INTERSECTION THEORY ON THE MODULI SPACE OF CURVES

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In this paper we describe a combinatorial method of computing intersection indices of divisors on the moduli space of curves with marked points. The corresponding generating function is the logarithm of the partition function for some (new?) matrix model. By Witten's conjecture [1], this partition function must be a τ-function for a hierarchy of Korteweg-de Vries equations (see [2]).

1. Notations. Let the integers g, n satisfy the inequalities

$$g \geqslant 0, \quad n > 0, \; 2 - 2g - n < 0.$$

Let $M_{g,n}$ be the moduli orbispace of nonsingular complete complex curves of genus g with n distinct marked points $x_1, \ldots, x_n$. Let $\overline{M}_{g,n}$ denote the Deligne–Mumford compactification [3]. Recall that $\overline{M}_{g,n}$ is the moduli space of connected complete curves with distinct marked points that satisfy the following conditions:

a) all singularities of the curve are simple selfintersections;

b) the marked points are nonsingular;

c) all components of the nonsingular affine curve obtained by removing the marked points and the selfintersection points are of hyperbolic type (i.e., have negative Euler characteristic).

Let $\mathscr{L}_{(i)}$, i = 1, ..., n denote the linear bundles over $\overline{M}_{g,n}$, where the fiber of $\mathscr{L}_{(i)}$ is the cotangent space to the curve at the point x. Consider a sequence of commuting symbols $\tau_0, \tau_1, \ldots$. Let $d_1, \ldots, d_n$ be nonnegative integers whose sum equals $\dim_{\mathbf{C}} \overline{M}_{g,n} = 3g - 3 + n$. Let $\langle \tau_{d_1} \ldots \tau_{d_n} \rangle$ denote the intersection index

$$\int_{\overline{M}_{g,n}} \prod_{i=1}^{n} c_1 (\mathscr{L}_{(i)})^{d_i},$$

and let $F(t_0, t_1, \ldots) = \sum \langle \tau_0^{n_0} \tau_1^{n_1} \ldots \rangle \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$ — a formal series in the variables $t_0, t_1, \ldots$.

2. The Main Result. Let Λ be an N × N positive definite Hermitian matrix, where N is arbitrary, and let $d\mu_\Lambda(X)$ denote the measure on the space of N × N Hermitian matrices given by the density

$$\text{const} \times \exp(-\operatorname{Tr} X^2 \Lambda / 2),$$

where the constant is chosen from the condition $\int d\mu_\Lambda(X) = 1$. Then for $t_i = -(2i-1)!! \times \operatorname{Tr} \Lambda^{-(2i+1)}$ the series $F(t_0, t_1, \ldots)$ is an asymptotic series for $\log \left(\int \exp(\sqrt{-1}/6 \times \operatorname{Tr} X^3)\, d\mu_\Lambda(X) \right)$, when $\Lambda \to +\infty$. There exists a universal, not depending on N mapping

$$I: \mathbf{Q}[x_1, x_3, x_5, \ldots] \to \mathbf{Q}[l_1, l_3, l_5, \ldots]$$

such that for any polynomial P,

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$$\int P(\mathrm{Tr}\, X, \mathrm{Tr}\, X^3, \ldots)\, d\mu_\Lambda(X) = I(P)(\mathrm{Tr}\, \Lambda^{-1}, \mathrm{Tr}\, \Lambda^{-3}, \ldots).$$

3. A Combinatorial Model for $M_{g,h}$. By a band graph we shall mean a finite undirected graph (possibly with loops and multiple arcs) in which for each vertex there is fixed a cyclic order on the set of edges entering the vertex. For example, any planar graph is automatically a band graph. To each band graph there corresponds an orientable surface with boundary: one has to replace edges by long oriented paper strips and glue these at vertices, observing the cyclic order. A band graph can be alternatively defined as a finite set X together with two permutations s_1, s_2 of the elements of X such that x_1 is an involution with no fixed points. X is the set of orientations of the edges of the graph, s_1 is the change of orientation, the vertices correspond to the orbits of the permutation s_2, and the components of the boundary of the surface correspond to the orbits of the permutation $s_1 s_2$. Let X_0 and X_1 denote the set of vertices and the set of edges of the graph, respectively. A graph with a metric is by definition a graph in which to each edge there is assigned a positive number (its length). To each band graph with a metric one can associate a noncompact surface with an almost everywhere defined flat metric; this surface is obtained by glueing to each side of any edge $x \in X$ of the graph a strip $]0, l(x)] \times [0, +\infty)$, where $l(x)$ is the length of x, and identifying the infinite sides of the strips for neighboring edges. On this surface there exists a unique complex structure compatible with the Riemannian metric; the surface can be obtained by removing a finite number of points from a complete complex curve.

Fix numbers g and n as in Sec. 1. Let $M_{g,n}^{\mathrm{comb}}$ denote the orbispace of equivalence classes of connected band graphs with a metric, with the property that each vertex is entered by at least three edges, and the corresponding surface with boundary has genus g and a boundary with n components, labeled by the numbers 1, ..., n. On $M_{g,n}^{\mathrm{comb}}$ there is a natural topology: if the length of a graph edge that is not a loop tends to 0, then in the limit one obtains a new graph with a contracted edge. $M_{g,n}^{\mathrm{comb}}$ consists of finitely many open cells, which correspond to distinct combinatorial types of graphs. The dimension of such a cell is equal to the number of edges of the graph. The cells that correspond to graphs in which any vertex is entered by exactly ζ edges have maximal dimension, namely, $6g - 6 + 3n$. Let $p_1, \ldots, p_n$ denote the sequence of the perimeters of the boundary components of the corresponding surface. Thus, we constructed mappings from $M_{g,n}^{\mathrm{comb}}$ into $M_{g,n}$ (the complex structure described above) and into $\mathbf{R}_+^n$ (the functions $p_1, \ldots, p_n$). From the results of Strebel (see [4] for $n \geq 2$ and [5] for $n = 1$) one derives the following

Assertion. The resulting mapping from $M_{g,n}^{\mathrm{comb}}$ into $M_{g,n} \times \mathbf{R}_+^n$ is a homeomorphism.

4. $BU(1)^{\mathrm{comb}}$. $BU(1)^{\mathrm{comb}}$ is defined to be the orbispace of finite sequences of real numbers $(l_1, \ldots, l_n)$, $n \geqslant 1$, $l_i > 0$, where two sequences are identified if one is obtained from another by a cyclic permutation. In other words $BU(1)^{\mathrm{comb}}$ is the moduli space of connected band graphs with a metric, with the property that each vertex is entered by exactly two edges and the connected components of the boundary of the corresponding surface (cylinder) are enumerated. $BU(1)^{\mathrm{comb}}$ is the union of the increasing sequence of subsets

$$BU(1)^{\mathrm{comb}}_{\leq N} = \{(l_1, \ldots, l_n) \mid 1 \leqslant n \leqslant N\}.$$

On each $BU(1)_{\leq N}^{\mathrm{comb}}$ one can introduce a natural topology, and $BU(1)^{\mathrm{comb}}$ is endowed with the inductive limit topology. The mapping

$$\pi\colon BU(1)^{\mathrm{comb}} \to \mathbf{R}_+,\ \pi((l_1, \ldots, l_n)) = l_1 + \ldots + l_n,$$

is proper on $BU(1)_{\leq N}^{\mathrm{comb}}$. Over $BU(1)^{\mathrm{comb}}$ there is a fiber bundle whose fibers are homeomorphic to S^1; specifically, the fiber over the point $(l_1, \ldots, l_n)$ is the boundary of the n-side polygon whose side lengths, in increasing order, are $l_1, \ldots, l_n$. It is not difficult to show that the total space of this fiber bundle is contractible, and hence its base is homotopy equivalent to $BU(1) \cong \mathbf{C}P^\infty$.

Let φ denote the natural mapping of $M_{g,n}^{\mathrm{comb}}$ into $[BU(1)^{\mathrm{comb}}]^n$ which sends each band graph with a metric into the n components of the boundary of the corresponding surface.

5. Compactification. The mapping φ is an embedding into $[BU(1)_{\leq N}^{\mathrm{comb}}]^n$ for N large enough (to the extent to which this is possible in the world of orbispaces). Let $\overline{M}_{g,n}^{\mathrm{comb}}$ denote the closure of the image of φ.

Assertion. $\overline{M}_{g,n}^{comb}$ is homeomorphic to $\overline{M}'_{g,n} \times \mathbf{R}_+^n$, where $\overline{M}'_{g,n}$ is the quotient space of $\overline{M}_{g,n}$ with respect to a certain equivalence relation, the fundamental class of $\overline{M}_{g,n}$ goes into the fundamental class of $\overline{M}'_{g,n}$, and the S^1-bundles associated to $\mathcal{L}(i)$ are induced under the map φ from $[BU(1)^{comb}]^n$.

The aforementioned equivalence relation on $\overline{M}_{g,n}$ is as follows: two stable curves with marked points are regarded as equivalent if there exists a homeomorphism between the curves that is complex-analytic on the components containing the marked points.

6. The Chern Classes on $BU(1)^{comb}$. Let $t_1, t_2, \ldots$ be a sequence of variables. Let $\alpha_k(t)$ denote the k-form $\sum_{1\leq i_1<i_2<\ldots<i_k} dt_{i_1} \wedge dt_{i_2} \wedge \ldots \wedge dt_{i_k}$. Choose the following coordinates on $BU(1)^{comb}$: $p, t_1, \ldots, t_n$, where $p = l_1 + \ldots + l_n$, $t_i = l_i/p$.

LEMMA. The form $\tilde{\alpha}_{2k}(t) = k! \times \alpha_{2k}(t_1, \ldots, t_{n-1})$ is correctly defined, closed, and represents the cohomology class $c_1{}^k$.

"Correctly defined" means that

a) $\alpha_{2k}(t_1, \ldots, t_{n-1}) = \alpha_{2k}(t_2, \ldots, t_n)$ if $t_1 + \ldots + t_n = 1$;

b) $\alpha_{2k}(t_1, \ldots, t_{n-2}, 0) = \alpha_{2k}(t_1, \ldots, t_{n-1})$.

7. A Poisson Structure on $M_{g,n}^{comb}$. Let π (resp. $\overline{\pi}$) denote the projection of $M_{g,n}^{comb}$ (resp. $\overline{M}_{g,n}^{comb}$) onto $\mathbf{R}_+^n$ which sends each band graph with a metric into the sequence $p_1, \ldots, p_n$ of perimeters of components of the boundary; π and $\overline{\pi}$ are trivial bundles with fibers homeomorphic to $M_{g,n}$ and $\overline{M}'_{g,n}$, respectively.

On the highest dimensional cells in $M_{g,n}^{comb}$ the 2-vector

$$\beta = \frac{1}{2} \sum_{x \in X} \frac{\partial}{\partial l(x)} \wedge \frac{\partial}{\partial l(s_2(x))}$$

gives a Poisson structure.

Assertion. $\operatorname{Ker} \beta = \pi^* T^* \mathbf{R}_+^n$.

Proof. Ker β is the space of functions f defined on the edges of the graph that satisfy the following condition: for any section similar to that shown in Fig. 1 one has that $f_1 + f_3 = f_2 + f_4$. Let us show that there exists a unique function g on the set of components of the boundary such that the value of f at any edge is equal to the sum of the values of g at the components of the boundary that are adjacent to the edge. On a section adjacent to an arbitrary vertex the function g is uniquely recoverable from f (see Fig. 2): $g_1 = (f_2 + f_3 - f_1)/2$. The condition given above guarantees that the value of g at any component of the boundary, recovered from a vertex adjacent to that component, does not depend on the choice of the vertex. This establishes the assertion, as well as the fact that $T^*\pi$: $T^* \mathbf{R}_+^n \to T^*$ (highest dimensional cell) is an embedding. Thus, in some sense, $\overline{M}_{g,n}^{comb}$ is a Poisson manifold whose symplectic leaves are the fibers of the projection $\overline{\pi}$. We will need an explicit formula for β^{-1} on Ker $T\pi$. Let $t_*^{(i)}$ denote the inverse image of the coordinates t_* under the mapping $M_{g,n}^{comb} \to BU(1)^{comb}$, corresponding to the boundary component with number i.

Assertion. On $\pi^{-1}((p_1, \ldots, p_n))$ one has $4\beta^{-1} = \sum_{i=1}^{n} p_i^2 \times \tilde{\alpha}_2(t_*^{(i)})$.

Assertion. The nondegenerate volume form $\exp(4\beta^{-1}) \times dp_1 \wedge \ldots \wedge dp_n$ on $\overline{M}_{g,n}^{comb}$ gives an orientation compatible on the one-dimensional cells.

From this it follows that for all $(p_*) \in \mathbf{R}_+^n$ the fiber $\overline{\pi}^{-1}((p_*))$ is a cycle homologous to $\overline{M}_{g,n}$.

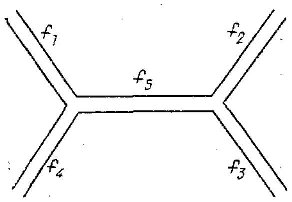

Fig. 1

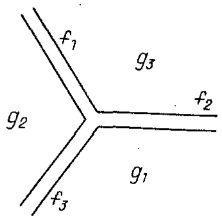

Fig. 2

8. The Volume of the Fiber. We have

$$\operatorname{vol}(\pi^{-1}(p_1,\dots,p_n)) = \int\limits_{\pi^{-1}((p_*))} \frac{(4\beta^{-1})^d}{d!} =$$

$$= \int\limits_{\pi^{-1}((p_*))} \frac{1}{d!} \times (p_1^2 c_1(\mathscr{L}_{(1)}) + \dots + p_n^2 c_1(\mathscr{L}_{(n)}))^d =$$

$$= \operatorname{sgn} \times \sum_{\substack{d_1,\dots,d_n \\ \sum d_i = d}} \frac{1}{d!} \times \prod_1^n p_i^{2d_i} \times \langle \tau_{d_1} \dots \tau_{d_n} \rangle \times \frac{d!}{\prod d_i!},$$

$$d = \dim_{\mathbf{C}} M_{g,n}.$$

Here sgn equals ±1 depending on whether the orientation derived from the symplectic structure is compatible with that derived from the complex structure or not. One can show that the sheaf $\mathscr{L}_{(1)} \otimes \dots \otimes \mathscr{L}_{(n)}$ is ample on $\overline{M}'_{g,n}$, and consequently sgn = +1 (the volume and the integral of a product of Chern classes are positive).

9. The Laplace Transformation. Let $\lambda_i > 0$, $i = 1, \dots, n$. Then

$$\int_0^\infty\!\!\int \dots \int \prod |dp_i| \times e^{-\sum \lambda_i p_i} \times \operatorname{vol}(\pi^{-1}((p_*)) =$$

$$= \sum_{(d_*)} \frac{\langle \tau_{d_1} \dots \tau_{d_n} \rangle}{\prod d_i!} \times \prod_{i=1}^n \int_0^\infty p_i^{2d_i} e^{-\lambda_i p_i}\, dp_i =$$

$$= \sum_{(d_*)} \langle \tau_{d_1} \dots \tau_{d_n} \rangle \times \prod_{i=1}^n \frac{(2d_i)!}{d_i!} \lambda_i^{-(2d_i+1)}.$$

Let us write the left-hand side of this chain of equalities in the form

$$\int\limits_{M_{g,n}^{\text{comb}}} e^{-\sum \lambda_i p_i} \times \rho \times \prod_{x \in X_1} |dl(x)|,$$

where ρ is the positive function on the highest dimensional cells given as the ratio of measures $\left(\prod |dp_i| \times \left|\frac{(4\beta^{-1})^d}{d!}\right|\right) : \prod\limits_{x \in X_1} |dl(x)|$. Clearly, ρ is constant on any cell and equals a positive rational number. The proof of the fact that Ker $\beta = \pi^* T^* \mathbf{R}_+^n$ works over the ring $\mathbf{Z}$ [1/2]; hence ρ is an invertible element in this ring, i.e., a power of 2. One can show that

$$\rho = 2^{2n+5g-5} = 4^d \times 2^{1-g} = 2^{d + \# X_1 - \# X_0}.$$

The integrand $\exp(-\Sigma \lambda_i p_i)$ equals

$$\prod_{\substack{x \in X_1 \\ x = i|j}} \exp(-l(x) \times (\lambda_i + \lambda_j)).$$

Therefore, we obtained the equality

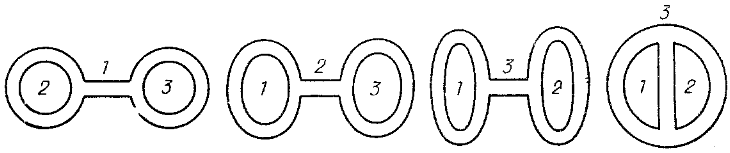

Fig. 3

$$\sum_{(d_*)} \langle \tau_{d_1} \dots \tau_{d_n} \rangle \times \prod_i (2d_i - 1)!! \times \lambda_i^{-(2d_i+1)} =$$
$$= 2^{-d} \times \sum_{(d_*)} \langle \tau_{d_1} \dots \tau_{d_n} \rangle \times \prod_i \frac{(2d_i)!}{d_i!} \lambda_i^{-(2d_i+1)} =$$
$$= 2^{-\# X_0} \times \sum_{\substack{\text{cells of highest} \\ \text{dimension}}} \prod_{\substack{\text{edges} \\ i \mid j}} \frac{2}{\lambda_i + \lambda_j}.$$

Example. g = 0, n = 3. In this case one has 4 cells of highest dimension (see Fig. 3).

In this case

$$\frac{2}{2\lambda_1(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)} + \frac{2}{2\lambda_2(\lambda_2+\lambda_3)(\lambda_2+\lambda_1)} + \frac{2}{2\lambda_3(\lambda_3+\lambda_1)(\lambda_3+\lambda_2)} + \frac{2}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)} = \frac{1}{\lambda_1\lambda_2\lambda_3}.$$

The only intersection index figuring in this example is $\langle \tau_0 \tau_0 \tau_0 \rangle = 1$.

10. A Matrix Model. Let Λ be a positive definite matrix, suppose Λ tends to $+\infty$, and let (λ_α), $\alpha \in A$, be the set of eigenvalues. Denote by t_k the expression

$$-(2k-1)!! \times \operatorname{Tr} \Lambda^{-(2k+1)}.$$

In the formula below summation is carried out over all g and n:

$$F(t_0, t_1, \dots) = \sum \langle \tau_{d_1} \dots \tau_{d_n} \rangle \times \frac{1}{n!} \times t_{d_1} \times \dots \times t_{d_n} =$$
$$= \sum_{n > 0;\, d_1, \dots, d_n} \frac{(-1)^n}{n!} \langle \tau_{d_1} \dots \tau_{d_n} \rangle \times \prod_i (2d_i - 1)!! \times \operatorname{Tr} \Lambda^{-(2d_i+1)} =$$
$$= \sum_{n > 0;\, \alpha_1, \dots, \alpha_n} \sum_{d_1, \dots, d_n} \frac{(-1)^n}{n!} \langle \tau_{d_1} \dots \tau_{d_n} \rangle \prod_i (2d_i - 1)!! \times \lambda_{\alpha_i}^{-(2d_i+1)} =$$
$$= \sum_{\text{graphs} \in \Gamma} \frac{(\sqrt{-1}/2)^{\# X_0}}{\# \operatorname{Aut}(\text{graph})} \times \prod_{\substack{\text{edges} \\ \alpha \mid\mid \alpha'}} \frac{2}{\lambda_\alpha + \lambda_{\alpha'}}.$$

In the last formula Γ denotes the set of equivalence classes of nonempty connected band graphs of valence 3, for which the components of the boundary of the corresponding surface are colored with colors from the set A. By the Feynman rules (see [6]), this sum is an asymptotic series for

$$\log \left(\int \exp(\sqrt{-1}/6 \times \operatorname{Tr} X^3)\, d\mu(X) \right),$$

where $d\mu(X)$ is the Gaussian measure on the space of Hermitian matrices of shape $A \times A$, such that $\langle X_{\alpha_1\alpha}, X_{\alpha_3\alpha_4} \rangle = \delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3} \times 2/(\lambda_{\alpha_1} + \lambda_{\alpha_2})$. It is readily seen that this is precisely the measure $d\mu_\Lambda(X)$ considered in Sec. 2, and we thus obtained a formula for the generating function F.

11. Other Moduli Spaces. The formula obtained above allows one to determine the value of the universal map I given in Sec. 2 on the polynomials x_3^k, k = 0, 1, ... The existence of I(P) for other monomials can be established according to the same scheme:

0) fix a sequence of nonnegative integers m_0, m_1, ... and another positive integer n;

1) consider the moduli space $M_{m_*,n}$ of band graphs with a metric with the property that there are exactly m_i vertices of degree $2i + 1$ and no vertices of even degree, and the components of the boundary of the corresponding surface are labeled by numbers from 1 to n;

2) define $\overline{M}$ as the closure of M in $[BU(1)_{\leq N}{}^{comb}]^n$, where N is sufficiently large;

3) introduce a Poisson structure β on M (to this end it is necessary to assume that all the vertices of the graph are of odd degree), the leaves of this Poisson structure are the fibres of the projection π, and the 2-form β^{-1} is given by the same formula as in Sec. 7;

4) check that $\overline{M}$, equipped with the orientation derived from the Poisson structure, is a cycle with closed support;

5) denote by $\langle\tau_{d_1} \ldots \tau_{d_n}\rangle_{m_0,m_1,\ldots}$ the rational number

$$\int_{\overline{M}_{m_*,n}} \prod_{i=1}^{n} c_1(\mathcal{L}_{(i)})^{d_i} \times \mathrm{fund}(\mathbf{R}_+^n),$$

[here fund(·) denotes the fundamental cohomology class with compact supports];

6) compute the ratio of the measures ρ (as in Sec. 9); this will be a constant on M, equal to $4^d \times 2^{1-g}$, where $d = 1/2(\dim M^{-n})$;

7) compute the Laplace transform of the volume of the fibers of the projection π in two ways:

a) as a polynomial whose coefficients are equal to $\langle\tau \ldots\rangle$,

b) as the sum over all components of M of the products $1/(\lambda_i + \lambda_j)$, taken over all edges;

8) now, as above, pass to the matrix model, taking the contribution of the integrals over $M_{m_*,n}$ with the weight $\prod_{j=0}^{\infty} s_j^{m_j}$, where s_j are new independent variables.

Then for $t_i = -(2i - 1)!! \times \mathrm{Tr}\,\Lambda^{-(2i+1)}$ we conclude that the expression

$$\sum_{n_*, m_*} \langle\tau_0^{n_0}\tau_1^{n_1}\ldots\rangle_{m_0, m_1,\ldots} \times \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!} \times \prod_{j=0}^{\infty} s_j^{m_j}$$

is an asymptotic series for

$$\log\left(\int \exp\left(\sum_{j=0}^{\infty} \sqrt{-1}\, s_j \times \left(\frac{-1}{2}\right)^j \times \mathrm{Tr}\,\frac{X^{2j+1}}{2j+1}\right) d\mu_\Lambda(X)\right).$$

12. Remarks and Conjectures. 1) For m = 0 the cycles M described in the preceding section lie in $\overline{M}_{g,n}{}^{comb}$. They are all even-dimensional, and the number of cycles in codimension 2N is equal to the number of partitions p(N). It is natural to expect that these cycles are images of cycles on $\overline{M}_{g,n}$ that are dual to all polynomials in the Chern classes of some vector bundle (or element of K_0) over $\overline{M}_{g,n}$. It is possible that this bundle is the tangent bundle.

2) The function F gives an asymptotic expansion at infinity for the matrix analogue of the Airy function:

$$\int \exp(\sqrt{-1}(\mathrm{Tr}\,X^3/6 - \mathrm{Tr}\,XY/2))\,dX =$$
$$= \int \exp(\sqrt{-1}(\mathrm{Tr}\,(X - Y^{1/2})^3/6 - \mathrm{Tr}\,(X - Y^{1/2})\,Y/2))\,dX =$$
$$= \exp\left(\frac{\sqrt{-1}}{3}\,\mathrm{Tr}\,Y^{3/2}\right) \times \int \exp(\sqrt{-1}(\mathrm{Tr}\,X^3/6 - \mathrm{Tr}\,X^2Y^{1/2}/2))\,dX.$$

3) Apparently the partition function of a matrix model depending on the parameters $s_0, s_1, \ldots$ is a τ-function in the variables $\mathrm{Tr}\, \Lambda^{-1}$, $\mathrm{Tr}\, \Lambda^{-3}/3, \ldots$.

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REGULAR FACTORIZATIONS OF A CHARACTERISTIC FUNCTION AND SINGULAR OPERATOR-FUNCTIONS

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UDC 517.9

This paper is devoted to a study of the regularity of factorizations of characteristic functions of bounded operators L with kernel imaginary parts. In the papers from the Odessa school on the operator theory it was demonstrated that an invariant subspace of an operator admits a certain nontrivial factorization of its characteristic function [1-3]. In the indicated papers there was also shown that, conversely, to every factorization of a characteristic function there corresponds a subspace which is invariant with respect to an operator that is distinguishable from L by an orthogonal component which, in turn, represents a self-adjoint operator. The resulting problem asks for a description of such factorizations, called regular, which generate subspaces invariant with respect to the same given operator. In [4] there were obtained necessary and sufficient conditions for the regularity of factorizations of contractive operator-functions. Below, there will be formulated criteria showing that a transition from the case of contractive operator-functions to the case of J-contractive operator-functions yields an additional source of irregularity due to the presence of a singular [5-7] (see the definition below) operator-function in each of the factors. From this criterion there follows immediately the regularity of the product of singular and regular [5-7] (see below) operator-functions and also, proven in the paper [8], the regularity of the canonical factorization of J-contractive functions into J-inner and J-outer factors.

Let E be a Hilbert space, $\Theta(\lambda)$: $E \to E$ be a J-contractive operator-function ($J - \Theta^*(\lambda) \times J\Theta(\lambda) \le 0$, $\mathrm{Im}\,\lambda > 0$, J: $E \to E$, $J = J^*$, $J^2 = I$). We will say that $\Theta(\lambda) \in \Omega_J{}^1$, if $\Theta(\lambda) \in \Omega_J$ [1] and $I - \Theta(\lambda) \in \sigma_1$ for all numbers $\lambda \in \mathbf{C}$ with sufficiently large moduli. In [1] there was shown that any operator function $\Theta(\lambda) \in \Omega_J$ is a characteristic function of some bounded operator L, acting in a Hilbert space H, $L = A + i\beta J\beta^*/2$, A: $H \to H$, $A = A^*$, β: $E \to H$, J: $E \to E$, $J^2 = I$. If $\Theta(\lambda) \in \Omega_J{}^1$, then, as it is easily seen, $\beta \in \sigma_2$. By using an argument, analogous to that in the paper [9], one can establish that $\Theta(\lambda) \in \Omega_J$ admits a representation (in the upper and lower semiplane) in the form of a relation between two bounded analytic operator-functions;

$$\Theta(\lambda) = \Theta_1^{-1}(\lambda)\, \Theta_2(\lambda),\ \mathrm{Im}\,\lambda > 0,$$
$$\Theta(\lambda) = \Theta_2^{-1}(\lambda)\, \Theta_1'(\lambda),\ \mathrm{Im}\,\lambda < 0. \qquad (1)$$

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