PROBLEM OF SEARCH OF THE MINIMUM OF ENTROPY IN INDETERMINATE EXTENSION PROBLEMS

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1. There are a large number of problems connected with spectral analysis: a) the Hamburger and the Stieltjes moments problems; b) the Caratheodory and the Nevanlinna–Pik problems; c) extension from the segment of a positive-definite function and of a correlation function of arc of a helix in a Hilbert space; d) search of the spectral functions of a canonical system of differential equations, etc., in which, in the matrix-valued case, the problem reduces to the determination of the special matrix-valued functions f(z) of the class $\mathbf{P}_n$ (or $\mathbf{C}_n$) that are holomorphic in $\Pi_r = \{z: \operatorname{Re} z > 0\}$ (in $K = \{z: |z| < 1\}$) of n-th order with $\operatorname{Re} f(z) \geq 0$. In addition, in the so-called completely indeterminate case, the discription of all the desired f(z) is given by the linear transformation

$$f_{\mathscr{E}}(z) = [a_{11}(z)\mathscr{E}(z) + a_{12}(z)]\,[a_{21}(z)\mathscr{E}(z) + a_{22}(z)]^{-1}, \tag{1}$$

where $\mathscr{E}(z)$ is an arbitrary matrix-valued function of the class $\mathbf{B}_n$, i.e., $\mathscr{E}(z)$ is a holomorphic matrix-valued function of n-th order in $\Pi_r(\mathbf{K})$ such that $\|\mathscr{E}(z)\| \leqslant 1$, and $A(z) = [a_{ik}(z)]_1^2$ is a (j, J)-inner matrix-valued function, i.e., A(z) is a meromorphic (j, J)-contractive matrix-valued function in $\Pi_r(\mathbf{K})$:

$$A^*(z)JA(z) \leqslant j, \quad J = \begin{bmatrix} 0 & -I_n \\ -I_n & 0 \end{bmatrix}, \quad j = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix},$$

which, in addition, has (j, J)-unitary boundary values a. e.

By a well-known integral formula, to each f(z) from $P_n(C_n)$ there corresponds a matrix-valued spectral measure σ_f such that

$$\int_{-\infty}^{\infty} (1+\lambda^2)^{-1}\|d\sigma_f(\lambda)\| < \infty \qquad \left(\int_{-\pi}^{\pi} \|d\sigma_f(\lambda)\| < \infty\right);$$

for the boundary values of f(z) a. e.

$$\pi \operatorname{Re} f(i\lambda) = \sigma_f'(\lambda) \qquad (2\pi \operatorname{Re} f(e^{i\lambda}) = \sigma_f'(\lambda)).$$

The following functional has a definite entropy sense [1-3]:

$$h(f;\zeta) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} |\lambda + i\zeta|^{-2} \operatorname{Re}\zeta \ln\det\sigma_f'(\lambda)\,d\lambda \qquad (\zeta \in \Pi_r)$$

$$\left(h(f;\zeta) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} |e^{i\lambda} - \zeta|^{-2})(1 - |\zeta|^2) \ln\det\sigma_f'(\lambda)\,d\lambda \qquad (\zeta \in \mathbf{K})\right),$$

it is considered usually for $\zeta = 1$ ($\zeta = 0$) and is denoted here in this case by h(f). It is also considered for the solution of the problem of extrapolation of stationary processes [4].

For an arbitrary (j, J)-inner matrix-valued function $A(z) = [a_{ik}(z)]_1^2$, the corresponding matrix-valued function $\chi(z) = -a_{22}^{-1}(z)a_{21}(z)$ is inner and there exists an inner matrix-valued function $\beta(z)$ such that $\beta(z)a_{22}(z)$ is an outer matrix-valued function [5]. Let us set

$$\Delta^{-1}(z) = \beta(z)\,[a_{22}(z)\,a_{22}^*(z) - a_{21}(z)\,a_{21}^*(z)]\,\beta^*(z).$$

<u>THEOREM 1.</u> Let $A(z) = [a_{ik}(z)]_1^2$ be an arbitrary (j, J)-inner matrix-valued function such that the corresponding matrix-valued function $\chi(z)$ is "pure," i.e., $\|\chi(z)\| < 1$ in $\Pi_r(\mathbf{K})$. Then for each fixed ζ from $\Pi_r(\mathbf{K})$

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$$\min_{\mathscr{E}\in B_n} h(f_{\mathscr{E}};\zeta) = -\frac{1}{2}\ln\det\Delta(\zeta),$$

where the minimum is attained for $\mathscr{E}(z)\equiv\chi^*(\zeta)$ and only for this case.

2. The family of the matrix-valued functions $f_{\mathscr{E}}(z)$ is obviously compact, i.e., each sequence of matrix-valued functions of this family contains a subsequence that converges in $\mathrm{U}_r(\mathbf{K})$. It turns out that for each compact family $\mathfrak{A}$ of the matrix-valued functions f(z) from $\mathbf{P}_n(\mathbf{C}_n)$ (finite or infinite) $\min\{h(f): f\in\mathfrak{A}\}$, denoted in the sequel by h $(\mathfrak{A})$, is always attained.

In the investigation of the singular problems, i.e., of the problems a) and b) with infinite number of data, the problems c), d), etc., it is usual, as a preliminary, to consider the one-parameter family of the corresponding "truncated" problems, to which correspond the families $\mathfrak{A}_x$ of matrix-valued functions from $\mathbf{P}_n(\mathbf{C}_n)$ that depend monotonically on the parameter x. In Theorem 2, formulated below, it is assumed that x runs over a certain ordered right-filtering set X.

THEOREM 2. Let $\mathfrak{A}_x$ $(x\in X)$ be compact families of matrix-valued functions from $\mathbf{P}_n(\mathbf{C}_n)$ such that $\mathfrak{A}_{x_2}\in\mathfrak{A}_{x_1}$ if $x_1 < x_2$. Then

$$h\left(\bigcap_{x\in X}\mathfrak{A}_x\right) = \inf_{x\in X} h(\mathfrak{A}_x) \quad \left(=\lim_{x\in X} h(\mathfrak{A}_x)\right).$$

3. By the Zasukhin–Krein theorem [4], the condition $h(f) < \infty$ is necessary and sufficient for the existence of an outer matrix-valued function $\varphi_f(z)$ such that $\varphi_f^*(z)\varphi_f(z) = \mathrm{Re}\, f(z)$ a. e. for the boundary values. Here $\varphi_f(z)$ is determined by f(z) up to a left constant unitary factor and $h(f;\zeta) = -\ln|\det\varphi_f(\zeta)|$.

THEOREM 3. Let the matrix-valued function $A(z) = [a_{ik}(z)]_1^2$ satisfy the conditions of Theorem 1 such that the corresponding function $\beta(z)$ is scalar. Then for each fixed $\zeta\in\mathrm{U}_r(\mathbf{K})$

$$\varphi^*_{f_{\mathscr{E}}}(\zeta)\,\varphi_{f_{\mathscr{E}}}(\zeta)\leqslant\Delta(\zeta),$$

where the sign of complete equality is attained here for the constant matrix-valued function $\mathscr{E}(z)\equiv\chi^*(\zeta)$ and only for this case.

4. We illustrate application of the above-formulated theorems to two examples.

1) The matrix Caratheodory problem (the matrix trigonometric problem of moments). In this problem, it is required to describe the set $\mathfrak{A}_m$ of the matrix-valued functions f(z) of the class $\mathbf{C}_n$ such that $f(z) = a_0 + a_1 z + \ldots + a_m z^m + O(z^{m+1})$, where a_i $(i = 0, 1, \ldots, m)$ are preassigned matrices. Let us set $a_{-i} = 0$ $(i = 0, 1, \ldots, m)$, $A = [a_{j-k}]_0^m$, and $T = 2\,\mathrm{Re}\,A$. We know that $\mathfrak{A}_m\neq\varnothing$ if and only if $T\geq 0$. In the completely indeterminate case, where $T>0$, $\mathfrak{A}_m$ is described by Eq. (1). We indicate explicit expressions for $a_{ik}(z)$. Let X_k and Y_k $(k = 0, 1, \ldots, m)$ be blocks that form the last n columns of the matrices T^{-1} and A^*T^{-1}, respectively, and $\tilde{X}_k$ and $\tilde{Y}_k$ be the blocks that form the first n columns of the matrices T^{-1} and AT^{-1}, respectively, and set

$$Q_m(z) = \left(\sum_{k=0}^m X_k z^k\right) X_m^{-1/2},\quad P_m(z) = \left(\sum_{k=0}^m Y_k z^k\right) X_m^{-1/2},\quad Q_m^{\sim}(z) = \left(\sum_{k=0}^m \tilde{X}_k z^k\right)\tilde{X}_0^{-1/2},$$

$$P_m^{\sim}(z) = \left(\sum_{k=0}^m \tilde{Y}_k z^k\right)\tilde{X}_0^{-1/2}.$$

The polynomial $\mathbf{Q}(z; a_0, a_1, \ldots, a_m) = \mathbf{Q}_m(z)$ of degree m is normalized and is orthogonal on the unit circle to all the polynomials $\mathbf{X}_{m-1}(z)$ of degree at most $m-1$ in the space of the matrix-valued polynomials $\mathbf{X}(z)$ with the matrix-valued metric defined by the formula

$$(\mathbf{X},\mathbf{Y}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{Y}^*(e^{i\lambda})\,d\sigma_f(\lambda)\,\mathbf{X}(e^{i\lambda}),$$

where f(z) is an arbitrary solution of the problem; and $Q_m^{\sim}(z) = z^m Q\left(\frac{1}{z}; a_0^*, \ldots, a_m^*\right)$ is the polynomial symmetric to it; it is orthogonal to all $z\mathbf{X}_{m-1}(z)$. The "conjugate" polynomials $\mathbf{P}_m(z)$ and $\mathbf{P}_m^{\sim}(z)$ have exactly the same sense if $f^{-1}(z)$ is considered in place of f(z). It turns out that

$$a_{11}(z) = z\,\mathbf{P}_m(z),\quad a_{12}(z) = \mathbf{P}_m^{\sim}(z),\quad a_{21}(z) = z\,\mathbf{Q}_m(z),\quad a_{22}(z) = -\mathbf{Q}_m^{\sim}(z).$$

These formulas are naturally simplified in the scalar case and were established in this case by A. P. Artëmenko and, independently, by Geronimus (see [6]). In the matrix (and even in the operator) case, various descriptions of the set $\mathfrak{A}_m$, however different from the one given here, were earlier obtained in [7, 8]. Since $\mathbf{Q}_{\tilde{m}}$ is an outer matrix-valued function, Theorem 3 is applicable, where

$$\Delta^{-1}(\zeta) = \Delta_m^{-1}(\zeta) = Q_m^{\sim}(\zeta)\,[Q_m^{\sim}(\zeta)]^* - |\zeta|^2\, Q_m(\zeta)\, Q_m^*(\zeta).$$

When $\zeta = 0$, for $f \in \mathfrak{A}_m$ we have

$$h(f) \geqslant \frac{1}{2} \ln \det \mathfrak{X}_0, \qquad \varphi_f^*(0)\, \varphi_f(0) \leqslant \mathfrak{X}_0^{-1},$$

the equalities here are attained for $f(z) = -\mathbf{P}_{\tilde{m}}(z)[\mathbf{Q}_{\tilde{m}}(z)]^{-1}$ and only for this case. The first inequality was established in [9] by using a flow of "entropy" investigations in spectral analysis [10], and the second inequality was established in [11]. Let us observe that a narrower class has been considered in these two works in place of $\mathfrak{A}_m$.

Applying Theorem 3 to the Caratheodory problem, we get a generalization of the Szegö asymptotic formula [12] to the matrix case.

2) The problem of extension from a segment of the Hermitian-positive functions. In this problem, for a preassigned Hermitian-positive continuous function $r_\tau(t)$ on $[-2\tau, 2\tau]$, it is required to describe the set of the Hermitian-positive functions $r(t)$ on $(-\infty, \infty)$ such that $r(t) = r_\tau(t)$ for $-2\tau \le t \le 2\tau$. We know [13, 14] that this set is always nonempty and, in the case where it contains more than one function $r(t)$, the family $\mathfrak{A}_\tau$ of the corresponding functions

$$f(z) = \int_0^{\infty} e^{-tz}\, r(t)\, dt \qquad (\operatorname{Re} z > 0) \tag{2}$$

coincides with the image of $\mathbf{P}_1$ under its mapping into itself by the linear transformation

$$f(z) = (w_{11}(iz)\,\omega(z) - iw_{12}(iz))\,(iw_{21}(iz)\,\omega(z) + w_{22}(iz))^{-1} \qquad (\omega \in \mathbf{P}_1), \tag{3}$$

where $\det[w_{ik}(\lambda)]_1^2 = 1$, $w_{ik}(\lambda)$ $(i, k = 1, 2)$ are entire real-valued functions of finite degree τ of the Cartwright class such that $[w_{ik}(0)]_1^2 = I_2$.

The substitution $\omega(z) = [1 - \mathscr{E}(z)]\,[1 + \mathscr{E}(z)]^{-1}$ transforms the transformation (3) into (1), so that

$$a_{21}(z) = \frac{1}{\sqrt{2}}\,[w_{22}(iz) - iw_{21}(iz)], \qquad a_{22}(z) = \frac{1}{\sqrt{2}}\,[w_{22}(iz) + iw_{21}(iz)].$$

It turns out that here $\beta(z) = e^{-\tau z}$. Therefore,

$$\Delta^{-1}(\zeta) = \Delta_\tau^{-1}(\zeta) = 2e^{-2\tau \operatorname{Re}\zeta}\, \operatorname{Im}\,[w_{21}(i\zeta)\,\overline{w_{22}(i\zeta)}] \qquad (\zeta \in \Pi_r).$$

Let us observe that in [3] merely the existence of $\min h(x)$ is proved under additional restrictions on $r_\tau(t)$.

Now if $r(t)$ is a continuous Hermitian-positive function on $(-\infty, \infty)$ such that for its restriction $r_\tau(t)$ to $[-2\tau, 2\tau]$, where τ is an arbitrary positive number, there is no unique Hermitian-positive extension, then by Theorem 3 for $f(z)$, obtained by Eq. (2), we get

$$h(f;\zeta) = \frac{1}{2} \ln \lim_{\tau \uparrow +\infty} \{2e^{-2\tau \operatorname{Re}\zeta}\, \operatorname{Im}\,[w_{21}^{(\tau)}(i\zeta)\,\overline{w_{22}^{(\tau)}(iz)}]\}.$$

The matrix-valued function $W(\lambda) = [w_{ik}(\lambda)]_1^2$ is the monodromy matrix of a certain Hamiltonian canonical system that admits an exact characterization. Analogous results are obtained in the consideration of the spectral functions of arbitrary Hamiltonian canonical systems of arbitrary phase dimension. From them, in particular, we obtain results about the entropy for strings, established in the fundamental monograph [15], and also a series of results from the interesting investigation [16] together with certain additions.

5. Formulas, analogous to those indicated here for the Caratheodory problem, were obtained earlier for the matrix Nehari problem [17]. Results, analogous to Theorems 1-3 with the replacement of $\operatorname{Re} f(z)$ by $I_q - f^*(z)f(z)$ are valid for the class $S_{p,q}$ of the matrix-valued functions $f(z)$ of order $p \times q$ that are measurable for $\operatorname{Re} z = 0$ $(|z| = 1)$ and for which $\|f(z)\| \le 1$. This enables us to apply the considered method to the Nehari problem and to a whole series of other problems that reduce to it.

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A CHARACTERISTIC OF SYMMETRICAL SPACES

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Poritskaya introduced in [1] for a symmetrical space E (all the functions and spaces considered below are given, unless otherwise stated, on [0, 1]; for definitions and terminology, see [1, 2]) the following numerical characteristic:

$$\Pi(E) = \inf \left\| \sum_{i=1}^{n} x_i \left(\frac{t - \tau_{i-1}}{\tau_i - \tau_{i-1}} \right) \varkappa_{[\tau_{i-1}, \tau_i]} \right\|_E, \tag{1}$$

where $\varkappa_e$ is the characteristic function of $e \subset [0, 1]$, and the infimum is taken with respect to $x_i \in E$, $\|x_i\|_E = 1$, τ_i, $0 = \tau_0 < \tau_1 < \ldots < \tau_n = 1$, and $n = 1, 2, \ldots$. It was shown there that the set $\{p \in [0, 1]: \lim\limits_{h \to 0} \|f(p + ht) - f(p)\|_E = 0\}$ of so-called Lebesgue–Banach E points of the function f has measure 1 for any $f \in E$ if, and only if, E is separable and $\Pi(E) > 0$. The purpose of this note is to clarify the independence of these conditions as well as the connection with other characteristics of E, and also to compute $\Pi(E)$ for the basic classes of symmetrical spaces.

Let an N-function $M(u)$ $(u > 0$; see [3]) and a nondecreasing concave function $\varphi(t)$ with $\varphi(0) = 0$ be given. We consider the space

$$L_{\varphi, M} = \left\{ f \in L^1, \ \|f\|_{\varphi, M} = \inf \left\{ \lambda, \int_0^1 M(f^*(s)/\lambda)\, d\varphi(s) \leqslant 1 \right\} < \infty \right\},$$

where, as usual, f^* is a decreasing permutation of $|f|$ (see [2]).

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