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Analytical multiple-scattering extension of the Mie theory: The LIDAR equation

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Abstract. Multiple scattering of light in aerosol media is described in a simple picture within the framework of Mie theory. Our approach leads to an analytical expression of the n-fold scattered electromagnetic field and then to an analytical derivation of multiple-scattering LIDAR equation form transport theory. This approach differs from both the descriptions of multiple scattering based on the approximation of radiative-tranfer theory and from statistical approaches mainly based on Monte-Carlo calculations. The physical quantities of interest are expressed by straightforward generalization of the corresponding single-scattering quantities. Therefore, the multiple-scattering contributions are calculated without losing the advantage of working with analytical expression in the frame of Mie theory.

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In a preceding paper [1], we have presented an extension of Mie theory for light scattering from spherical particles, which includes the effects of multiple scattering. In this model, the spherical wave scattered by the first particle is considered as the field impinging on the second one. This procedure is repeated at all scattering orders. Following analytically this process step by step we construct the n-fold electromagnetic field scattered by an ensemble of particles. The multiple-scattering process is described without losing the advantage of working with analytical expressions in the framework of Mie theory and the physical quantities of interest, the backscattering and extinction coefficients, are expressed by a straightforward generalization of the corresponding single-scattering quantities. In [1], the multiple-scattering contributions to the radiation diffused by the whole medium are calculated from the light scattered once by an arbitrary but fixed particle of the aerosol, i.e., from a point-like source. This example has mainly the character of an illustration of how the quantities introduced in Mie theory for single-scattering events change when the incident field is not a plane wave but the single-scattered field itself, and when the scattering involves not only one particle but the whole medium. These results, beside their illustrative character, are useful in order to consider the modifications of single scattering due to the surroundings. However, the more interesting effects are global effects like the scattering of light propagating through an aerosol as it is the case in atmospheric physics or in optical depth calculations for suspension of particles. In order to make the previous results suitable for treating these problems, we have to consider backscattering and extinction or scattered-field intensity from the ensemble of particles in an aerosol considered as source for multiple scattering without isolating one of them as it has been done in [1].

In this paper, we consider the propagation case which corresponds to the scattering of a pulsed beam through an aerosol as it is realized in LIDAR applications in atmospheric physics, where information on the diffusion at different depths in the aerosol is considered. The LIDAR returns from an atmospheric aerosol or from clouds are currently described by the LIDAR equation in the single-scattering approximation [2]. In this paper, we present a generalization of the LIDAR equation including multiple-scattering effects which is based on the approach presented in [1].

The paper is organized as follows. In Sect. 1 we give a short recollection of the results of [1], which are of relevance in the present context. In Sect. 2 we present a generalization of the LIDAR equation which contains multiple-scattering contributions, and in Sect. 3, we discuss the applications of this equation to the specific numerical experiment described in the common comparative paper included in this issue [3].

1 Multiple-scattering contributions to the intensity

In this section, without giving an exhaustive description of our approach already published [1], we report some of the results which are relevant in the present context.

The single-scattering results of Mie theory [4] have been generalized by introducing the scattering coefficients of order n. In the single-scattering regime, particles scatter the impinging field independently and the scattering coefficients $a_l^{(1)}$ and $b_l^{(1)}$ are given by the standard expressions [5]

$$a_l^{(1)}(x) = \frac{m\psi_l(mx)\,\psi_l'(x) - \psi_l'(mx)\,\psi_l(x)}{m\psi_l(mx)\,\zeta_l^{(1)\prime}(x) - \psi_l'(mx)\,\zeta_l^{(1)}(x)}, \quad (1)$$

$$b_l^{(1)}(x) = \frac{m\psi_l'(mx)\,\psi_l(x) - \psi_l(mx)\,\psi_l'(x)}{m\psi_l'(mx)\,\zeta_l^{(1)}(x) - \psi_l(mx)\,\zeta_l^{(1)\prime}(x)}, \quad (2)$$

where m is the index of refraction of the sphere, $x = 2\pi r/\lambda$ is the particle-size parameter, r is the particle radius, λ the wavelength and ψ_l, ψ_l', $\zeta_l^{(1)}$ $\zeta^{(1)\prime}$ are the Riccati-Bessel and Hankel functions and their derivatives, respectively. Double scattering implies that the light scattered by a first particle is rescattered by a second one, and vice versa. In [1], the scattering coefficients for double scattering, denoted by $a_l^{(2)}$ and $b_l^{(2)}$, are obtained from the boundary conditions on the second particle using the single-scattered field as input. The boundary conditions on the surface of the second particle lead to expressions for $a_l^{(2)}$ and $b_l^{(2)}$ analogous to those valid for single scattering:

$$a_l^{(2)}(x_1, x_2, q) = a_l^{(1)}(x_1)\frac{m\psi_l(mx_2)\,\zeta_l^{(1)\prime}(q) - \psi_l'(mx_2)\,\zeta_l^{(1)}(q)}{m\psi_l(mx_2)\,\zeta_l^{(1)\prime}(x_2) - \psi_l'(mx_2)\,\zeta_l^{(1)}(x_2)}$$

$$\times \frac{x_2}{q} = a_l^{(1)}(x_1)\,\gamma_l(x_2, q), \quad (3)$$

$$b_l^{(2)}(x_1, x_2, q) = b_l^{(1)}(x_1)\frac{m\psi_l'(mx_2)\,\zeta_l^{(1)}(q) - \psi_l(mx_2)\,\zeta_l^{(1)\prime}(q)}{m\psi_l'(mx_2)\,\zeta_l^{(1)}(x_2) - \psi_l(mx_2)\,\zeta_l^{(1)\prime}(x_2)}$$

$$\times \frac{x_2}{q} = b_l^{(1)}(x_1)\,\delta_l(x_2, q), \quad (4)$$

where $q = |q|$ is the interparticle distance normalized with the factor $\lambda/2\pi$, and the dependence on the size parameters x_1 and x_2 for different particles is explicitly written down. The quantities $a_l^{(2)}$ and $b_l^{(2)}$ are the corrections to Mie theory due to double scattering. The correction factors γ_l and δ_l are functions of the inverse distance and they decrease to a negligible value when the interparticle distance becomes very large, i.e., in the single-scattering limit [1]. In analogy, we consider the light scattered by the second particle as the incoming field for the scattering on the third particle. The same calculation will give the third-order corrections to the scattering coefficients. A simple recurrence relation leads to a generalization of scattering coefficients for the n th-scattering order. The coefficients $a_l^{(n)}$ and $b_l^{(n)}$ have the same structure as those for double scattering. They are expressed by the single-scattering coefficients $a_l^{(1)}$ and $b_l^{(1)}$, which depend on the first particle-size parameter x_1 times the product of the correction factors γ_l or δ_l, respectively, which in turn, depend only on the j-th particle size parameter x_j and on the normalized interparticle distance q. Each term γ_l or δ_l can vary between one (in the case of superposed particles, i.e., for $q = x$) and zero (in the case of a very large distance). Therefore, the products defining the coefficients $a_l^{(n)}$ and $b_l^{(n)}$ contain a natural convergency of the correction factors as a function of the scattering order [1].

The multiple-scattering contributions for an aerosol of N particles, are calculated by integrating over the continuous distance-distribution function $\varrho(q)$ which reflects the random spatial positions of the particles in the aerosol giving the successive n-th scattering orders, i.e., for each scattering order, the sum over all scattering paths between the particles with respect to the position of the point-like source and over the particle-size distribution $g(x)$. For instance, for the case of non-correlated scatterers in an homogeneous medium and a punctual light source, we obtain

$$a_l^{(n)}(\mathbf{q}_1) = \sum_{m=2}^{n} \int\int a_l^{(1)}(x)\,\gamma_l^{(m)}(x, \mathbf{q}_1, q)\,\varrho(q)\,g(x)\,\mathrm{d}q\,\mathrm{d}x, \quad (5)$$

where $|\mathbf{q}_1|$ is the distance of the particle giving the first scattering, i.e., the point-like source, with respect to a fixed reference frame. In (5), the multidimensional integral over the interparticle distances, which is usually encountered in other methods, is reduced to a one-dimensional integration over the distance-distribution function q. In the case of a microscopic homogeneity of the medium, this is a consequence of the recurrence relation defining the n th order scattering coefficients. In the case of an incident beam, the distance $|\mathbf{q}_1|$ has to be averaged over the total width of the medium L.

The same holds for $b_l^{(n)}$. Expression (5) contains the main result of [1].

The explicit expression of the components of the amplitude of the n- fold scattered field $\mathbf{E}^{(n)}$ is constructed as in Mie theory and depends on the point-like particle positions through the coefficients $a_l^{(n)}$ and $b_l^{(n)}$. As an example, we report the explicit form of the θ-component of $\mathbf{E}^2$ in spherical coordinates. It reads

$$E_\theta^{(2)}(R, \mathbf{q}_1; \theta, \Phi) = \mathrm{i}\frac{\cos\Phi}{|\mathbf{R} - \mathbf{q}_1|}\sum_l \Big[\zeta_l^{(1)\prime}(|\mathbf{R} - \mathbf{q}_1|)\,a_l^{(2)}(\mathbf{q}_1) \times (P_l^{(1)\prime}(\cos\theta)\cos\theta\sin\theta - \zeta_l^{(1)}(|\mathbf{R} - \mathbf{q}_1|)\,b_l^{(2)}(\mathbf{q}_1)\,P_l^{(1)}(\cos\theta)\frac{\cos\theta}{\sin\theta}\Big], \quad (6)$$

where $a^{(n)}(\mathbf{q}_1)$ and $b^{(n)}(\mathbf{q}_1)$ are the n-th order coefficients introduced above, $P_l^{(1)}(\cos\theta)$ is the Legendre polynomial of order l and $R = |\mathbf{R}|$ is the distance to an arbitrary point in space normalized by the factor $\lambda/2\pi$. Expression (6) represents the amplitude of the field scattered two times by the whole medium when one considers as incoming field the radiation scattered by one single particle, i.e., a point-like light source represented here by the particle with index 1. In the case of an incident beam, a sum over the positions $\mathbf{q}_1$ of the particles responsible for the first scattering in the laboratory reference frame is to be considered.

The amplitude of the total scattered field will be

$$\mathbf{E} = \sum_j \mathbf{E}^{(j)}, \tag{7}$$

where $\mathbf{E}^{(j)}$ are the contributions of the scattering process of order j to the field. It is important to remark that the convergency of the products of $\gamma_l^{(n)}$ and $\delta_l^{(n)}$ of the multiple scattering correction factors as a function of the scattering orders (n) in the expression of $a_l^{(n)}$ and $b_l^{(n)}$ coefficients leads to a convergency of the sum in (7) as a function of the multiple-scattering contributions to the scattered field.

From the expression of the total scattered field, we derive the expressions of the attenuation and backscattering of light by the medium, including the effects of multiple scattering. The flow of energy is calculated as usual from the Poynting vector for the incident, the internal and the scattered fields. However, in this case, the scattered field contains the multiple-scattering contributions as well. This is the only difference which exists with respect to the expressions for the extinction and the backscattering in the single-scattering case. The sum of the n-th dimensionless averaged quantities

$$\beta^{(n)}(\mathbf{q}_1, \lambda) = \frac{\lambda^2}{2\pi} \sum_{l=1}^{\infty} (2l+1)\,\mathrm{Re}\,[a_l^{(n)}(\mathbf{q}_1, \lambda)\, a_l^{(n-1)*}(\mathbf{q}_1, \lambda) + b_l^{(n)}(\mathbf{q}_1, \lambda)\, b_l^{(n-1)*}(\mathbf{q}_1, \lambda)]\,, \tag{8}$$

$$\tau^{(n)}(\mathbf{q}_1, \lambda) = \frac{\lambda^2}{2\pi} \left| \sum_{l=1}^{\infty} (2l+1)(-1)^n [a_l^{(n)}(\mathbf{q}_1, \lambda) - b_l^{(n)}(\mathbf{q}_1, \lambda)] \right|^2 \tag{9}$$

give the total backscattering and the optical depth of the whole medium due to multiple scattering alone, always in the case of a point-like source.

Here, we are interested in the multiple-scattering contributions to the light intensity impinging on an aerosol. The intensity is easily found from the expression of the amplitude of the n-fold scattered field and we can quote here its expressions for the n th scattering order

$$I_\theta^{(n)}(R, \mathbf{q}_1; \theta, \Phi) = \left| \frac{1}{|\mathbf{R}-\mathbf{q}_1|} \sum_l \left[\zeta_l^{(1)\prime}(|\mathbf{R}-\mathbf{q}_1|)\, a_l^{(2)}(\mathbf{q}_1)\, P_l^{(1)\prime}(\cos\theta)\cos\theta\sin\theta - \zeta_l^{(1)}(|\mathbf{R}-\mathbf{q}_1|)\, b_l^{(2)}(\mathbf{q}_1) \times P_l^{(1)}(\cos\theta)\frac{\cos\theta}{\sin\theta} \right] \right|^2 \cos^2\Phi\,. \tag{10}$$

An analogous expression holds for I_Φ with $\sin^2\Phi$ instead $\cos^2\Phi$ appearing in it. The expression for the total scattered intensity up to all scattering orders is given by

$$I_s = \sum_n (I_\theta^{(n)} + I_\Phi^{(n)})\,. \tag{11}$$

This implies that no interference effects between the contributions from the fields at different scattering orders are considered. The angular coordinates θ, Φ in (6) and (10) refer to a reference frame placed in the center of the particle. In order to use the expression for the intensity in a propagation context, we have to express it in the laboratory reference frame, i.e., in a fixed direction θ', Φ', and to account for the extinction. This will be done in the next section.

2 Derivation of the LIDAR equation including multiple scattering

The stationary propagation of radiation in an aerosol is best described by a transport equation for the intensity of the electromagnetic field [6]. This is written in general as

$$\frac{\mathrm{d}I}{\mathrm{d}R} = -\sigma I + J\,, \tag{12}$$

where σ is the attenuation coefficient including single-scattering processes only. The source term J accounts for the contributions of scattering from different particles in the direction of propagation and, therefore, includes, in principle, also multiple scattering effects. It is generally expressed as

$$J(R, \theta', \Phi') = \sigma \int\int \mathrm{d}\theta\,\mathrm{d}\Phi\, P(\theta, \Phi; \theta', \Phi')\, I(R, \theta, \Phi)\,, \tag{13}$$

where $P(\theta, \Phi; \theta', \Phi')$ is the function describing the transposition from the angular direction θ, Φ to θ', Φ'. In the following, we will set $\theta' = \Phi' = 0$ for simplicity. When only single scattering is considered, the term J vanishes and the integral form of the LIDAR equation follows from (12) by formal integration, where the LIDAR pulse length determines the integration range. The backscattering contributes via the initial condition for (12) and the geometrical and physical properties of the receiver are considered. In order to evaluate the contribution of the source term (13) to the transport equation, we consider the propagation of a beam of intensity I through an aerosol from the point q to a point $q+dq$. The attenuation of the aerosol, will be given by (9) including multiple scattering, for a set of point-like sources placed in a tiny slab of with dq. However, the position of the point-like sources appears now as a variable quantity in the expression for the extinction. The contribution to the intensity scatted from all particles in the direction $\mathbf{R}$ will be given by the sum of all scattering events described by (10), and must also account for the attenuation due to the multiple-scattering extinction. An average over the positions q_1 of the point-like source has to be considered by integration over the length of the medium. Therefore, from (10), we introduce the ansatz

$$I(R, \theta, \Phi) = \sum_n \int I^{(n)}(R, \mathbf{q}_1, \theta, \Phi) \exp\left[-\int_0^{|\mathbf{R}-\mathbf{q}_1|} \tau_1(x)\,\mathrm{d}x \right] \mathrm{d}\mathbf{q}_1\,, \tag{14}$$

where τ_t is the total attenuation coefficient including single- and multiple-scattering contributions, as given by (9), which has to be integrated over the distance between a generic point-like source position $\mathbf{q}_1$ and the observation point $\mathbf{R}$. Substitution of (14) into (13) leads to an expression for the source term J which contains the effects of multiple scattering explicitelly. The function $P(\theta, \Phi)$ in (13) is responsible for the transformation of the angular variables in (10) to the laboratory reference frame as it projects the generic direction in which light is scattered into a fixed direction. We formally integrate the transport equation obtaining

$$I(R) = I_0 \exp[-\tau_1(R)] + \tau_1(R) \int dR' \exp[-\tau_1(R-R')] \int d\theta \, d\Phi \, P(\theta, \Phi) I_s(R', \theta, \Phi). \quad (15)$$

Here I_0 is the backscattered intensity due to single scattering alone. This expression is a general result which allows to evaluate the scattering intensity at a given point and for a given direction $\mathbf{R}$. In the next section, we will use it in order to discuss ground-based LIDAR returns from homogeneous clouds including multiple-scattering effects.

3 Numerical results

Equation (15) allows to calculate the signals which are sent to a receiver from an aerosol. An interesting example of these signals are the LIDAR returns from clouds. In this section, we give some results for ground-based LIDAR return from homogeneous clouds. The general form of (15) will be simplified on the basis of the geometry of the receiver and of the physical characteristics of the LIDAR pulses. In a first approximation, the integrals over the angles are simplified as follows. We assume that the contributions of the source term J to the transport equation arise from a small angle around the direction corresponding to an angle between $\mathbf{R}$ and $\mathbf{q}$ of 180°. This approximation is justified for ground-based measurements when the distance between the receiver and the scattering medium is small. Therefore, we eliminate the integral over the angles in (15) and replace it by the value of the integrand function for $\theta = \pi$ and $\varphi = \pi/2$ multiplied by a small angular width $\Delta\theta$ which corresponds to the angular aperture of the receiver. The space integral is now performed numerically by using a physical discretization length which is given by the laser-pulse length. Within this assumption, the intensity due to multiple scattering is replaced by the total backscattering coefficient, defined in (8). This gives, in fact, the intensity scattered for $\theta = \pi$, which is required here. Furthermore, it depends on the point-like source position as already pointed out in Sect. 1. We then assume that on a region, whose dimension are given by the laser-pulse length, the integral over the point-like source positions may be replaced by the value of the intensity for $\mathbf{R} = \mathbf{q}$. This amounts to assume that the attenuation function $\exp[-\tau_t(q)]$ does not vary very much in the interval which is considered, i.e., on the layer defined by the LIDAR pulse length. This approximation is justified for homogeneous scattering media. Within these assumptions, we obtain from (15)

$$I(R) = I_0 \exp[-\tau_t(R)] + \tau_t(R) \, \Delta\theta \int dR' \exp[-\tau_t(R-R')] \exp[-\tau_t(R') \, \beta_\lambda(R')] \, . \quad (16)$$

The remaining integral in (16) is performed numerically. Results for a C1 cloud are shown in the comparative paper in this issue [3].

4 Conclusion

The main feature of our approach is the analytical generalization of the scattered electromagnetic field to include the contributions of multiple-scattering processes. The basic expressions are derived in a very general context without using far-field or small-angle approximations. Moreover, the general expression of the field accounts for interference effects between the contributions from the fields at different scattering orders. In atmospheric applications, and, in particular, in the case of the LIDAR experiment considered for the comparative study defined in [3], the total scattered intensity has been calculated in the framework of some approximations compatible with the particular experiment and the major assumptions used by other groups. The multiple-scattering LIDAR equation has been introduced like an ansatz from transport theory. The results, which are in good agreement with the other proposed methods, are valid in the framework of these approximations. They have mainly an illustrative character of how our approach can work in some specific cases. However, they leave the way open to more general and accurate calculations. In a general way, our approach is not affected by some of the approximation errors, which may appear in purely numerical approaches. The multiple-scattering contributions are calculated avoiding the convergence and stability problems often encountered in atmospheric applications. These problems occur due to the large number of events that have to be taken into account, or when large optical depths and/or low signal-to-noise levels exists.

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