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SKLYANIN ELLIPTIC ALGEBRAS

A. V. Odesskii and B. L. Feigin

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In [2, 3] Sklyanin has constructed the family of the algebras $A(\mathscr{E}, \tau)$, parametrized by the set of pairs $(\mathscr{E}, \tau)$, where $\mathscr{E}$ is an elliptic curve and τ is a point on it. This family has the following properties:

1. The algebra $A(\mathscr{E}, \tau)$ is graded, $\dim A(\mathscr{E}, \tau)_i = 0$ for $i < 0$, and $\dim A(\mathscr{E}, \tau)_i = C_{i+3}{}^i$. The algebra $A(\mathscr{E}, \tau)$ is generated by the four-dimensional space $A(\mathscr{E}, \tau)_1$ and quadratic relations: the six-dimensional space

$$\operatorname{Ker} (A(\mathscr{E}, \tau)_1 \otimes A(\mathscr{E}, \tau)_1 \to A(\mathscr{E}, \tau)_2).$$

The algebra $A(\mathscr{E}, 0)$ is isomorphic to the algebra of polynomials in four variables.

2. Let the symbol Γ_n denote the finite Heisenberg group, i.e., the group generated by elements x, y, and ε and the relations $x^n = y^n = \varepsilon^n = 1$, $x\varepsilon = \varepsilon x$, $y\varepsilon = \varepsilon y$, $xy = \varepsilon yx$. The group Γ_4 acts by graduation-preserving automorphisms on the algebra $A(\mathscr{E}, \tau)$. The space $A(\mathscr{E}, \tau)_1$ is an irreducible representation Γ_4.

3. Let $\mathbf{C}[V]$ be the ring of polynomials generated by the space V and $a \in \operatorname{End} V$. Let us form the semidirect product of $\mathbf{C}[t]$ and $\mathbf{C}[V]$. This is the algebra generated by its subalgebra $\mathbf{C}[V]$ and the element t and the relations $tv = (av)t$, where v runs over V. Let $\mathbf{C}[V, a]$ denote the subalgebra of $\mathbf{C}[t] \ltimes \mathbf{C}[V]$ generated by the subspace $\mathbf{C}\cdot 1 \oplus tV$; $\mathbf{C}[V, a]$; it is called the algebra of skew polynomials.

Let τ be a point of fourth order on $\mathscr{E}$. Let us identify the group of points of fourth order on $\mathscr{E}$ with the quotient of Γ_4 modulo the center. Let $X(\tau)$ be a lifting of τ in Γ_4. The algebra $A(\mathscr{E}, \tau)$ is isomorphic to the algebra $\mathbf{C}[A(\mathscr{E}, \tau)_1, X(\tau)]$.

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4. The set of the graded $A(\mathscr{E}, \tau)$-modules $M = \oplus M_i$, $i > 0$, such that $\dim M_i = 1$ for all i and M is generated by $M_1 \subset M$, is parametrized by points of the curve $\mathscr{E}$ if τ is not a point of fourth order.

5. If τ is not a point of finite order, then the center of the algebra $A(\mathscr{E}, \tau)$ is generated by two elements of second degree. By continuity, these elements exist for all $\tau \in \mathscr{E}$. For $\tau = 0$, the corresponding two polynomials are equations for $\mathscr{E} \subset \mathbf{C}P^3$. The level curves of these polynomials are symplectic leaves of the algebra $A(\mathscr{E}, \tau)$, considered as a deformation of the algebra $A(\mathscr{E}, 0)$.

In this article we construct a family of associative algebras, denoted by $Q_{n,k}(\mathscr{E}, \tau)$, with analogous properties. Here $\mathscr{E}$ is an elliptic curve, $\tau \in \mathscr{E}$; and n and k are relatively prime numbers such that $k < n$. Thus, the family of algebras depends on two continuous parameters. In addition, $A(\mathscr{E}, \tau) = Q_{4,1}(\mathscr{E}, \tau)$. The algebras with p^2 generators, constructed by Cherednikov [6], correspond to the case $n = p^2$, $k = p - 1$.

Following [2], we will call a graded associative algebra $B = \oplus B_i$, $i \geq 0$, such that $B_0 \cong \mathbf{C}$, B is generated by B_1 with quadratic relations, and $\dim B_i = C_{i+n-1}{}^i$, where $n = \dim B_1$, a Sklyanin algebra. This means that $B = T(B)/I(S)$, where $T(B) = \mathbf{C} \oplus B_1 \oplus (B_1 \oplus B_1)\ldots$ is a tensor algebra and $I(S)$ is the two-sided ideal generated by a space of quadratic relations $S \subset B_1 \oplus B_1$, such that $\dim S = n(n-1)/2$. It is necessary that the dimension of B_i is equal to the dimension of the space of homogeneous polynomials of degree i in n variables.

We will call the set of all, up to isomorphism, linear modules the characteristic variety ch(B) of the Sklyanin algebra B. A module $M = M_1 \oplus M_2 \oplus \ldots$ is said to be linear if $\dim M_i = 1$ for all i and M is graded and is generated by M_1. Let $\theta(M) = \theta(M)_1 \oplus \theta(M)_2 \oplus \ldots$ denote a linear module such that $\theta(M)_i = M_{i+1}$. We get a mapping θ: $ch(B) \to ch(B)$. Set $V = \{v \in B_1, vM_1 - 0\}$. Then $\dim V = n - 1$. Let $S(M) = V^{\perp} \subset B_1{}^*$ denote the annihilator of V. The mapping $M \to S(M) \in CP^{n-1}$ is an embedding for Sklyanin algebras of general position. Thus, the structure of a projective algebraic variety is defined on ch(B). In this case, θ is an algebraic automorphism of ch(B).

If B is a deformation of a polynomial ring, then the cone over ch(B) coincides with the join of homogeneous two-dimensional symplectic leaves, defined by the deformation.

We assume that the Sklyanin algebra B is associated with an elliptic curve $\mathscr{E}$ if the group $\mathscr{E}$ acts on its characteristic variety ch(B) and the mapping θ: $ch(B) \to ch(B)$ is the translation by an element $\tau \in \mathscr{E}$.

Let N be an algebraic variety, ξ be a linear fibering on N, and Z be a subvariety of $N \times N$. Let $Z_0 = N$, $Z_1 = Z$, and Z_k be the set of the collections $(n_1, \ldots, n_k)$, such that $n_i \in N$ and $(n_j, n_{j+1}) \in Z$ for all i and j. Let ξ_k be the fibering $\overset{k}{\boxtimes}\xi$ on N^k, restricted to $Z_k \subset N^k$. It is clear that there exists a mapping $Z_{i+j} \to Z_i \times Z_j$ for all i and j that induces a homomorphism $H^0(\xi_i) \oplus H^0(\xi_j) \cong H^0(\xi_i \boxtimes \xi_j) \to H^0(\xi_{i+j})$. These homomorphisms define the structure of a graded algebra on the space $\bigoplus_i H^0(\xi_i) = A(N, Z, \xi)$.

If B is a Sklyanin algebra, $R \subset ch(B) \times ch(B)$ is the graph of the mapping θ, and ξ is the image of the standard fibering on the projective space $P(B_1{}^*)$ under the mapping S: $ch(B) \to P(B_1{}^*)$, then there exists a graded homomorphism of algebras $B \to A(ch\,B, R, \xi)$.

The problem of reconstruction of a Sklyanin algebra from its characteristic variety N, the mapping θ: $N \to N$, and the fibering ξ arises naturally. It is clear that by far for an arbitrary triple (N, θ, ξ) there exists an algebra and there may be several such algebras. In this article, we construct algebras whose characteristic varieties are products of identical elliptic curves and varieties close to them.

In Sec. 1 we introduce notation for θ-functions of order n, define Sklyanin algebras, and describe the connection of these algebras with the Belavin R-matrices. In Sec. 2 we give an explicit construction for the algebras $Q_n(\mathscr{E}, \tau) = Q_{n,1}(\mathscr{E}, \tau)$. In Sec. 3 we describe certain properties of the algebras $Q_{n,k}(\mathscr{E}, \tau)$. Let $\mathfrak{g} = \mathfrak{gl}_n \otimes \mathbf{C}\,[t, t^{-1}]$ be a loop algebra and set $\mathfrak{g}^{(k)} = \mathfrak{g}/(t-1)^k\mathfrak{g}$. Then $\dim \mathfrak{g}^{(k)} = kn^2$. In [5, 6] algebras A and $A^{(k)}$ ($k = 1, 2, \ldots$), such that A is an elliptic deformation of the universal enveloping algebras $U(\mathfrak{g})$, $A^{(k)}$ is an elliptic deformation of $U(\mathfrak{g}^{(k)})$ and $A^{(k)}$ is a homomorphic image of A for all k, are constructed. There exists a comultiplication Δ: $A \to A \oplus A$, with respect to which A is a Hopf algebra. Let $J^{(k)}$ be the kernel of the mapping $A \to A^{(k)}$. Then $\Delta(J^{(k_1+k_2)}) \subset J^{(k_1)} \otimes A + A \otimes J^{(k_2)}$. This means that there exists a homomorphism of algebras $A^{(k_1+k_2)} \to A^{(k_1)} \otimes A^{(k_2)}$. Thus, the tensor product of a representation of the algebra $A^{(k_1)}$ with a representation of the algebra $A^{(k_2)}$ is a representation of $A^{(k_1+k_2)}$. In this article we construct a family of coproducts on the algebras $Q_{n,k}(\mathscr{E}, \tau)$ that is a generalization of the indicated family. Let us observe that $A^{(d)} = Q_{n^2d,nd-1}(\mathscr{E}, \tau)$ in our notaton.

A part of results of this article is contained in the preprint [8] of the authors.

SECTION 1

1. Let $\mathscr{E}$ be an elliptic curve, ξ be a positive linear fibering on $\mathscr{E}$ and $n = \dim \times H^0(\xi)$. We know (see [1]) that the translation by a point of order n transforms ξ into an isomorphic fibering. In addition, an irreducible projective representation of the group $\mathscr{E}_n$ of points of n-th order is realized on the space $H^0(\xi)$. This means that the Heisenberg group Γ_n (the central extension of $\mathscr{E}_n$) acts on $H^0(\xi)$ in an irreducible manner. Let us identify $\mathscr{E} \cong C/\Gamma$, where Γ is the lattice generated by 1 and η, such that $\operatorname{Im}(\eta) > 0$. The space $H^0(\xi)$ is identified with the space $\Theta_n(\Gamma)$ of the functions that are holomorphic on the whole complex plane and satisfy the relations $f(z + 1) = f(z)$, $f(z + \eta) = -e^{-2\pi i n z} \times f(z)$. Let us identify the group $\mathscr{E}_n$ with $(1/n)\Gamma/\Gamma$. The images of $1/n$ and η/n form a basis in $\mathscr{E}_n$. Fixing a basis in $\mathscr{E}_n$, we can choose a basis in $\Theta_n(\Gamma)$. Let us define a collection of functions $\Theta_\alpha(z)$, $\alpha \in \mathbf{Z}_n$, by means of the functional equation $\theta_\alpha(z + 1/n) = \exp(2\pi i\alpha/n) \times \theta_\alpha(z)$, $n = 2P(2\ell + 1)$, $\theta_\alpha(z + \eta/n) = \exp(-2\pi iz - \pi i/2P + ((n - 1)\pi i\eta)/n)\theta_{\alpha+1}(z)$. The functions $\theta_\alpha(z) \in \Theta_n(\Gamma)$ are called the θ-functions of n-th order. The zero set of the function $\theta_\alpha(z)$ is $\{-(\alpha/n)\eta + (m/n) + \Gamma | m \in \mathbf{Z}\}$. It is easily verified that $(*)\theta_\alpha(-z) = -e^{-2\pi i n z}\theta_{-\alpha}(z)$, if n is odd, and $\theta_\alpha(-z) = -\exp[-2\pi inz + (\alpha\pi i)/(2P^{-1})]\theta_{-\alpha}(z)$, if n is even.

2. We fix a natural number $n \geq 3$ and let $k \in \mathbf{Z}_n$ be a residue modulo n, invertible in the ring $\mathbf{Z}_n$ [i.e., H.C.F. (n, k) = 1]. The algebra $Q_{n,k}(\mathscr{E}, \tau)$, where $\tau \in \mathscr{E}$, is defined as the algebra with the n generators x_i, $i \in \mathbf{Z}_n$ and the relations

$$\sum_{r \in \mathbf{Z}_n} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{j-i-r}(-\tau)\,\theta_{rk}(\tau)} x_{k(j-r)} x_{k(i+r)} = 0.$$

Here $i, j \in \mathbf{Z}_n$. These relations can be written in a somewhat simpler form by using $(*)$, but the form of expression would be different for odd and even n.

Remarks. 1. It follows from the transformation formulas for θ-functions that for $\tau \in \mathscr{E}_n$ the algebra $Q_{n,k}(\mathscr{E}, \tau)$ is isomorphic to an algebra of skew polynomials. In particular, $Q_{n,k}(\mathscr{E}, 0)$ is simply a ring of polynomials in n variables. The algebra $Q_{n,n-1}(\mathscr{E}, \tau)$ is an algebra of polynomials for each τ.

2. Setting $\deg x_i = i$, we get a $\mathbf{Z}_n$-graduation on the algebra $Q_{n,k}(\mathscr{E}, \tau)$. Moreover, the algebra $Q_{n,k}(\mathscr{E}, \tau)$ is invariant with respect to the automorphism $x_i \to x_{i+1}$. Therefore, the group Γ_n acts on the algebra $Q_{n,k}(\mathscr{E}, \tau)$ by automorphisms (its generators act by the formula $x_i \to \varepsilon^i x_i$, $x_i \to x_{i+1}$, where ε is a primitive root of degree n of 1). Indeed, the space of relations is invariant with respect to the action of Γ_n.

3. It is easily verified that $Q_{n,k}(\mathscr{E}, \tau) \cong Q_{n,k'}(\mathscr{E}, \tau)$, where $kk' = 1$ in the ring $\mathbf{Z}_n$. If $\tau \notin \mathscr{E}_n$, then there are no other isomorphisms between the algebras $Q_{n,k}(\mathscr{E}, \tau)$.

3. The algebras $Q_{n,k}(\mathscr{E}, \tau)$ are closely connected with the Belavin elliptic R-matrices (see [6]). Again let (n, k) be a pair of relatively prime numbers such that $k < n$. We will identify k with the corresponding residue modulo n. The Zamolodchikov algebra $Z_{n,k}(\mathscr{E}, \tau)$ is defined as the algebra with an infinite number of generators $x_i(u)$, where $i \in \mathbf{Z}_n$, $u \in \mathbf{C}$, and the relations

$$K x_{ki}(u)\, x_{kj}(v) = \sum_{r \in \mathbf{Z}_n} \frac{\theta_{j-i+r(k-1)}(v-u+\tau)}{\theta_{rk}(\tau)\,\theta_{j-i-r}(v-u)} x_{k(j-r)}(v)\, x_{k(j+r)}(u).$$

Here

$$K = \frac{\theta_1(0)\ldots\theta_{n-1}(0)\,\theta_0(v-u+\tau)\ldots\theta_{n-1}(v-u+\tau)}{\theta_0(\tau)\ldots\theta_{n-1}(\tau)\,\theta_0(v-u)\ldots\theta_{n-1}(v-u)}.$$

Let $V(u)$ denote the space with a basis $\{x_i(u),\ i \in \mathbf{Z}_n\}$. We can consider the above relations as the matrix of the operator

$$R^{\tau}_{n,k}(u, v):\ V(u) \otimes V(v) \to V(v) \otimes V(u).$$

The group of translations $\{U_s,\ s \in \mathbf{C}\}$, $U_s(x_i(u)) = x_i(u + s)$ acts on the algebra $Z_{n,k} \times (\mathscr{E}, \tau)$ by automorphisms. Let us identify the space $V(u)$ with $V(0) \cong V$ by means of the mapping U_{-u}. The matrix $R_{n,k}{}^{\tau}(u, v)$: $V \otimes V \to V \otimes V$ satisfies the triangle equality (see [4]) and is called a Belavin R-matrix. The kernel of the operator $R_{n,k}{}^{\tau}(u, u + \tau)$ is a space of relations in the algebra $Q_{n,k}(\mathscr{E}, \tau)$.

Let there be given an R-matrix $R(u, v)$: $V \otimes V \to V \otimes V$. It is clear that the matrices $R^*(u, v)$: $V^* \otimes V^* \to V^* \otimes V^*$ and $R \otimes R^*(u, v)$: $(V \otimes V^*) \otimes (V \otimes V^*) \to (V \otimes V^*) \otimes (V \otimes V^*)$ also satisfy the triangle equality. The Zamolodchikov algebra, corresponding to the matrix $R \otimes R^*(u, v)$, is called the algebra of L-operators. Let us identify $V(u) \otimes V(u)^* \cong \mathrm{Hom} \times (V(u), V(u))$. Let $L_{ij}(u) \in \mathrm{Hom}(V(u), V(u))$ be a matrix element. By [4] the algebra of L-operators is a Hopf algebra with the comultiplication $\Delta L_{ij}(u) = \sum_l L_{il}(u) \otimes L_{lj}(u)$.

Thus, we can multiply representations of the algebra of L-operators tensorially. There exists a natural representation in the space $V(u)$. The operator $R(u, v)$: $V(u) \otimes V(v) \to V(v) \otimes V(u)$ commutes with the action of the algebra of L-operators. If $\det R(u, v) \neq 0$, then the representations $V(u) \otimes V(v)$ and $V(v) \otimes V(u)$ are isomorphic,

$$\det R^{\tau}_{n,k}(u, v) = \left(\frac{\theta_0(v-u-\tau)\ldots\theta_{n-1}(v-u-\tau)}{\theta_0(v-u+\tau)\ldots\theta_{n-1}(v-u+\tau)}\right)^{\frac{n(n-1)}{2}},$$

i.e., the determinant is equal to zero, if $v - u = \tau + x$, and has a pole if $v - u = -\tau + x$, where $x \in \mathscr{E}_n$. Thus, there exists a nontrivial subrepresentation of $\Lambda^2(u)$ in the space $V(u) \otimes V(u + \tau)$: the kernel of the operator $R_{n,k}{}^{\tau}(u, u + \tau)$. The range of this operator is the representation $S^2(u) \subset V(u + \tau) \otimes V(u)$. In addition, $V(u + \tau) \otimes V(u)/S^2(u) \cong \Lambda^2(u)$, $V(u) \otimes V(u + \tau)/\Lambda^2(u) \cong S^2(u)$, $\dim \Lambda^2(u) = n(n - 1)/2$, $\dim S^2(u) = n(n- 1)/2$.

By [6] there exists a symmetrization operator

$$\textstyle\sum_s (u):\ V(u) \otimes V(u+\tau) \otimes \ldots \otimes V[u + (s-1)\tau] \to V[u + (s-1)\tau] \otimes \ldots \otimes V(u).$$

Let $S^s(u) = \mathrm{Im} \sum_s (u)$; then $\dim S^s(u) = n(n - 1)\ldots(n + s - 1)/s!$ and $S^s(u)$ is a representation of the algebra of L-operators. Let $S^s = S^s(0) \cong U_{-u}S^s(u)$.

Let us set $A_s = V((s - 1)\tau) \otimes\ldots\otimes V(\tau) \otimes V(0)$. Let us define the structure of a graded coalgebra on the space $A = \bigoplus_s A_s$ by the formula

$$\Delta(r_1 \otimes \ldots \otimes r_s) = \sum_p U_{(p-s)\tau}(r_1 \otimes \ldots \otimes r_p) \otimes (r_{p+1} \otimes \ldots \otimes r_s).$$

If is clear that the coalgebra A is dual to the free algebra with n generators. The subspace $\overline{S} = \bigoplus_s S^s \subset A$ is a subcoalgebra. Let $Q_{n,k}(\mathscr{E}, \tau) = \overline{S}^*$ be the dual algebra. The algebra $\Lambda_{n,k}(\mathscr{E}, \tau) = \mathrm{Ext}_{Q_{n,k}(\mathscr{E},\tau)}(\mathbf{C}, \mathbf{C})$ is dual to $Q_{n,k}(\mathscr{E}, \tau)$. The algebra $\Lambda_{n,k}(\mathscr{E}, \tau)$ can be obtained analogously if in place of the matrix $R_{n,k}{}^{\tau}(u, v)$ we use the matrix $R_{n,k}{}^{\tau}(u, v)$.

The range of the operator $R_{n,k}{}^{\tau}(u, u + \tau)^*$: $V^*(u + \tau) \otimes V^*(u) \to V^*(u) \otimes V^*(u + \tau)$ is a representation of the algebra of L-operators of dimension $n(n - 1)/2$. This is the analog of the second outer power. In the module $B_s(u) = V^*(u) \otimes V^*(u + \tau) \otimes\ldots\otimes V^*(u + (s - 1)\tau)$ we can find a submodule $\Lambda^s(u)$ of dimension $n(n - 1)\ldots(n - s + 1)/s!$; $\Lambda^s(u)$ is the generalized s-th outer power. The coalgebra $B = \bigoplus_s b_s$ has a subalgebra $\overline{\Lambda} = \bigoplus_s \Lambda^s$. The dual algebra of $\overline{\Lambda}$ is $\Lambda_{n,k}(\mathscr{E}, \tau)$.

SECTION 2

Let $\overset{s}{\boxtimes}\xi$ be the s-th tensor power of the fibering ξ, i.e., the fibering on $\mathscr{E}^s$, $H^0 \times (\overset{s}{\boxtimes}\xi) \cong \otimes^s V$ and $\dim V = n$. Let D_s denote the space of sections of the fibering $\overset{s}{\boxtimes}\xi$ that have the following properties:

a) A section $f \in D_s$ vanishes on $(x_1, \ldots, x_s)$ if $x_\beta - x_\alpha = (n - 2(\beta - \alpha))\tau$ for certain $\beta > \alpha$. In other words, the divisor of the section f is represented in the form $N^\tau + L^\tau$, where $N^\tau = \sum N_{\alpha,\beta}{}^\tau$, $N_{\alpha,\beta}{}^\tau = \{(x_1, \ldots, x_s) | x_\beta - x_\alpha = (n - 2(\beta - \alpha))\tau\}$.

b) The divisor L^τ is symmetric with respect to the twisted action of the permutation group S_s on $\mathscr{E}^s$. Under this action the transposition $(\alpha, \alpha + 1)$ transforms the collection $(x_1, \ldots, x_\alpha, x_{\alpha+1}, \ldots, x_s)$ into $(x_1, \ldots, x_{\alpha+1} + 2\tau, x_\alpha - 2\tau, \ldots, x_s)$. Let $M_{\alpha,\beta}$ be the set of fixed points of the transposition (α, β). Then $L^\tau = \sum_{\alpha<\beta} pM_{\alpha,\beta} + Q$, where p is an even number and $M_{\alpha,\beta}$ does not occur in Q.

Proposition 1. The dimension of the space D_s is equal to the dimension of the space of the skew-symmetric sections of the fibering $\overset{s}{\boxtimes}\xi$.

Indeed, the skew-symmetric sections are defined so that their divisor has the form $D = D' + qM$, where q is odd, D' is symmetric, $M = N^0$, and D' and M do not have any common component. We write D in the form $D = (D' + (q - 1)M) + M$. Let us observe that N^τ is the translation of the divisor of M by the element $(0, (n - 2)\tau, \ldots, (s - 1)(n - 2)\tau)$. Let us set $D^\tau = [D' + (q - 1)M|^\tau + N^\tau$, where $(D' + (q - 1)M)^\tau$ is the translation of $D' + (q - 1)M$ by the element $(0, -2\tau, \ldots, -2(s - 1)\tau)$. It is easily verified that D and D^τ are equivalent. Thus, D^τ is the divisor of a certain section from the space D_s. We can construct a skew-symmetric section with respect to an element from D_s analogously.

Using Proposition 1, we can construct a basis in D_s.

Let $0 \le i_1 < i_2 < \ldots < i_s < n$. Let us set

$$\theta_{i_1,\ldots i_s}(z_1, \ldots, z_s) = \frac{\mathrm{Alt}\,(\theta_{i_1}(z_1)\,\theta_{i_2}(z_2 - 2\tau)\ldots\theta_{i_s}(z - 2(s-1)\tau))}{\prod_{\alpha<\beta} \theta(z_\beta - z_\alpha - z(\beta - \alpha)\tau)} \prod_{\alpha<\beta} \theta(z_\beta - z_\alpha - (n - 2(\beta - \alpha))\tau).$$

The alternation is carried out with respect to the indices $i_1, \ldots, i_s$; $z_1, \ldots, z_s$ are the coordinates on the universal covering of the variety $\mathscr{E}^s$ and $\theta(z)$ is a θ-function of first order such that $\theta(0) = 0$.

Let $V = H^0(\xi)$, $\overset{s}{\otimes} V \cong H^0(\overset{s}{\boxtimes}\ \xi)$ and $T(V) \cong \underset{s}{\oplus}\ H^0(\overset{s}{\boxtimes}\ \xi)$ be the tensor algebra generated by V. On T(V) we introduce the structure of a coalgebra with the comultiplication $H^0(\overset{i}{\boxtimes}\xi) \to \oplus\ H^0(\overset{p}{\boxtimes}\ \xi) \oplus H^0(\overset{i-p}{\boxtimes}\xi)$.

Proposition 2. The subspace $D = \oplus D_s \subset T(V)$ is a subalgebra in T(V).

Let $f \in D_s$. Let us set $\bar{f}(x_1, \ldots, x_{\bar{s}}) = f(x_1, \ldots, x_{\bar{s}}, a_{\bar{s}+1}, \ldots, a_s)$.

Here $a_{\bar{s}+1}, \ldots, a_s$ is a collection of points of $\mathscr{E}$ and $\bar{f}$ is a section of the fibering $\overset{s}{\boxtimes}\xi$. To prove Proposition 2 it is sufficient to show that $\bar{f} \in D_{\bar{s}}$. Indeed, it follows from a) and b) that the divisor of the section $\bar{f}$ is decomposed into two parts N' and L', where L' is determined by condition a) and N' is symmetric with respect to the twisted action of the symmetric group. The last assertion is valid since $N' = N_1 + N_2$, where $N_1 = \{(x_1, \ldots, x_{\bar{s}})\,|\,\exists\alpha, \beta\colon\ x_\alpha = a_\beta - (n - 2(\beta - \alpha))\tau\}$ and $N_2 = (f) \cap (\mathscr{E}^{\bar{s}} \times (a_{\bar{s}+1}, \ldots, a_s))$. Both divisors have the desired symmetry.

On the other hand, we will consider $D_2 \subset V \oplus V$ as the space of quadratic relations in the algebra T(V) and let (D_2) be the ideal generated by D_2.

Proposition 3. $Q_n(\mathscr{E}, \tau) \overset{\mathrm{def}}{=} Q_{n,1}(\mathscr{E}, \tau) = T(V)/(D_2)$. The dual algebra of the coalgebra D is $\Lambda_n(\mathscr{E}, \tau) \underset{\mathrm{def}}{=} \Lambda_{n,1}(\mathscr{E}, \tau)$.

To prove Proposition 3 it is necessary to verify that the formulas from Sec. 1 (relations in $Q_n(\mathscr{E}, \tau)$) and $\{\theta_{i,j}(z_1, z_2)\}$ give the same subspace in the space of θ-functions of two variables. It is easily seen that this is true if $\tau \in \mathscr{E}_n$. The desired identity for gneral τ is a relation between θ-functions of order at most 2n. It is identically fulfilled since it is valid at n^2 points.

We have shown, in particular, that the relations from Sec. 1 really define a Sklyanin algebra. We describe the symplectic leaves of the algebra $Q_n(\mathscr{E}, \tau)$. Let $\mathscr{E} \subset CP^{n-1}$ be the embedding defined by the linear system ξ. The join of all j-dimensional planes, passing through j + 1 points of $\mathscr{E}$, in CP^{n-1} is called the variety of j-chords. Let K_j be the cone over the variety of j-chords (K_0 is the cone over $\mathscr{E}$). Then $\dim K_j = 2(j + 1)$, $K_0 \subset K_1 \subset K_2 \subset \ldots \subset K_j \subset \ldots$, and K_j is the variety of singularities of K_{j+1}, if $\dim D_{j+1} < n$.

The homogeneous symplectic leaves of $Q_n(\mathscr{E}, \tau)$ are K_j, where $0 \le j < (n/2) - 1$. A maximal homogeneous leaf has dimension n − 1 if n is odd and n − 2 if n is even.

SECTION 3

We describe some properties of the algebra $Q_{n,k}(\mathscr{E}, \tau)$ for general τ.

1. $Q_{n,k}(\mathscr{E}, \tau) \cong Q_{n,k'}(\mathscr{E}, \tau)$, where $kk' \equiv 1 \bmod n$. There is no other isomorphism between the algebras of this family.

2. Let $c = \mathrm{HCF}(n, k+1)$. Then the center of the algebra $Q_{n,k}(\mathscr{E}, \tau)$ is a ring of polynomials in c elements, the degree of each of which is n/c. The action of $\mathbf{Z}_n \times \mathbf{Z}_n$ on the algebra $Q_{n,k}(\mathscr{E}, \tau)$ reduces to the irreducible projective action of $\mathbf{Z}_c \times \mathbf{Z}_c$ on the space of generators of the center. In particular, $Q_{n,n-1}(\mathscr{E}, \tau)$ is commutative.

3. We describe the characteristic variety of the algebra $Q_{n,k}(\mathscr{E}, \tau)$. Let $n/k = n_1 - (1/(n_2 - \ldots - (1/n_p))$, where $n_i \geq 2$ for $1 \leq i \leq p$. It is clear that such an expansion in a continued fraction exists and is unique. It is easily verified that $n/k' = n_p - (1/n_{p-1} \ldots - (1/n_1))$, where $kk' \equiv 1 \bmod n$, $k' < n$. It is also clear that

$$n = \det\begin{pmatrix} n_1 & -1 & 0 & 0 \\ -1 & n_2 & -1 & 0 \\ 0 & -1 & n_3 & 0 \\ 0 & 0 & 0 & n_p \end{pmatrix}, \quad k = \det\begin{pmatrix} n_2 & -1 & 0 \\ -1 & n_3 & 0 \\ 0 & 0 & n_p \end{pmatrix}, \quad k' = \det\begin{pmatrix} n_1 & -1 & 0 \\ -1 & n_2 & 0 \\ 0 & 0 & n_{p-1} \end{pmatrix}.$$

In the sequel we will use the notation

$$d(m_1, \ldots, m_q) = \det\begin{pmatrix} m_1 & -1 & 0 & 0 \\ -1 & m_2 & -1 & 0 \\ 0 & -1 & m_3 & 0 \\ 0 & 0 & 0 & m_q \end{pmatrix}$$

$$(d(m) = m,\ d(m_1, m_2) = m_1 m_2 - 1 \quad \text{etc.}).$$

Let $\mathscr{E}_1 = \mathscr{E}_2 = \ldots = \mathscr{E}_p = \mathscr{E}$ be p copies of the curve $\mathscr{E}$. Let us define a fibering ξ_i $(1 \leq i \leq p)$ on the curve $\mathscr{E}_i$ such that ξ_1 has $n_1 + 1$ sections, ξ_p has $n_p + 1$ sections, and ξ_i, $1 < i < p$, has $n_i + 2$ sections. The fibering $\mathscr{E}^p = \mathscr{E}_1 \times \ldots \times \mathscr{E}_p$ (the product of fiberings) is defined on $\bar{\xi} = \xi_1 \boxtimes \cdots \boxtimes \xi_p$. Let $\Delta_{i,i+1}$ $(1 \leq i < p)$ be the divisor on $\mathscr{E}^p$ that consists of the points $(z_1, z_2, \ldots, z_p)$, such that $z_i + z_{i+1} = 0$.

Let ξ be the fibering on $\mathscr{E}^p$ obtained from $\bar{\xi}$ by the subtraction of the divisor $\Delta_{1,2} + \Delta_{2,3} + \ldots + \Delta_{p-1,p}$. Then ξ has n sections and the restriction of ξ to $\mathscr{E}_i$ has n_i sections. In general, the restriction of ξ to $\mathscr{E}_i \times \mathscr{E}_{i+1} \times \ldots \times \mathscr{E}_j$ has $d(n_i, n_{i+1}, \ldots, n_j)$ sections.

The fibering ξ defines a mapping of $\mathscr{E}^p$ into $\mathbf{C}P^{n-1}$, and the range is invariant with respect to a certain irreducible projective action of $\mathbf{Z}_n \times \mathbf{Z}_n$, since the group of translations on $\mathscr{E}^p$, preserving ξ, is isomorphic to $\mathbf{Z}_n \times \mathbf{Z}_n$. This range is the characteristic variety for the algebra $Q_{n,k}(\mathscr{E}, \tau)$. The dimension of the characteristic variety is equal to p. If $n_i > 2$ for $1 \leq i \leq p$, then the described mapping is an embedding and the characteristic variety is isomorphic to $\mathscr{E}^p$. If some of n_i are equal to 2, then the characteristic variety is covered in unramified manner by a product of the curves $\mathscr{E}$ and projective spaces, where the dimension of each space is equal to $n_i = 2$, running in succession. For example, if $p = 4$ and $(n_1, n_2, n_3, n_4) = (3, 2, 2, 3)$, then the characteristic variety is equal to $\mathscr{E} \times \mathbf{C}P^2 \times \mathscr{E}/\mathbf{Z}_3^2$. The translation on the characteristic variety is equal to $(\tau_1, \ldots, \tau_p)$, where $\tau_1 = (n - k - 1)\tau$, $\tau_p = (n - k' - 1)\tau$; for $1 < i < p$, $\tau_i = (n - d(n_1, \ldots, n_{i-1}) - d(n_{i+1}, \ldots, n_p))\tau$.

4. The algebras $Q_{n,k}(\mathscr{E}, \tau)$ are connected with each other for various n and k. We need some notation to describe this connection. Let ξ be a fibering on $\mathscr{E}$. Let $L(\xi)$ denote the space of sections of ξ.

Let $\dim L(\xi) = m$ and set $L_m(\mathscr{E}, \tau) = \mathbf{C} \oplus L(\xi) \oplus L(\xi^{\tau} \oplus \xi) \oplus L(\xi^{2\tau} \oplus \xi^{\tau} \oplus \xi) \oplus \ldots$, where $\xi^{i\tau}$ is the fibering obtained from ξ by translation by $i\tau$. The natural bilinear mappings $L(\xi^{i\tau} \oplus \ldots \oplus \xi) \oplus L(\xi^{j\tau} \oplus \ldots \oplus \xi) \to L(\xi^{(i+j)\tau} \oplus \ldots \oplus \xi)$ define on $L_m(\mathscr{E}, \tau)$ the structure of a graded associative algebra with unity, and $\dim L(\xi^{i\tau} \oplus \ldots \oplus \xi) = i + 1)m$. The irreducible projective action of $\mathbf{Z}_m \times \mathbf{Z}_m$ on $L(\xi)$ can be extended to automorphisms of the algebra $L_m(\mathscr{E}, \tau)$.

Let $A^{(i)} = \mathbf{C} \oplus A_1^{(i)} \oplus A_2^{(i)} \oplus \ldots$ be graded associative algebras with unity whose automorphism groups contain $\mathbf{Z}_m \times \mathbf{Z}_m$, acting projectively on $A_1^{(i)}$. Then $(A^{(1)} \oplus \ldots \oplus A^{(t)})^{Z_m^2}$ is also a graded associative algebra with unity. It consists of the elements $A^{(1)} \oplus \ldots \oplus A^{(t)}$, invariant with respect to $\mathbf{Z}_m^2$.

There exist the following graded homomorphisms of algebras:

a) $Q_{n,k}(\mathscr{E}, \tau) \to (L_{nk}(\mathscr{E}, ((n-k-1)/k)\tau) \oplus Q_{k,k_3}(\mathscr{E}, ((n/k)\tau)))^{Z_k^2}$, where $k_3 = d(n_3, n_4, \ldots, n_p)$;

b) $Q_{n,k}(\mathscr{E}, \tau) \to (L_{nk'}(\mathscr{E}, ((n-k'-1)/k')\tau) \oplus Q_{k',k_{p-2}'}(\mathscr{E}, ((n/k')\tau)))^{Z_{k'}^2}$, where $k_{p-2}' = d(n_1, n_2, \ldots, n_{p-2})$;

c) $Q_{n,k}(\mathscr{E}, \tau) \to \left(Q_{a,\alpha}\left(\mathscr{E}, \frac{n}{a}\tau\right) \otimes L_{nab}\left(\mathscr{E}, \frac{n-a-b}{ab}\tau\right) \otimes Q_{b,\beta}\left(\mathscr{E}, \frac{n}{b}\tau\right)\right)^{Z_{ab}^2}$, where $a = d(n_1, \ldots, n_{i-1})$, $\alpha = d(n_1, \ldots, n_{i-2})$, $b = d(n_{i-1}, \ldots, n_p)$, $\beta = d(n_{i+2}, \ldots, n_p)$ $(1 < i < p, d(\emptyset) \overset{\text{def}}{=} 1)$.

5. There exists a method to obtain other formulas, analogous to the ones described in Part 4c), for $Q_{n,k}(\mathscr{E}, \tau)$. To show this, let us observe that

$$n = d(n_1, \ldots, n_p) = d(n_1, \ldots, n_{i-1}, n_i + 1, 1, n_{i+1} + 1, n_{i+2}, \ldots, n_p).$$

Let $(N_1, \ldots, N_q)$ $(q > p)$ be the sequence obtained from $(n_1, \ldots, n_p)$ by applying the above-described procedure a certain number of times. Then $n = d(N_1, \ldots, N_q)$ and the formula from Part 4c) remains valid after the replacement of $(n_1, \ldots, n_p)$ by $(N_1, \ldots, N_q)$.

6. Formula 4c) is of special interest in the case $n = a + b$, since in this case the algebra $L_{nab}\left(\mathscr{E}, \frac{n-a-b}{ab}\tau\right)$ is commutative and we get a comultiplication $Q_{n,k}(\mathscr{E}, \tau) \to Q_{a,\alpha}\left(\mathscr{E}, \frac{n}{a}\tau\right) \otimes Q_{b,\beta}\left(\mathscr{E}, \frac{n}{b}\tau\right)$ that depends on a parameter. This is possible when $(n_1, \ldots, n_p)$ is replaced by $(N_1, \ldots, N_i, 1, N_{i+1}, \ldots, N_q)$ and $n = d(N_1, \ldots, N_i) + d(N_{i+1}, \ldots, N_q)$.

Each such comultiplication has the form

$$Q_{(a+b)n,\,(a+b)p-1} \to Q_{an,\,ap-1} \otimes Q_{bn,\,bq-1},$$

where $q \equiv -(p/(ap-1)) \bmod n$, HCF $(ap-1, n)$ = HCF $((a+b)p-1, n)$ = HCF $(p, n) = 1$. In particular, there exists a comultiplication

$$Q_{(d_1+d_2)n^2,\,(d_1+d_2)np-1} \to Q_{d_1n^2,\,d_1np-1} \otimes Q_{d_2n^2,\,d_2np-1}, \quad \text{HCF } (n, p) = 1.$$

We call $Q_{n,k}(\mathscr{E}, \tau)$ indecomposable if there exists no such comultiplication for it. It is easy to prove that the decomposition into irreducible factors is unique. The algebra $Q_{n^2,np-1}(\mathscr{E}, \tau)$ is decomposed into $\varphi(n)$ irreducible factors, where $\varphi(n)$ is the number of the numbers that are less than n and are relatively prime to it, and the decompositions for different p are obtained from each other by cyclic permutation [HCF $(p, n) = 1$]. For example, $Q_{9,2} \to Q_6 \oplus Q_3$, $Q_{16,3} \to Q_8 \oplus Q_{8,5}$, $Q_{25,4} \to Q_{10} \oplus Q_{5,3} \oplus Q_5 \oplus Q_{5,2}$.

7. The indicated formulas enable us in principle to find all symplectic leaves of the algebra $Q_{n,k}(\mathscr{E}, \tau)$ since the leaves of the algebras $Q_n(\mathscr{E}, \tau)$ are known. In particular, it follows from formulas 4a) and b) that $Q_{n,k}(\mathscr{E}, \tau)$ has the leaves $K(\mathscr{E} \times \mathbf{C}P^{k-1}/\mathbf{Z}_k^2)$ and $K(\mathscr{E} \times \mathbf{C}P^{k'-1}/\mathbf{Z}_{k'}^2)$ and from 4c) that $K(\mathbf{C}P^{a-1} \times \mathscr{E} \times \mathbf{C}P^{b-1}/\mathbf{Z}_{ab}^2)$ is a leaf.

We describe homogeneous leaves for the case of the algebra $Q_{4d,2d-1}(\mathscr{E}, \tau)$. In this case, for $1 \le j \le d$ the join of the 2j-dimensional leaves consists of j components. Let us denote them by M_{ij}, where $1 \le i \le j$. Here

$$\dim M_{ij} = d + j, \quad \deg M_{ij} = \frac{4d(d+j-1)!}{(2i-1)!\,(d-j)!\,(2j-2i+1)!}.$$

The inclusion relation between leaves is described by the formulas

$$M_{ij} = M_{i,j+1} \cap M_{i+1,j+1}, \quad M_{ij} \cap M_{i+p,j} = M_{ij} \cap M_{i+1,j} \cap \cdots \cap M_{i+p,j} = M_{i,j-p}.$$

In addition

$$M_{ij} \cong M_{j+1-i,j} \cong K(\mathbf{C}P^{2i-2} \times \mathscr{E} \times \mathbf{C}P^{d-j-1} \times \mathscr{E} \times \mathbf{C}P^{2j-2i}/\mathbf{Z}^2_{(2i-1)(d-j)(2j-2i+1)})$$

(if $1 < i < j$, then this is a variety in $\mathbf{C}^{4d}$ with self-intersections and K(N) denotes the cone over $N \subset \mathbf{C}P^{4d-1}$). In particular, the characteristic variety of the algebra $Q_{4d,2d-1} \times (\mathscr{E}, \tau)$ is isomorphic to $\mathscr{E} \times \mathbf{C}P^{d-2} \times \mathscr{E}/\mathbf{Z}_{d-1}^2$. Let us consider, in particular, the case d = 2, i.e., the algebra $Q_{8,3}(\mathscr{E}, \tau)$. The center of this algebra is generated by four elements of second degree. The restriction of the automorphism group of the algebra $Q_{8,3}(\mathscr{E}, \tau)$ to this space gives there an irreducible projective representation of $\mathbf{Z}_4 \times \mathbf{Z}_4$. Let C_i^τ, $i \in \mathbf{Z}_4$ be the central elements and C_i be the corresponding polynomials for $\tau = 0$. The homogeneous leaves are given by the equations $\{C_i = 0 \mid i \in \mathbf{Z}_4\}$. This variety, whose degree is $2^4 = 16$, decomposes into two isomorphic components of degree 8 (M_{12} and M_{22} in our notation). Each of these components is the cone over $\mathscr{E} \times \mathbf{C}P^2/\mathbf{Z}_3^2$ and is a homogeneous four-dimensional leaf. In addition, $M_{11} = M_{12} \cap M_{22}$ is the cone over $\mathscr{E} \times \mathscr{E}$ and is a join of homogeneous two-dimensional leaves. The algebra $Q_{8,3}(\mathscr{E}, \tau)$ has no other homogeneous leaves. Now we describe the nonhomogeneous leaves. Let us consider the algebra $Q_4(\mathscr{E}, 2\tau) \oplus Q_4(\mathscr{E}, 2\tau)$. It is generated by the generators $\{p_i, q_i | i \in \mathbf{Z}_4\}$, where $p_i q_j = q_j p_i$ and $\{p_i\}$, $\{q_i\}$ separately generate the algebra $Q_4(\mathscr{E}, 2\tau)$. There exists a family of homomorphisms (see 4c) $Q_{8,3}(\mathscr{E}, \tau) \to Q_4(\mathscr{E}, 2\tau) \otimes Q_4(\mathscr{E}, 2\tau)$ parametrized by the element $z \in \mathscr{E}$, or a homomorphism $Q_{8,3}(\mathscr{E}, \tau) \to L_8(\mathscr{E}) \otimes Q_4(\mathscr{E}, 2\tau) \otimes Q_4(\mathscr{E}, 2\tau)$, where $L_8(\mathscr{E})$ is the commutative algebra generated by θ-functions of order 8. This homomorphism is an embedding. In other words, the elements $x_i = \theta_i(z)p_0q_{-2i} + \theta_{i+2}(z)p_1q_{-2i+1} + \theta_{i+4}(z)p_2q_{-2i+2} + \theta_{i+6}(z)p_3q_{-2i+3}$, where $i \in \mathbf{Z}_8$, $2i \in \mathbf{Z}_4$ is the corresponding element under the homomorphism $\mathbf{Z}_8 \to \mathbf{Z}_4$, define a subalgebra of $L_8(\mathscr{E}) \otimes Q_4(\mathscr{E}, 2\tau) \otimes Q_4(\mathscr{E}, 2\tau)$ that is isomorphic to $Q_{8,3}(\mathscr{E}, \tau)$. In addition, $C_{2i}^{\tau} = \theta_i(z)\theta_{i+4}(z)\theta_{i+4}(z)C_1'^{\tau}C_{4i+1}''^{\tau} + \theta_{i+2}(z)\theta_{i+6}(z)C_0'^{\tau}C_{4i}''^{\tau}$, where $C_j'^{\tau}$ and $C_j''^{\tau}$ are central elements of the algebras $Q_4(\mathscr{E}, 2\tau)$, $j \in \mathbf{Z}_2$. Therefore, the general nonhomogeneous leaf of the algebra $Q_{8,3}(\mathscr{E}, \tau)$, defined by the equations $\{C_i = \alpha_i \mid i \in \mathbf{Z}_4\}$, where α_i are such that this variety is nonsingular, is the direct product of two nonsingular nonhomogeneous leaves of the algebra $Q_4(\mathscr{E}, 2\tau)$. If α_i are such that the variety $\{C_i = \alpha_i \mid i \in \mathbf{Z}_4\}$ is singular, then its variety of singularities is also a leaf (or a join of leaves). Let us consider the join of all leaves, obtained in this manner. This variety consists of the following components:

1) $x_1 = x_3 = x_5 = x_7 = 0$, $x_0 = x_4$, $x_2 = x_6$ and three more components, obtained from this by the action of $\mathbf{Z}_2^2 = \mathbf{Z}_3^2/\mathbf{Z}_4^2$. Each point of these components is an 0-dimensional leaf.

2) $x_i = \theta_i(z)q_{-2i} + \theta_{i+4}(z)q_{-2i+2}$, $i \in \mathbf{Z}_8$. This five-dimensional component is the join of the spaces $\mathbf{C}^4$ (with the coordinates q_i, $i \in \mathbf{Z}_4$), parametrized by the point $z \in \mathscr{E}$. Each of these $\mathbf{C}^4$ is a join of two-dimensional leaves. These are defined by the equations $\{C_0'' = \beta_0, C_1'' = \beta_1\}$. The leaves that belong to the component being described are precisely the leaves of the algebra $Q_4(\mathscr{E}, 2\tau)$, numbered by the point $z \in \mathscr{E}$.

The algebra $Q_{8,3}(\mathscr{E}, \tau)$ has no other leaves. The component 2 contains two-dimensional homogeneous leaves. We have $\beta_0 = \beta_1 = 0$ on them. If $\beta_0 \neq 0$ or $\beta_1 \neq 0$, then the corresponding leaves are nonhomogeneous. If β_0 and β_1 are such that the corresponding leaf is singular, then the singularities also belong to the component 1. This is precisely the whole of the component 1.

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