\Phi = 0$.

Thus, the structure of the algebra $\hat{H}$ is described completely by:

THEOREM 3.11. The mapping π is an isomorphism of the algebra $\hat{H}$ onto the algebra of formal differential operators of the form $\sum_{i=0}^{\infty} \Phi_i \cdot (t \cdot \partial/\partial q)^i$ (the Φ_i are formal functions of q and z) that send the subalgebra $\hat{A}$ into operators of zero order.

To prove that the mapping is an epimorphism it is enough to construct functions $\mathcal{P}_j$ such that $\pi(\mathcal{P}_j) = t \cdot \partial/\partial q_j$. These can be easily chosen in the form $\mathcal{P} = \sum \Psi_i(q, z) \circ x_i + \Phi(q, z)$, where $x_i \in \mathfrak{g}$ [then $\pi(\mathcal{P}) = \sum \Psi_i (q, z) \circ \pi(x_i) + \Phi(q, z)$, where the $\pi(x_i)$ are known operators of the first order]. The conclusion that π is a monomorphism follows from the next assertion, whose proof is omitted.

Proposition 3.12. a) If the orbit Ω is defined by the equation $z = z_0$, then the kernel of the representation $\pi_{t,\Omega}$ is the ideal in $\hat{H}$ generated by $z - z_0$.

b) The intersection of the kernels of $\pi_{t,\Omega}$ (with respect to all Ω) is zero. QED.

4. Polarizations

In this section we are going to prove the various properties of polarizations used earlier. We discuss the case of points in general position, i.e., points lying in the complement of some submanifold in $\mathfrak{g}^*$ (depending on the problem being considered). The set of these points is open and dense in $\mathfrak{g}^*$.

The proof of Theorem 2.2 is based on the following important result.

Proposition 4.1 (see [11]). If $\mathfrak{g}$ is an algebraic Lie algebra whose radical is nilpotent, then there exists in the space $\mathfrak{g}^*$ an open everywhere dense G-invariant set consisting of closed orbits of maximal dimension.

COROLLARY 4.2. If $\mathfrak{g}$ is an algebraic Lie algebra whose radical is nilpotent, then any polarization of points in general position in $\mathfrak{g}^*$ satisfies Pukanszky's condition.

For suppose that f lies in a closed orbit and that $\mathfrak{p}$ is a polarization of f; then the intersection of the orbit with $f + \mathfrak{p}^{\perp}$ is closed in $f + \mathfrak{p}^{\perp}$. But this intersection contains a fiber Pf dense in $f + \mathfrak{p}^{\perp}$. QED.

Proof of the First Part of Theorem 2.2. Let $\mathfrak{g}$ be a Lie algebra; then $[\mathfrak{g}, \mathfrak{g}]$ is an algebraic Lie algebra whose radical is nilpotent. The theorem holds for it. We take in $\mathfrak{g}$ a chain of ideals

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_n \subset \mathfrak{g}_{n-1} \subset \ldots \subset \mathfrak{g}_1 \subset \mathfrak{g}_0 = \mathfrak{g}$$

such that $\dim \mathfrak{g}_i/\mathfrak{g}_{i+1} = 1$. Let $f \in \mathfrak{g}^*$, and f_i be the restriction of f to $\mathfrak{g}_i$, where f_n is a point in general position in $\mathfrak{g}_n^*$. We can choose a polarization $\mathfrak{p}$ of f such that $\mathfrak{p} \cap \mathfrak{g}_i$ is a polarization of f_i for all i (see [7]). We prove by induction on $\dim \mathfrak{g}_i$ that the polarization $\mathfrak{p}_i = \mathfrak{p} \cap \mathfrak{g}_i$ satisfies Pukanszky's condition.

Inductive Transition from $\mathfrak{g}_1$ to $\mathfrak{g}$. Let π: $\mathfrak{g}^* \to \mathfrak{g}_1^*$ be a projection; then $\pi(f) = f_1$. Since $\mathfrak{p}_1 \subset \mathfrak{p}$ we have $\pi(f + \mathfrak{p}^{\perp}) \subset f_1 + \mathfrak{p}_1^{\perp}$. Therefore, if l is some point of $f + \mathfrak{p}^{\perp}$, then by the inductive hypothesis its projection is carried into f_1 by an element from P_1. Hence there is an element p such that $pl - f_{|\mathfrak{g}_1} = 0$, i.e. ($\mathfrak{g}_1$ – an ideal of codimension 1) $\chi = pl - f$ is a character of g, and we need to send $f + \chi$ into f.

Note that p sends $f + \mathfrak{p}^{\perp}$ into itself; therefore, $pl \in f + \mathfrak{p}^{\perp}$. Hence the functional χ vanishes on $\mathfrak{p}$, and consequently it is representable in the form $\chi(x) = f([a, x])$, where $a \in \mathfrak{p}$. (Since $\mathfrak{p}$ is a Lagrange subspace of $\mathfrak{g}$ relative to the form $x, y \to f([x, y])$, every functional that vanishes on $\mathfrak{p}$ has this form.)

The inductive step is completed by:

LEMMA. Let $a \in \mathfrak{g}$, $f \in \mathfrak{g}^*$, and $\chi(x) = f([a, x])$ be a character on $\mathfrak{g}$. Then $\exp a$ sends f into $f + \chi$.

For, $(\exp a)f = f + af + {}^1/_2 a^2 f + \ldots = f + \chi + {}^1/_2 a\chi + \ldots$. Since χ is a character, it is mapped into itself under the action of a Lie algebra. Therefore, in the last sum there are only two summands. QED.

We do not give the proof of the second part of Theorem 2.2 because it is too involved.

The subsequent arguments are based on the following result of Duflo [5].

Proposition 4.3. Let P be a polarization of $f \in \mathfrak{g}^*$. If it satisfies Pukanszky's condition, then any torus (algebraic) in P is conjugate to a torus lying in the stabilizer G(f) of f.

Proof. Under the action of the linear manifold $f + \mathfrak{p}^{\perp}$ the torus T must have a fixed point. By Pukanszky's condition there is an element from P that sends this point into f. Therefore, its stabilizer, which contains T, is sent by conjugation into G(f). QED.

COROLLARY 4.4 (See [5]). Let $\mathfrak{g}$ be algebraic and f be a regular point in $\mathfrak{g}^*$. A solvable polarization of f satisfying Pukanszky's condition is admissible (see Lemma 2.4).

Let P be a solvable polarization; clearly, it is enough to verify the condition for admissibility for semisimple elements, which by Proposition 4.3 can be assumed to lie in G(f). The algebra $\mathfrak{g}(f)$ of the group G(f) is commutative [1]; therefore, for $x \in \mathfrak{g}(f)$, the nonzero eigenvalues of adx on $\mathfrak{p}$ and $\mathfrak{p}/\mathfrak{g}(f)$ coincide. To compare the eigenvalues on $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}/\mathfrak{g}(f)$ we note that the form x, y → f([x, y]) defines a nonsingular duality between these spaces, and for $x \in \mathfrak{g}(f)$ the operator adx preserves this form. QED.

COROLLARY 4.5. Let $\mathfrak{g}$ be arbitrary. Every regular point in $\mathfrak{g}^*$ has an admissible polarization.

For the proof we consider the algebraic hull $\tilde{\mathfrak{g}}$ of the algebra $\mathfrak{g}$. Let $\tilde{f}$ be a point in general position in $\tilde{\mathfrak{g}}^*$, and f be its restriction to $\mathfrak{g}$. We saw in the proof of Theorem 2.2 that we can construct a polarization $\mathfrak{p}$ of $\tilde{f}$ satisfying Pukanszky's condition such that $\mathfrak{p} = \tilde{\mathfrak{p}} \cap \mathfrak{g}$ is a polarization of f. Then (see Corollary 4.4), for all $x \in \mathfrak{p}$ the nonzero eigenvalues of adx in $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{p}}$ are of opposite sign. But adx sends $\tilde{\mathfrak{g}}$ into $\mathfrak{g}$, therefore we can replace $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ by $\mathfrak{g}/\mathfrak{p}$ and $\tilde{\mathfrak{p}}$ by $\mathfrak{p}$. Thus, $\mathfrak{p}$ is admissible.

Now let f be an arbitrary point of $\mathfrak{g}^*$. There is a sequence f_i of points in general position that converges to f. Every point f_i has an admissible polarization $\mathfrak{p}_i$. By going over to a subsequence, we may assume that $\mathfrak{p}_i$ converges to some subalgebra $\mathfrak{p}$. If f is regular, then by dimensionality arguments $\mathfrak{p}$ is a polarization of f.

Let $x \in \mathfrak{p}$, and $Q_{\mathfrak{p}}(t)$ and $Q_{\mathfrak{g}/\mathfrak{p}}(t)$ be characteristic polynomials of the operator adx in $\mathfrak{p}$ and $\mathfrak{g}/\mathfrak{p}$, respectively. The admissibility of $\mathfrak{p}$ means that $Q_{\mathfrak{p}}(-t) = t^r \cdot Q_{\mathfrak{g}/\mathfrak{p}}(t)$. This property is preserved when we take the limit.

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