Functional Models for Representations of Current Algebras and Semi-Infinite Schubert Cells

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UDC 512.81

§1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional simply laced simple Lie algebra over $\mathbb{C}$, let $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra, i.e., the one-dimensional central extension of the current algebra $\mathfrak{g}^{S^1} = \dots + \mathfrak{g}t^{-2} + \mathfrak{g}t^{-1} + \mathfrak{g} + \mathfrak{g}t + \mathfrak{g}t^2 + \dots$, and let K be a central element of $\hat{\mathfrak{g}}$. We will deal with integrable representations of the Lie algebra $\hat{\mathfrak{g}}$ from the category $\mathcal{O}$ of representations with highest weight, where K acts by a scalar k (the number k is called the central charge or the level of a representation). An integrability criterion can be stated as follows. Let us fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Let α be an arbitrary root of the Lie algebra $\mathfrak{g}$, let $e(\alpha)$ be a nonzero element of $\mathfrak{g}$ from the root space attached to α, and let us put $e_i(\alpha) = e(\alpha) \cdot t^i$ and $S_i^{(k+1)}(\alpha) = \sum_{i_1+\dots+i_{k+1}=i} e_{i_1}(\alpha) \dots e_{i_{k+1}}(\alpha)$. The infinite expressions $S_i^{(k+1)}(\alpha)$, $i \in \mathbb{Z}$, act on representations of $\hat{\mathfrak{g}}$ from the category $\mathcal{O}$, and a representation V of level k is integrable if and only if $k \in \mathbb{Z}$, $k \geq 0$, and all expressions $S_i^{(k+1)}(\alpha)$ have the zero action on V. In other words, the elements $S_i^{(k+1)}(\alpha)$ generate a two-sided ideal in the completed universal enveloping algebra $\widetilde{U}(\hat{\mathfrak{g}})$, and this ideal annihilates the integrable representations of level k.

Now let us restrict ourselves to the vacuum irreducible representation V_k of level k. Let v be the vacuum vector of V_k. Then $\hat{\mathfrak{g}}^{\text{in}} v = 0$, where $\hat{\mathfrak{g}}^{\text{in}} = \mathfrak{g} + \mathfrak{g}t + \mathfrak{g}t^2 + \dots$. Denote by $\hat{\mathfrak{n}}_+ = \dots + \mathfrak{n}_+ t^{-1} + \mathfrak{n}_+ + \mathfrak{n}_+ t + \dots \subset \hat{\mathfrak{g}}$ the Lie algebra of currents with values in the positive nilpotent subalgebra of $\mathfrak{g}$. The main role in our investigation is played by the subspace $W = U(\hat{\mathfrak{n}}_+)v \subset V_k$. The space W can be identified with the quotient space $U(\hat{\mathfrak{n}}_+)/I_k$, where I_k is a left ideal of $U(\hat{\mathfrak{n}}_+)$. The structure of this ideal is described by the following theorem.

Theorem 1.1.1. *$I_k = U(\hat{\mathfrak{n}}_+)\hat{\mathfrak{n}}_+^{\text{in}} + J_k$, where $\hat{\mathfrak{n}}_+^{\text{in}} = \hat{\mathfrak{n}}_+ \cap \hat{\mathfrak{g}}^{\text{in}}$, and J_k is a two-sided ideal generated by the elements $S_i^{(k+1)}(\alpha_j)$ (the expressions from J_k are finite modulo $\hat{\mathfrak{n}}_+^{\text{in}}$), where α_j are the simple roots of the Lie algebra $\mathfrak{g}$, $j = 1, \dots, l$, and $l = \operatorname{rank} \mathfrak{g}$.*

This theorem provides us with a rather cumbersome construction of the dual space of W (see Construction 1.1.2). First we describe this construction in the simplest case $\mathfrak{g} = \mathfrak{sl}_2$.

Let $\Omega^1\mathbb{C} = \mathbb{C}[x]\,dx$ be the space of polynomial 1-forms on a line. The symmetric power $S^n\Omega^1\mathbb{C}$ of it is realized in the space of expressions $f(x_1, \dots, x_n)\,dx_1 \dots dx_n$, where $f(x_1, \dots, x_n)$ is a symmetric polynomial. Let us define the "restricted symmetric power" of the space $\Omega^1\mathbb{C}$ as the subspace $S^n_{(k+1)}\Omega^1\mathbb{C} \subset S^n\Omega^1\mathbb{C}$ that consists of the expressions $f(x_1, \dots, x_n)\,dx_1 \dots dx_n$ such that the polynomial f vanishes for $x_1 = x_2 = \dots = x_{k+1}$. It is clear that $S^\bullet_{(k+1)}\Omega^1\mathbb{C}$ is a commutative coalgebra. We claim that $S^\bullet_{(k+1)}\Omega^1\mathbb{C} \simeq W^*$.

This result can be used to describe the irreducible representation V_k as a linear space. Recall that, in V_k, there is a family $\{v_n,\ n \in \mathbb{Z}\}$ of so-called extremal vectors. The translation subgroup $\mathbb{Z}$ of the affine Weyl group of $\widehat{\mathfrak{sl}}_2$ acts on V_k, and $\{v_n\}$ is the orbit of the vacuum vector under this action. Consider the family of subspaces $W_n = U(\hat{\mathfrak{n}}_+)v_n$; we have $W_{n_1} \simeq W_{n_2}$, and the isomorphism is given by the action of an element of the affine Weyl group. On the other hand, there is a sequence of embeddings

$$\dots \hookrightarrow W_1 \hookrightarrow W = W_0 \hookrightarrow W_{-1} \hookrightarrow W_{-2} \hookrightarrow \dots,$$

and V_k is the inductive limit of this sequence. Informally, this means that it is possible to define the "semi-infinite restricted symmetric powers" of the space $\Omega^1(S^1)$ of 1-forms on a circle, so that the space $W_{-\infty} = V_k$ is dual to $\bigoplus_{i\in\mathbb{Z}} S^{\frac{\infty}{2}+i}_{(k+1)}(\Omega^1(S^1))$. In some sense, these "semi-infinite restricted symmetric

Moscow Mathematical Institute. Independent Moscow University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 28, No. 1, pp. 68–90, January–March, 1994. Original article submitted July 25, 1993.

0016–2663/94/2801–0055 $12.50

powers" are very close to spaces of semi-infinite exterior forms, but the construction of "symmetric powers" is less transparent. The character formula for the space V_k arising from the "semi-infinite realization" of the representation coincides with the "parafermionic" formula of Lepowsky and Primc [3]. The relationships between the semi-infinite construction and parafermion algebras are still a riddle for us.

Now we pass to the case of an arbitrary Lie algebra $\mathfrak{g}$.

Construction 1.1.2. Consider the $\mathbb{Z}^l$-graded vector space $M = \bigoplus M_{m_1,\dots,m_l}$, $m_i \in \mathbb{Z}$, $m_i \geq 0$,

$$M_{m_1,\dots,m_l} = \Big\{ f(x_1(\alpha_1),\dots,x_{m_1}(\alpha_1);\, x_1(\alpha_2),\dots,x_{m_2}(\alpha_2);\,\dots;\, x_1(\alpha_l),\dots,x_{m_l}(\alpha_l)) \times \prod_{i'<j'} (x_i(\alpha_{i'}) - x_j(\alpha_{j'}))^{-1} \cdot \prod_{i,j} dx_i(\alpha_j) \Big\}.$$

Here f is a polynomial in the variables $x_i(\alpha_j)$ symmetric with respect to each group of variables $\{x_i(\alpha_1)\},\dots,\{x_i(\alpha_l)\}$. The space $M_{m_1,\dots,m_l}$ can be considered as a component of the "extended symmetric power" of the space $F = M_{1,0,\dots,0} \oplus M_{0,1,0,\dots,0} \oplus \cdots \oplus M_{0,0,\dots,0,1}$. We call the space M "extended," because M is larger than the symmetric algebra of F: the expressions from M may have poles of order one on diagonals $x_i(\alpha_{i'}) = x_j(\alpha_{j'})$. Now we add the "Serre relations." Let $A = (A_{ij})$ be the Cartan matrix of $\mathfrak{g}$. Let us introduce the subspace $\bar{M} = \bigoplus \bar{M}_{m_1,\dots,m_l}$ of M, where $\bar{M}_{m_1,\dots,m_l} \subset M_{m_1,\dots,m_l}$ consists of the expressions in which f vanishes provided $x_1(\alpha_i) = x_2(\alpha_i) = \cdots = x_{1-A_{ij}}(\alpha_i) = x_1(\alpha_j)$ for some $1 \leq i, j \leq l$, $i \neq j$. We claim that $\bar{M}$ is naturally isomorphic to the dual space of $U(\hat{\mathfrak{n}}_+)/U(\hat{\mathfrak{n}}_+)\hat{\mathfrak{n}}_+^{\mathrm{in}}$. Finally, let us describe $W^* = (U(\hat{\mathfrak{n}}_+)/I_k)^*$. It is a graded subspace of $\bar{M}$, $W^*_{m_1,\dots,m_l} \subset \bar{M}_{m_1,\dots,m_l}$, and an element of $\bar{M}$ belongs to W^* if f satisfies the following additional condition: for each $1 \leq i \leq l$ the polynomial f vanishes for $x_1(\alpha_i) = x_2(\alpha_i) = \cdots = x_{k+1}(\alpha_i)$.

The "functional realization" of the space W^* thus obtained leads to a character formula for this space. Let L_0 be the energy operator. First, let us consider the case $k = 1$. Then

$$\mathrm{Tr}(q^{L_0})|_W = \sum_{m_1,\dots,m_l \geq 0} \frac{q^{\frac{1}{2}\sum A_{ij} m_i m_j}}{(q)_{m_1} \cdots (q)_{m_l}}. \tag{1.1.3}$$

Here (A_{ij}) is the Cartan matrix. For a general k the formula has the same form, but l is replaced by $l \cdot k$, and (A_{ij}) by the quadratic form with the matrix $A \otimes \widetilde{B}_k^{-1}$, where A is the Cartan matrix of $\mathfrak{g}$, and $\widetilde{B}_k$ is the symmetrized Cartan matrix B_k (cf. 2.7.3). Note that formulas of this kind appeared in the papers [11–14], where they described the character of the space of quasi-particles in the thermodynamic Bethe Ansatz.

The same scheme as in the case $\mathfrak{g} = \mathfrak{sl}_2$ leads to a description of the space V_k, given the description of W^*, and to the following character formula for V_k:

$$\mathrm{ch}\, V_k = \frac{1}{(q)_\infty^l} \sum_{\substack{N_1^{(i)} \geq \cdots \geq N_k^{(i)} \in \mathbb{Z} \\ i=1,\dots,l}} \frac{q^{\frac{1}{2}\sum_{i,j,p} A_{ij} N_p^{(i)} N_p^{(j)}} z_1^{\sum_p N_p^{(1)}} \cdots z_l^{\sum_p N_p^{(l)}}}{\prod_{i=1}^l \prod_{p=1}^{k-1} (q)_{N_p^{(i)} - N_{p+1}^{(i)}}} \tag{1.1.4}$$

(where the powers of $z_1,\dots,z_l$ correspond to the weights with respect to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and the powers of q are eigenvalues of L_0). This formula describes the decomposition of the space V_k into irreducible representations with respect to the homogeneous Heisenberg subalgebra $\hat{\mathfrak{h}}$.

Finally, note that the above results can be generalized to the case of a non-simply-laced Lie algebra $\mathfrak{g}$. In Theorem 1.1.1, one should replace $S_i^{(k+1)}(\alpha_j)$ by $S_i^{(\varkappa k+1)}(\alpha_j)$ if the root α_j is $\sqrt{\varkappa}$ times shorter than a long root ($\varkappa = 1$, 2, or 3). The same change (the replacement of $k+1$ by $\varkappa k + 1$) must be done in Construction 1.1.2. The character formula for V_k suggests the idea (according to a remark of E. B. Vinberg) that $\mathfrak{g}$ is realized as the fixed-point algebra of a diagram automorphism of a simply laced

Lie algebra $\mathfrak{g}_1$ [4], and the representation V_k of $\hat{\mathfrak{g}}$ is realized as a subspace in a representation of $\hat{\mathfrak{g}}_1$. For example, here is the character formula for $\mathfrak{g}$ of type B_2:

$$\operatorname{ch} V_k = \frac{1}{(q)_\infty^2} \sum_{\substack{N_1^{(1)} \ge \dots \ge N_k^{(1)} \in \mathbb{Z} \\ N_1^{(2)} \ge \dots \ge N_{2k}^{(2)} \in \mathbb{Z}}} \frac{q^{\sum_{p=1}^{k} N_p^{(1)2} - \sum_{p=1}^{k} N_p^{(1)}(N_{2p-1}^{(2)} + N_{2p}^{(2)}) + \sum_{p=1}^{2k} N_p^{(2)2}} z_1^{\sum_{p=1}^{k} N_p^{(1)}} z_2^{\sum_{p=1}^{2k} N_p^{(2)}}}{\prod_{p=1}^{k-1} (q)_{N_p^{(1)} - N_{p+1}^{(1)}} \prod_{p=1}^{2k-1} (q)_{N_p^{(2)} - N_{p+1}^{(2)}}}. \tag{1.1.5}$$

In this paper only the case $\mathfrak{g} = \mathfrak{sl}_2$ is discussed in some detail (see §2). The general case is much more technical, and we hope to tell more about it in our forthcoming paper. Note that the notion of semi-infinite restricted symmetric powers is not fully developed here even for $\mathfrak{g} = \mathfrak{sl}_2$. This notion deserves an individual investigation, and we hope to realize it in the future.

The second subject of the paper is related to the geometry of the flag manifold of the Lie algebra $\hat{\mathfrak{g}}$. Let $F = \widehat{G}/\mathbf{B}_+$ be the flag manifold, $\mathbf{1}$ the unit coset, and M the closure of the orbit $\widehat{N}_+ \cdot \mathbf{1}$, where the subgroup $\widehat{N}_+ \subset \widehat{G}$ consists of the currents taking values in the unipotent subgroup $N_+ \subset G$ with the Lie algebra $\mathfrak{n}_+$. The irreducible integrable representation V_λ with highest weight λ is realized in the dual space of the space of sections of a holomorphic line bundle L_λ on F. The space $W = U(\hat{\mathfrak{n}}_+)v$ (v being the vacuum vector) is dual to the space $H^0(M, L_\lambda)$. Hence, we can use geometric methods when we deal with W. In §3, in the case $\mathfrak{g} = \mathfrak{sl}_2$, we apply the holomorphic Lefschetz fixed point formula to determine the character of $H^0(M, L_\lambda)$. The variety M is nonsingular in the case $\mathfrak{g} = \mathfrak{sl}_2$. (As is well known, applying the same method for the full flag manifold, instead of M, one obtains the Weyl character formula.) We also write down a Demazure type character formula for W and obtain the same result.

Thus, we have two character formulas for W: the first formula is a consequence of the functional realization in the space of symmetric polynomials, and the second one is given by the Lefschetz or Demazure formula. A comparison of these two expressions gives the Rogers–Ramanujan identities (for $k = 1$) and the Gordon identities (for a general k).

In §4 we discuss the case $\mathfrak{g} = \mathfrak{sl}_3$. For $\mathfrak{g} = \mathfrak{sl}_3$ the variety M is singular, so the fixed-point formula is much more complicated. We are unable to write down the whole formula, but we state a conjecture, which implies that the specialization $(\operatorname{Tr} q^{L_0}|_W)$ of the second character formula for W coincides with the Kac formula for the character of the vacuum irreducible representation of the algebra $\widehat{\mathfrak{sl}}_2$ with the same central charge.

An extended version of this text is published as a preprint [10].

We are grateful to V. A. Fateev for helpful discussions with the second author about the relationships of our subject with the theory of S-matrices arising in deformations of conformal field theories.

We dedicate this paper to Izrail Moiseevich Gel′fand on the occasion of his 80th birthday.

§2. The Functional Model: the Case $\mathfrak{g} = \mathfrak{sl}_2$

2.1. Notation. The Lie algebra $\mathfrak{sl}_2$ has the standard basis e, f, h, and the Lie algebra $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[[t, t^{-1}] \oplus \langle K \rangle$ has the basis consisting of $e_i = e \otimes t^i$, $f_i = f \otimes t^i$, $h_i = h \otimes t^i$, $i \in \mathbb{Z}$, and the central element K. In this basis the bracket is given by the formulas

$$[K, e_i] = [K, f_i] = [K, h_i] = [e_i, e_j] = [f_i, f_j] = 0, \quad [h_i, e_j] = 2e_{i+j},$$
$$[h_i, f_j] = -2f_{i+j}, \quad [e_i, f_j] = h_{i+j} + iK\delta_{i,-j}, \quad [h_i, h_j] = 2iK\delta_{i,-j}.$$

In the triangular decomposition $\widehat{\mathfrak{sl}}_2 = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ we have $\mathfrak{h} = \langle h_0, K \rangle$, $\mathfrak{n}_+ = \langle e_i, f_i, h_i : i > 0 \rangle + \langle e_0 \rangle$, and $\mathfrak{n}_- = \langle e_i, f_i, h_i : i < 0 \rangle + \langle f_0 \rangle$. The root vectors corresponding to the simple roots are e_0 and f_1. The algebra $\widehat{\mathfrak{sl}}_2$ is a graded Lie algebra: $\deg e_i = \deg f_i = \deg h_i = i$ and $\deg K = 0$; the degree of an element γ is called the energy of γ. The affine Weyl group $W_{\text{aff}} = \mathbb{Z}_2 \ltimes \mathbb{Z}$ consists of the integral shifts on a real line T_n, $n \in \mathbb{Z}$, and of the reflections S_n, $n \in \mathbb{Z}$, with respect to the points $n/2$. The reflection S_i, $i > 0$, corresponds to the root vector f_i, and the reflection S_{-i}, $i \ge 0$, corresponds to the vector e_i.

The weights $\boldsymbol{\lambda}$ of graded representations of $\widehat{\mathfrak{sl}}_2$ are given by triples of numbers $\boldsymbol{\lambda} = (m, \lambda, k)$, where m is the energy, $\lambda = \boldsymbol{\lambda}(h_0)$, and $k = \boldsymbol{\lambda}(K)$. The action of the Weyl group on the set of weights is given by the formulas

$$T_n \cdot (m, \lambda, k) = (m - \lambda n - kn^2, \lambda + 2kn, k), \qquad S_0 \cdot (m, \lambda, k) = (m, -\lambda, k). \tag{2.1.1}$$

In particular, T_n acts on the root vectors as follows:

$$T_n(e_i)T_{-n} = (e_{i-2n}), \qquad T_n(h_i)T_{-n} = (h_i) \quad (i \neq 0), \qquad T_n(f_i)T_{-n} = (f_{i+2n}). \tag{2.1.2}$$

The Virasoro algebra acts on the algebra $\widehat{\mathfrak{sl}}_2$ and the integrable representations of it belonging to the category $\mathcal{O}$:

$$\begin{gathered} L_i = \frac{1}{2(k+2)} : \sum_{\alpha+\beta=i} (e_\alpha f_\beta + f_\alpha e_\beta + h_\alpha h_\beta/2) :, \\ [L_i, e_j] = je_{i+j}, \quad [L_i, f_j] = jf_{i+j}, \quad [L_i, h_j] = jh_{i+j}. \end{gathered} \tag{2.1.3}$$

Let us also introduce the "half-sum of positive roots of $\widehat{\mathfrak{sl}}_2$" $\boldsymbol{\rho} = (0, 1, 2)$.

2.2. The basic representation of $\widehat{\mathfrak{sl}}_2$ is the irreducible representation V with highest weight $\boldsymbol{\lambda}_0 = (0, 0, 1)$. It is the quotient module of the Verma module $M_{\boldsymbol{\lambda}_0}$ with vacuum vector $\bar{v}$ by the maximal submodule $M_{S_0 * \boldsymbol{\lambda}_0} + M_{S_1 * \boldsymbol{\lambda}_0}$ (here the action of an element $\omega \in W_{\text{aff}}$ on a weight $\boldsymbol{\lambda}$ is defined by the formula $\omega * \boldsymbol{\lambda} = \omega \cdot (\boldsymbol{\lambda} + \boldsymbol{\rho}) - \boldsymbol{\rho}$). The corresponding singular vectors in $M_{\boldsymbol{\lambda}_0}$ are $f_0 \bar{v}$ and $e_{-1}^2 \bar{v}$. Denote the image of $\bar{v}$ under the projection $M_{\boldsymbol{\lambda}_0} \to V$ by v.

Let $\hat{\mathfrak{n}} = \langle e \rangle \otimes \mathbb{C}[[t, t^{-1}]$ be the abelian subalgebra in $\widehat{\mathfrak{sl}}_2$ with the basis e_i, $i \in \mathbb{Z}$. Define the *principal subspace* $W \subset V$ by $W = U(\hat{\mathfrak{n}})v$. In fact, since $e_i v = 0$ for $i \geq 0$, only the algebra $U(\hat{\mathfrak{n}}^{\text{out}}) = \mathbb{C}[e_{-1}, e_{-2}, \dots]$ acts nontrivially on v. Hence, $W = (\mathbb{C}[e_{-1}, e_{-2}, \dots]/I)v$, where I is an ideal in $\mathbb{C}[e_{-1}, e_{-2}, \dots]$. We know that $e_{-1}^2 \in I$.

Theorem 2.2.1. *The ideal I is generated by the polynomials $S_{-k} = \sum e_i e_{-k-i}$, $k \geq 2$.*

This theorem will be proved in §3. Now we explain only the reason for $S_{-k} \in I$. From the explicit formula (2.1.3) it follows that $L_{-1} v = 0$. Hence, $(k-2)! S_{-k} v = \pm[(\operatorname{ad} L_{-1})^{k-2}(e_{-1}^2)]v = 0$.

Remark 2.2.2. In general, the infinite expressions $S_m = \sum_{i+j=m} e_i e_j$, which are coefficients of the formal series $(\sum_{i \in \mathbb{Z}} e_i z^i)^2 = e(z)^2$, act on arbitrary representations of $\widehat{\mathfrak{sl}}_2$ from the category $\mathcal{O}$. It is known that if the central charge is equal to 1, then all S_m act trivially on an integrable representation: $e(z)^2 = 0$ (cf. §2.4).

2.3. The space W is the direct sum of its weight components: $W = \bigoplus W_{(n,\lambda,1)}$. In order to evaluate its formal character $\operatorname{ch}(W)$, which is equal to $\sum q^i z^j \dim W_{(-i,2j,1)}$ by definition, we now introduce a convenient description of the dual space W^*, and this will be called the functional model.

The vector space $\hat{\mathfrak{n}}^{\text{out}} = \langle e_i \rangle_{i<0}$ consists of "singular currents" $\varphi(x) \otimes e$ with values in the subalgebra $\mathfrak{n}_+ \subset \mathfrak{sl}_2$, where $\varphi(x)$ is a polynomial in x^{-1} without a constant term. The dual space $(\hat{\mathfrak{n}}^{\text{out}})^*$ is naturally identified with the space of polynomial 1-forms $\Omega^1\mathbb{C}$ (with gradation $\deg x^n dx = n + 1$). Hence, $U(\hat{\mathfrak{n}}^{\text{out}})^* \simeq \bigoplus_{k \geq 0} (S^k \hat{\mathfrak{n}}^{\text{out}})^* \simeq \bigoplus_{k \geq 0} S^k(\Omega^1\mathbb{C})$, where $S^k(\Omega^1\mathbb{C})$ is the space of the expressions $f(x_1, \dots, x_k)\, dx_1 \dots dx_k$ such that $f(x_1, \dots, x_k)$ is a symmetric polynomial, and different dx_i commute. We will call $S^k\Omega^1\mathbb{C}$ "the space of k particles." The pairing of $S^k\Omega^1\mathbb{C}$ with $S^k\hat{\mathfrak{n}}^{\text{out}}$ is given by the formula

$$\langle f\, dx_1 \dots dx_k, (\varphi_1 \otimes e) \cdot \ldots \cdot (\varphi_k \otimes e) \rangle = \operatorname{Res}_{x_1 = \dots = x_k = 0}(f(x_1, \dots, x_k)\varphi_1(x_1) \dots \varphi_k(x_k)\, dx_1 \dots dx_k). \tag{2.3.1}$$

The space $W^* = \bigoplus_k W^* \cap S^k(\Omega^1\mathbb{C})$ is a subspace of $S^\bullet(\Omega^1\mathbb{C})$, and Theorem 2.2.1 easily implies that

$$W^* \cap S^k(\Omega^1\mathbb{C}) = \{f(x_1, \dots, x_k)\, dx_1 \dots dx_k : f = 0 \text{ if } x_1 = x_2\}.$$

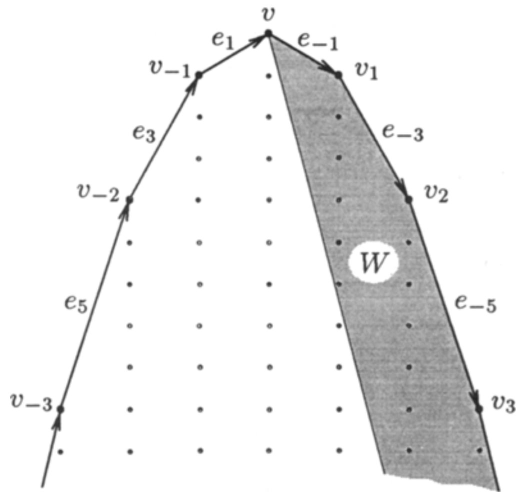

Fig. 1

(In terms of particles: the function must be zero if two particles coincide.) Thus,

$$W^* = \bigoplus_{k=0}^{\infty} W_k^*,$$

where

$$W_k^* = \left\{ g(x_1, \dots, x_k) \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^{k} dx_i,\ g(x_1, \dots, x_k) \text{ is a symmetric polynomial} \right\}; \quad (2.3.2)$$

$$\operatorname{ch} W = \sum_{k=0}^{\infty} \operatorname{ch} W_k^* = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(1-q)(1-q^2)\dots(1-q^k)}. \quad (2.3.3)$$

For $z = 1$ and $z = q$, these are the left-hand sides of the Rogers–Ramanujan identities.

2.4. Now let us apply Theorem 2.2.1 to describe the whole representation space V. Let $n \in \mathbb{Z}$, and let $v_n = T_n v$ be the nth extremal vector of this representation. By (2.1.1), the weight of v_n is equal to $(-n^2, 2n, 1)$. Consider the space $W_n = T_n W = U(\hat{\mathfrak{n}}) v_n$. Obviously, $\dots \subset W_2 \subset W_1 \subset W_0 = W \subset W_{-1} \subset \dots$ and $V = \bigcup W_n = \varinjlim W_{-N}$ (see. Fig. 1). From Theorem 2.2.1 and formula (2.1.2) it follows that $W_n = (\mathbb{C}[e_{-2n-1}, e_{-2n-2}, \dots]/I_n) v_n$, where I_n is the ideal generated by the polynomials S_m, $m \le -4n-2$. The space $W_0 = (\mathbb{C}[e_{-1}, e_{-2}, \dots]/(e_{-1}^2, e_{-1}e_{-2}, 2e_{-1}e_{-3} + e_{-2}^2, \dots))v$ is embedded in $W_{-1} = (\mathbb{C}[e_1, e_0, e_{-1}, \dots]/(e_1^2, e_1 e_0, 2e_1 e_{-1} + e_0^2, \dots))v_{-1}$ via the homomorphism of $\mathbb{C}[e_i]$-modules that maps v to $e_1 v_{-1}$. Furthermore, $e_1 v_{-1} = e_1 e_3 v_{-2} = e_1 e_3 e_5 v_{-3} = \dots$. Any vector of the space V is a finite linear combination of expressions of the form $e_{i_1} e_{i_2} \dots e_{i_k} v_{-N} = e_{i_1} e_{i_2} \dots e_{i_k} e_{2N+1} v_{-N-1} = \dots$ for N sufficiently large. Now let N *tend to infinity*, i.e., let us substitute the expression $e_{2N+1} e_{2N+3} e_{2N+5} \dots v_{-\infty}$ (in an absolutely formal way) instead of v_{-N}. We obtain the following description of the basic representation.

Theorem 2.4.1. *Let $\widetilde{V}$ be the vector space with the basis consisting of infinite "monomials" $m = e_{i_1} e_{i_2} \dots e_{2N+1} e_{2N+3} \dots v_{-\infty}$ such that, starting from some position, the successive symbols e_i are indexed by successive odd numbers $i = 2N+1, 2N+3, 2N+5, \dots$, and it is assumed that*

(i) *different e_i commute (i.e., $e_{i_1} e_{i_2} \dots e_{i_k} e_{i_l} \dots v_{-\infty} = e_{i_1} e_{i_2} \dots e_{i_l} e_{i_k} \dots v_{-\infty}$);*

(ii) *if a symbol e_{i_k} with the index $i_k \ge 2N$ appears before the stable part $e_{2N+1} e_{2N+3} \dots v_{-\infty}$ of a "monomial" $m = e_{i_1} e_{i_2} \dots e_{2N+1} e_{2N+3} \dots v_{-\infty}$, then $m = 0$ ("repulsion rule").*

The infinite expressions $S_m = \sum_{\alpha+\beta=m} e_\alpha e_\beta$, $m \in \mathbb{Z}$, act from the left on the space $\widetilde{V}$. Let $V = \widetilde{V}/(S_m)\widetilde{V}$ be the quotient space by this action. Then in the space V the basic representation of $\widehat{\mathfrak{sl}}_2$ is realized, and the elements e_i act in an obvious way (by multiplication from the left).

Remark 2.4.2. The symbol $v_{-\infty}$ stands for the "extremal vector at infinity." The vector v_{-N} is annihilated by the subalgebra $\mathfrak{n}_+^{-N} = T_{-N}\mathfrak{n}_+T_N = \langle f_i, i > -2N\,;\ h_i, i > 0\,;\ e_i, i \geq 2N\rangle$. As N tends to infinity, the subalgebra $\mathfrak{n}_+^{-N}$ tends to $\mathfrak{n}_+^{-\infty} = \langle f_i, i \in \mathbb{Z}\,;\ h_i, i > 0\rangle$. Therefore, it is natural to consider that the "vector" $v_{-\infty}$ is annihilated by the subalgebra $\mathfrak{n}_+^{-\infty}$. But, in fact, $v_{-\infty}$ is annihilated by the larger subalgebra $\hat{\mathfrak{b}} = \{f_i, h_i, i \in \mathbb{Z}\}$. We will try to explain this viewpoint in the next subsection.

2.5. It is natural to try to construct the action of $\widehat{\mathfrak{sl}}_2$ on the space V of Theorem 2.4.1 independently of the preceding exposition. The action of the part $\langle h_i, i > 0\rangle$ of the algebra $\mathfrak{n}_+^{-\infty}$ is reconstructed automatically from the condition $h_i(v_{-\infty}) = 0$. For example,

$$\begin{aligned} h_i(e_1e_3e_5\dots v_{-\infty}) &= [h_i, e_1]e_3e_5\dots v_{-\infty} + e_1h_ie_3e_5\dots v_{-\infty} \\ &= 2e_{i+1}e_3e_5\dots v_{-\infty} + 2e_1e_{i+3}e_5\dots v_{-\infty} + \dots + e_1e_3e_5\dots h_iv_{-\infty}\,. \end{aligned}$$

All summands, except for the latter, are zero by the "repulsion rule" from Theorem 2.4.1, and the latter one vanishes because of the equality $h_iv_{-\infty} = 0$. (Of course, such "reasoning" simply expresses the fact that $h_iv_{-N} = 0$ for N sufficiently large.) Similarly, the action of the operators $f_i \in \mathfrak{n}_+^{-\infty}$ reduces to the action of h_j, due to the condition $f_iv_{-\infty} = 0$.

Thus, it remains to construct the action of h_i for $i < 0$. A surprising (though easily explainable) fact is that the operators h_i also act by the rule $h_iv_{-\infty} = 0$ in this case. (In other words, the symbol $v_{-\infty}$ in the notation of the "monomials" can be omitted; thus, the construction 2.4.1 is somewhat close to the constructions of semi-infinite forms.)

Let us give an example. A calculation gives $h_{-1}v = e_0v_{-1}$ in the basic representation. On the other hand,

$$h_{-1}(e_1e_3e_5\dots v_{-\infty}) = 2e_0e_3e_5\dots v_{-\infty} + 2e_1e_2e_5e_7\dots v_{-\infty} + \dots + e_1e_3\dots h_{-1}v_{-\infty}\,.$$

From the relations $S_3e_5e_7\dots v_{-\infty} = S_7e_7e_9\dots v_{-\infty} = \dots = 0$ it follows that

$$e_0e_3e_5\dots v_{-\infty} = -e_1e_2e_5e_7\dots v_{-\infty} = e_1e_3e_4e_7e_9\dots v_{-\infty} = \dots\,.$$

Thus, if we put $2e_0e_3e_5\dots v_{-\infty} = a$, then

$$h_{-1}(e_1e_3e_5\dots v_{-\infty}) = a - a + a - a + \dots + e_1e_3e_5\dots h_{-1}v_{-\infty}\,.$$

It is natural to consider that the sum of the series $a - a + a - a + \dots$ is equal to $a/2$, which is just equal to $e_0e_3e_5e_7\dots v_{-\infty} = e_0v_{-1} = h_{-1}v$. Therefore, we must realize that the operator h_{-1} obeys the rule $h_{-1}v_{-\infty} = 0$.

2.6. It is much easier to work with the principal space W and with the space V from Theorem 2.4.1 if bases of monomials are chosen in these spaces. A monomial $e_{i_1}\dots e_{i_n}v \in W$ is said to be *reduced* if $i_1 < i_2 < \dots < i_n < 0$ and, moreover, $i_{k+1} - i_k \geq 2$ for all k. A "monomial" $m = e_{i_1}e_{i_2}e_{i_3}\dots v_{-\infty}$ is said to be *reduced* if $i_{k+1} - i_k \geq 2$ for all k.

Proposition 2.6.1. (a) *The reduced monomials form a basis of the space* W.
(b) *The reduced "monomials" form a basis of* V.

Part (a) can be easily deduced from Theorem 2.2.1; (b) follows trivially from (a).

From Proposition 2.6.1(a) one can get another proof of formula (2.3.3) for the character of the space W.

Now we evaluate the character of V. Let us introduce the standard notation

$$(q)_k = (1-q)(1-q^2)\dots(1-q^k) \quad \text{and} \quad (q)_\infty = \prod_{i=1}^{\infty}(1-q^i)\,.$$

Then we have

$$\operatorname{ch} W_{-N} = \operatorname{ch}(T_{-N}W) = T_{-N}\operatorname{ch} W = \sum_{n=-N}^{\infty}\frac{q^{n^2}z^n}{(q)_{n+N}} \quad (\text{see } (2.1.1)),$$

$$\operatorname{ch} V = \lim_{N\to\infty}\operatorname{ch} W_{-N} = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty} q^{n^2}z^n. \tag{2.6.2}$$

We have obtained the well-known character formula for the boson realization of the basic representation.

2.7. The results of 2.2–2.6 can be easily generalized to other irreducible integrable representations of $\widehat{\mathfrak{sl}}_2$ with highest weight. We will present the corresponding statements here.

In this subsection, V is an irreducible representation of $\widehat{\mathfrak{sl}}_2$ with highest weight $\boldsymbol{\lambda} = (0, l, k)$, where k and l are integers, $0 \le l \le k$, v is the vacuum vector, and $W = U(\hat{\mathfrak{n}})v = (\mathbb{C}[e_{-1}, e_{-2}, \dots]/I)v$.

Theorem 2.2.1′. *The ideal I is generated by the polynomials e_{-1}^{k+1-l} and*

$$S_{-i}^{(k+1)} = \sum_{\alpha_1+\alpha_2+\dots+\alpha_{k+1}=-i} e_{\alpha_1} e_{\alpha_2} \dots e_{\alpha_{k+1}}, \qquad i \ge k+1.$$

The proof of the theorem will be given in 3.4.

Remark 2.2.2′. $e(z)^{k+1} = 0$.

One has the following analog of (2.3.2):

$$W^* = \bigoplus_{m=0}^{\infty} W_m^*,$$

where

$$\begin{aligned} W_m^* \simeq \{ & f(x_1, \dots, x_m)\, dx_1 \dots dx_m, f \text{ is symmetric}, \\ & f = 0 \text{ if } x_1 = x_2 = \dots = x_{k+1} \text{ (for } k+1 \le m) \\ & \text{and if } x_1 = x_2 = \dots = x_{k-l+1} = 0 \text{ (for } k-l+1 \le m)\}. \end{aligned} \tag{2.3.2′}$$

(The function vanishes if $k+1$ particles coincide or $k+1-l$ particles coincide with zero.) The character formula for W is given by

$$\operatorname{ch} W = \sum_{m=0}^{\infty} \sum_{\substack{N_1 \ge \dots \ge N_k \ge 0 \\ N_1+\dots+N_k=m}} \frac{z^{m+l/2} q^{N_1^2+\dots+N_k^2+N_{k-l+1}+\dots+N_k}}{(q)_{N_1-N_2}(q)_{N_2-N_3}\dots(q)_{N_{k-1}-N_k}(q)_{N_k}}. \tag{2.3.3′}$$

We will give a sketch of the proof of this statement.

For simplicity, we will assume $l = 0$.

Theorem 2.7.1. *The character of the space $W_m^* = \{f(x_1,\dots,x_m)\, dx_1 \dots dx_m,$ f is a symmetric polynomial, $f = 0$ for $x_1 = x_2 = \dots = x_{k+1}\}$ with gradation $\deg(x_1^{t_1} \dots x_m^{t_m}\, dx_1 \dots dx_m) = \sum t_i + m$ is equal to*

$$\operatorname{ch} W_m^* = \sum_{\substack{N_1 \ge \dots \ge N_k \ge 0 \\ N_1+\dots+N_k=m}} \frac{q^{N_1^2+\dots+N_k^2}}{(q)_{N_1-N_2}(q)_{N_2-N_3}\dots(q)_{N_{k-1}-N_k}(q)_{N_k}}.$$

Sketch of the proof. Let $p = (p_1, \dots, p_s)$ be a partition of the number m, $p_1 \ge \dots \ge p_s > 0$. Let $U_p = \{f(x_1, \dots, x_m)\, dx_1 \dots dx_m$, f is symmetric, and if we have simultaneously $x_1 = x_2 = \dots = x_{p_1}$, $x_{p_1+1} = \dots = x_{p_1+p_2}$, $x_{p_1+p_2+1} = \dots = x_{p_1+p_2+p_3}, \dots, x_{p_1+\dots+p_{s-1}+1} = \dots = x_m$, then $f(x_1, \dots, x_m) = 0\} \subset S^m \Omega^1 \mathbb{C}$. Define a filtration of $S^m \Omega^1 \mathbb{C}$ by the subspaces $F_p = \bigcap_{p' \ge p} U_{p'}$, where the set of partitions of m is ordered lexicographically. Clearly, $W_m^* = U_{p(k)} = F_{p(k)}$, where $p(k) = (k+1, 1, 1, \dots, 1)$.

The associated graded quotient space $(\operatorname{Gr} F)_p$ can be identified with the space of polynomial forms $\varphi(z_1, \dots, z_s)(dz_1)^{p_1} \dots (dz_s)^{p_s}$ of the variables $z_1 = x_1 = x_2 = \dots = x_{p_1}$, $z_2 = x_{p_1+1} = \dots = x_{p_1+p_2}$, $\dots$, $z_s = x_{p_1+\dots+p_{s-1}+1} = \dots = x_m$ (the particles have combined into s groups, p_i particles in the ith group), such that

(1) if $p_i = p_j$, then φ is symmetric with respect to z_i and z_j;

(2) φ vanishes on the diagonal $z_i = z_j$ with some multiplicity $\varkappa_{ij}$.

The multiplicity $\varkappa_{ij}$ can be calculated by forgetting all variables x_t except for the two groups of variables corresponding to z_i and z_j. Let $p_i \geq p_j$. Considering the component $(\operatorname{Gr} F)_{(p_i,p_j)}$ for the filtration F on the space of symmetric functions of $p_i + p_j$ variables one can easily see that $\varkappa_{ij} = 2p_j$ ($= \deg \operatorname{Sym} \prod_{t=1}^{p_j} (x_t - x_{p_j+t})^2$, Sym is the symmetrization over $x_1, \ldots, x_{p_i+p_j}$). Thus,

$$(\operatorname{Gr} F)_p \simeq \Big\{ \varphi(z_1, \ldots, z_s) \prod_{i<j} (z_i - z_j)^{2p_j} (dz_1)^{p_1} \ldots (dz_s)^{p_s}, \text{ where, for each } r \in \mathbb{N}, \ \varphi \text{ is symmetric with respect to the variables } z_i \text{ for which } p_i = r \Big\}.$$

Let n_r be the number of summands of the partition p equal to r. Then

$$\operatorname{ch}(\operatorname{Gr} F)_p = \frac{q^{\sum_{i<j} 2p_j + \sum_i p_i}}{\prod_r (q)_{n_r}} = \frac{q^{\sum_r N_r^2}}{(q)_{N_1-N_2}(q)_{N_2-N_3} \cdots (q)_{N_r-N_{r+1}} \cdots},$$

where $N_r = n_r + n_{r+1} + \ldots$ are the summands of the partition that is transposed to p.

Summing over p and m, we get the "Gordon identity for $k = \infty$":

$$\operatorname{ch} S^* \Omega^1 \mathbb{C} = \frac{1}{(q)_\infty} = \sum_{\substack{(N_1 \ldots N_r \ldots 0\,0\,0 \ldots) \\ N_1 \geq \cdots \geq N_t = 0}} \frac{q^{\sum_r N_r^2}}{(q)_{N_1-N_2}(q)_{N_2-N_3} \cdots}. \tag{2.7.2}$$

The desired character of W_m^* is obtained by summing of $\operatorname{ch}(\operatorname{Gr} F)_p$ over all $p < p(k)$, i.e., $p_i \leq k$ for all i. In the partition transposed to p we have $N_{k+1} = N_{k+2} = \cdots = 0$, and this gives the formula from Theorem 2.7.1.

Remark 2.7.3. The matrix of the quadratic form

$$\sum_{r=1}^{k} (n_r + n_{r+1} + \cdots + n_k)^2 = \sum_r r n_r^2 + \sum_{r<t} 2 r n_r n_t$$

is inverse to the symmetrized Cartan matrix

$$\widetilde{B}_k = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \ldots & 0 & -1 & 1 \end{pmatrix}.$$

Therefore, if we put

$$\Psi_A(q) = \sum_{n_1, \ldots, n_k \geq 0} \frac{q^{\sum A_{ij} n_i n_j}}{(q)_{n_1} \cdots (q)_{n_k}},$$

then we see that we have proven the relation $\operatorname{ch} W(q,1) = \Psi_{\widetilde{B}_k^{-1}}(q)$.

Theorem 2.4.1$'$. *The irreducible representation of $\widehat{\mathfrak{sl}}_2$ with highest weight λ is realized in the quotient space $\widetilde{V}/(S_i^{(k+1)})_{i \in \mathbb{Z}} \widetilde{V}$, where $\widetilde{V}$ is the space with the basis consisting of "monomials"*

$$e_{i_1} e_{i_2} \ldots e_{2N}^{l} e_{2N+1}^{k-l} e_{2N+2}^{l} e_{2N+3}^{k-l} \cdots$$

such that

(i) *different e_i commute,*

(ii) *if a "monomial" $m = e_{i_1} e_{i_2} e_{i_3} \ldots$ contains a symbol e_{i_j} with $i_j \geq 2N$ (resp., $i_j \geq 2N+1$) before the stable part of the form $e_{2N}^{l} e_{2N+1}^{k-l} \cdots$ (resp., $e_{2N+1}^{k-l} e_{2N+2}^{l} \cdots$), then $m = 0$.*

The elements $e_i \in \widehat{\mathfrak{sl}}_2$ act on V by left multiplication.

Proposition 2.6.1′. *We say that a monomial $e_{i_1}\dots e_{i_m}v \in W$ ($e_{i_1}e_{i_2}\dots \in V$) is reduced if $i_1 \le \dots \le i_m < 0$, $i_{m-k+l} < -1$, and $i_{j+k} - i_j \ge 2$ for all j (respectively, $i_1 \le i_2 \le \dots$ and $i_{j+k} - i_j \ge 2$ for all j). Then the reduced monomials form a basis of W (respectively, V).*

Finally, we obtain the following character formula for the representation V:

$$\operatorname{ch} V = \frac{1}{(q)_\infty} \sum_{N_1 \ge \dots \ge N_k \in \mathbb{Z}} \frac{q^{N_1^2+\dots+N_k^2+N_{k-l+1}+\dots+N_k} z^{N_1+\dots+N_k+l/2}}{(q)_{N_1-N_2}(q)_{N_2-N_3}\dots(q)_{N_{k-1}-N_k}}. \tag{2.6.2′}$$

Remark 2.7.4. One can easily see that the character formula just obtained coincides with the "parafermionic" formula from [3]; for the case $l = 0$ it has the form

$$\sum_{j=0}^{k-1}\sum_{i=0}^{\infty} q^{i-j^2/k} \dim V_{(-i,2j,1)} = \frac{1}{(q)_\infty} \Psi_{A_{k-1}^{-1}}(q)$$

in the notation (2.7.3) (A_{k-1}^{-1} is the inverse matrix to the Cartan matrix A_{k-1}).

§3. Semi-Infinite Schubert Cells: the Case $\mathfrak{g} = \mathfrak{sl}_2$

3.1. Let $G = SL(2, \mathbb{C})$, let $\widehat{G} = \widetilde{SL}(2, \mathbb{C}[t, t^{-1}]])$ be the central extension (with the help of $\mathbb{C}^*$) of the G-valued current group, let $\mathbf{B}_+$ be the Borel subgroup in $\widehat{G}$ with Lie algebra $\mathfrak{b}_+$, and let $F = \widehat{G}/\mathbf{B}_+$ be the flag manifold of $\widehat{G}$. We will consider F as an infinite-dimensional complex algebraic variety.

The irreducible integrable representation V of the Lie algebra $\hat{\mathfrak{g}}$ with highest weight $\boldsymbol{\lambda}$ is realized in the space $H^0(F, L_{\boldsymbol{\lambda}})^*$, where $L_{\boldsymbol{\lambda}} = \widehat{G} \times_{\mathbf{B}_+} \mathbb{C}_{(-\boldsymbol{\lambda})}$ is the holomorphic Borel–Weil line bundle on F [5, 6].

To the principal subspace $W \subset V$ there corresponds the *principal subvariety* $M = \overline{\widehat{N} \cdot \mathbf{1}} \subset F$, which is the closure of the orbit of the unit coset under the action of the group $\widehat{N}$ of currents with the values in the group of upper triangular matrices from $SL(2, \mathbb{C})$. The inclusion map $W \to V$ is dual to the restriction map for sections $H^0(F, L_{\boldsymbol{\lambda}}) \to H^0(M, L_{\boldsymbol{\lambda}})$. To verify this statement, we note that $M = \varinjlim M_n$, where $M_n = \overline{\mathbf{B}_+^n \cdot \mathbf{1}}$, $\mathbf{B}_+^n = T_n \mathbf{B}_+ T_{-n}$ (the limit is taken in the sense of algebraic geometry), and a similar fact for the finite-dimensional variety M_n (and the Lie algebra $\mathfrak{b}_+^n$ instead of $\hat{\mathfrak{n}}$) does not differ from the well-known theorem for the flag manifold of a finite-dimensional complex semisimple Lie group [15, 16]. Our statement is now obtained by passing to the limit over n.

The same reasoning verifies that the higher cohomology groups of M with coefficients in $L_{\boldsymbol{\lambda}}|_M$ vanish and that the Atiyah–Bott–Lefschetz fixed-point formula, for the action of the maximal torus $\mathbf{T} = \mathbb{T}\times\mathbb{T}\times\mathbb{T}$ of the group $\mathbb{T} \ltimes \widehat{SU}(2)$, which is the "compact form" of $\mathbb{C}^* \ltimes \widehat{G}$, can be applied to $(M, L_{\boldsymbol{\lambda}}|_M)$.† (Here the extra factor $\mathbb{C}^*$ corresponds to the gradation according to energy on V and to the letter q in character formulas.)

Combining the Lefschetz formula with the triviality of $H^i(M, L_{\boldsymbol{\lambda}}|_M)$ for $i > 0$, we obtain the following character formula for W:

$$\operatorname{ch} W = \sum_{\omega \in W_{\mathrm{aff}} \cap M} \frac{e^{i\omega\cdot\boldsymbol{\lambda}}}{\prod_{\mu \text{ is a weight of } T_\omega M} (1 - e^{i\mu})}. \tag{3.1.1}$$

(Recall that the Weyl group $W_{\mathrm{aff}} = N(\mathbf{T})/\mathbf{T}$ is included in $F \simeq (\mathbb{T} \ltimes \widehat{SU}(2))/\mathbf{T}$.)

3.2. Theorem 3.2.1 (the structure of the variety M).

(1) *M is nonsingular.*

(2) $M \cap W_{\mathrm{aff}} = \{T_n : n \ge 0, S_n : n > 0\}$.

(3) *The set of weights for the action of the maximal torus $\mathbf{T}$ on the tangent space of M at a point $\omega \in W_{\mathrm{aff}} \cap M$ is a subset of the set of roots of $\hat{\mathfrak{g}}$. The corresponding root vectors are*

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†Despite the fact that the varieties M_n are singular, one can write an analog of the Atiyah–Bott–Lefschetz formula for them. See 3.5.

for $\omega = T_n$, $n \geq 0$: $\{e_{-i},\ i \geq 2n+1;\ f_i,\ n+1 \leq i \leq 2n;\ h_{-i},\ 1 \leq i \leq n\}$;
for $\omega = S_n$, $n > 0$: $\{e_{-i},\ i \geq 2n;\ f_i,\ n \leq i \leq 2n-1;\ h_{-i},\ 1 \leq i \leq n-1\}$.

(4) *The stratification of the flag manifold by the orbits of the group* $\mathbf{N}_-$ *induces a stratification of* M: $M = \bigsqcup_{\omega \in W_{\mathrm{aff}} \cap M} Y_\omega$, $Y_\omega = \mathbf{N}_-\omega \cap M$, *and*

(i) Y_ω *is a contractible subvariety,* $\operatorname{codim} Y_{T_n} = \operatorname{codim} Y_{S_n} = n$;

(ii) Y_{T_n} *and* $Y_{S_{n+1}}$ *are* n*-parametric families of* $\widehat{N}$*-orbits of codimension* $2n$ *and* $2n+1$ *respectively, and a transverse subvariety to the family* Y_ω $(\omega = T_n$ *or* $S_{n+1})$ *is given by the formula*

$$(d_1, \dots, d_n) \mapsto \begin{pmatrix} (1 + d_1 t^{-1} + \dots + d_n t^{-n})^{-1} & 0 \\ 0 & 1 + d_1 t^{-1} + \dots + d_n t^{-n} \end{pmatrix} \cdot \omega;$$

(iii) $\overline{Y}_{T_n} = \bigcup_{m \geq n} Y_{T_m} \bigcup_{m > n} Y_{S_m}$ *and* $\overline{Y}_{S_n} = \bigcup_{m \geq n} Y_{S_m} \bigcup_{m \geq n} Y'_{T_m}$, *where* Y'_{T_m} *is a subfamily of* $\widehat{N}$*-orbits in* Y_{T_m} *of codimension* 1 *that consists of orbits with parameter* $d_m = 0$.

To prove the theorem, let us choose the following representatives of cosets $\omega \in W_{\mathrm{aff}} = N(\mathbf{T})/\mathbf{T}$ (and denote them by the same letter ω): $T_n = \begin{pmatrix} t^{-n} & 0 \\ 0 & t^n \end{pmatrix}$, $S_n = \begin{pmatrix} 0 & -t^{-n} \\ t^n & 0 \end{pmatrix}$. The manifold F is covered by coordinate charts $F = \bigcup_{\omega \in W_{\mathrm{aff}}} U_\omega$, $U_\omega = \omega \mathbf{N}_- \cdot \mathbf{1} \simeq \mathbf{N}_-$. A direct computation in coordinates U_ω proves all statements of the theorem.

For example, let $\omega = T_n = \begin{pmatrix} t^{-n} & 0 \\ 0 & t^n \end{pmatrix}$. An element $\omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \omega \mathbf{N}_- \cdot \mathbf{1}$, where $a = 1 + a_1 t^{-1} + a_2 t^{-2} + \dots$, $b = b_1 t^{-1} + b_2 t^{-2} + \dots$, $c = c_0 + c_1 t^{-1} + \dots$, $d = 1 + d_1 t^{-1} + \dots$, and $ad - bc = 1$, belongs to $\widehat{N} \cdot \mathbf{1}$ if and only if, for some Laurent series $p = p_1 t^{-1} + p_2 t^{-2} + \dots$, the matrix

$$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-n} & 0 \\ 0 & t^n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} at^{-n} + ct^n p & bt^{-n} + dt^n p \\ ct^n & dt^n \end{pmatrix} \tag{3.2.2}$$

lies in $\mathbf{B}_+$. If $n < 0$, then the series $dt^n = t^n + d_1 t^{n-1} + \dots$ cannot belong to $\mathbb{C}[t]$; therefore, $\widehat{N} \cdot \mathbf{1} \cap U_\omega = \varnothing$. Now let us discuss the case $n = 1$ in detail, in order to illustrate the phenomenon of imaginary roots appearing in the tangent space $T_\omega M$. The condition "matrix (3.2.2) lies in $\mathbf{B}_+$" means in this case that we have

$$c = c_0, \qquad d = 1 + d_1 t^{-1},$$
$$t^{-1} + a_1 t^{-2} + a_2 t^{-3} + \dots + c_0(p_1 + p_2 t^{-1} + \dots) = c_0 p_1,$$
$$b_1 t^{-2} + b_2 t^{-3} + \dots + (t + d_1)(p_1 t^{-1} + p_2 t^{-2} + \dots) = p_1.$$

If $c_0 = 0$, then the third equation cannot hold, and p does not exist. But if $c_0 \neq 0$, then the third equation determines p_2, p_3, ... uniquely, e.g., $p_2 = -1/c_0$. Equating the coefficients of t^{-1} in the fourth equation, we get $p_1 d_1 = -p_2 = 1/c_0$. Therefore, $d_1 \neq 0$ is also necessary. Conversely, if $c_0 \neq 0$ and $d_1 \neq 0$, then let us define p_i by the formulas

$$p_1 = \frac{1}{c_0 d_1}, \qquad p_2 = -\frac{1}{c_0}, \qquad p_i = -\frac{a_{i-2}}{c_0} \quad \text{for } i = 3, 4, \dots. \tag{3.2.3}$$

Then the matrix (3.2.2) has the form $\begin{pmatrix} c_0 p_1 & x \\ c_0 t & d_1 + t \end{pmatrix}$, and its determinant $c_0 p_1 (d_1 + t) - x c_0 t$ is equal to 1, whence $x = p_1$ and the matrix lies in $\mathbf{B}_+$, q.e.d.

Thus, we have

$$\widehat{N} \cdot \mathbf{1} \cap U_{T_1} = \left\{ T_1 \cdot \begin{pmatrix} 1 + a_1 t^{-1} + \dots & b_1 t^{-1} + \dots \\ c_0 & 1 + d_1 t^{-1} \end{pmatrix} \cdot \mathbf{1} : c_0, d_1 \neq 0 \right\},$$

$$M \cap U_{T_1} = \left\{ T_1 \cdot \begin{pmatrix} a & b_1 t^{-1} + \dots \\ c_0 & 1 + d_1 t^{-1} \end{pmatrix} \cdot \mathbf{1} \right\} = \left\{ \begin{pmatrix} a & b_1 t^{-3} + b_2 t^{-4} + \dots \\ c_0 t^2 & 1 + d_1 t^{-1} \end{pmatrix} \cdot T_1 \right\}.$$

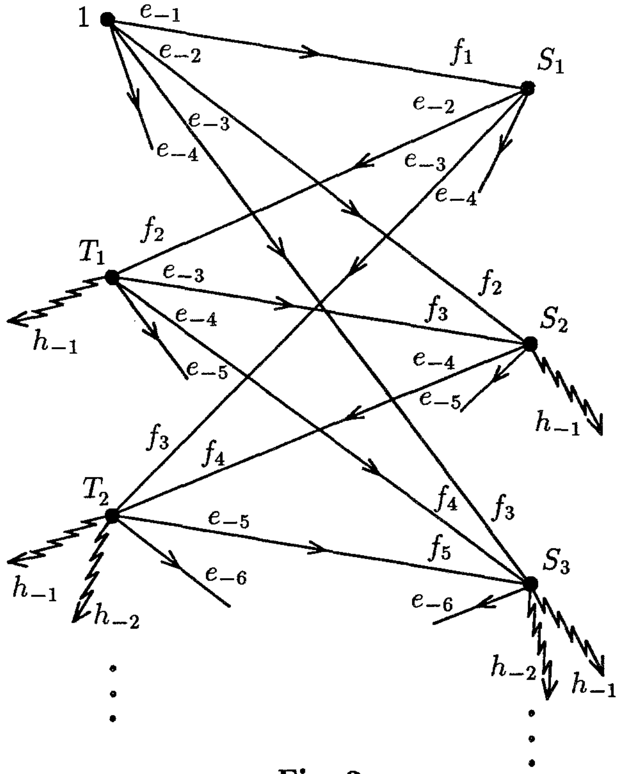

Fig. 2

In particular, $M \cap U_{T_1}$ is nonsingular; moreover, we have proved part (3) for $\omega = T_1$.

The hyperplane $\{c_0 = 0\} \subset M \cap U_{T_1}$ is

$$Y_{T_1} = \bigcup_{d_1 \in \mathbb{C}} \widehat{N} \cdot \begin{pmatrix} (1+d_1 t^{-1})^{-1} & 0 \\ 0 & 1+d_1 t^{-1} \end{pmatrix} \cdot T_1 .$$

The hyperplane $\{d_1 = 0\}$ is the intersection $U_{T_1} \cap \overline{Y}_{S_1}$; this can be seen easily, taking into account that the action of a copy of $SL(2,\mathbb{C})$ with Lie algebra $\langle e_{-2}, f_2, h_0 - 2c\rangle \subset \hat{\mathfrak{g}}$ on the variety F induces a holomorphic embedding $SL(2,\mathbb{C})/B_+ \simeq \mathbb{C}P^1 \hookrightarrow M$ which maps $0 \in \mathbb{C}P^1$ to T_1, z to $\begin{pmatrix} 1 & 0 \\ zt^2 & 1 \end{pmatrix} \cdot T_1$, and ∞ to $S_2 T_1 = S_1$.

The latter observation suggests a way of illustrating concisely the information of Theorem 3.2.1 by a picture. In Fig. 2, the intervals of straight lines symbolically represent the projective lines $\mathbb{C}P^1 \subset M$, generated by the action of e_{-i}, that join the points ω and $S_i\omega$ of $W_{\text{aff}} \cap M$.

3.3. Combinatorial consequences of Theorem 3.2.1. Substituting the results of Theorem 3.2.1 into the formula (3.1.1), we obtain

$$\begin{aligned} \operatorname{ch} W = {} & \sum_{n=0}^{\infty} \frac{e^{i(T_n \cdot \boldsymbol{\lambda})}}{(1-q)\dots(1-q^n)(1-(q^{n+1}z)^{-1})\dots(1-(q^{2n}z)^{-1})(1-q^{2n+1}z)\dots} \\ & + \sum_{n=1}^{\infty} \frac{e^{i(S_n \cdot \boldsymbol{\lambda})}}{(1-q)\dots(1-q^{n-1})(1-(q^n z)^{-1})\dots(1-(q^{2n-1}z)^{-1})(1-q^{2n}z)\dots} \\ = {} & \frac{1}{\prod_{i=1}^{\infty}(1-q^i z)} \sum_{n=0}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}} z^n [e^{iT_n \cdot \boldsymbol{\lambda}} - e^{iS_{n+1} \cdot \boldsymbol{\lambda}} q^{2n+1} z] \prod_{i=1}^{n} \frac{1-q^i z}{1-q^i} . \end{aligned} \tag{3.3.1}$$

A comparison of this formula with (2.3.3′) gives some interesting combinatorial identities. We write them down for $z = 1$:

Theorem 3.3.2. (a) (Euler pentagonal theorem)

$$\prod_{m=1}^{\infty}(1-q^m)=\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{3n^2+n}{2}}.$$

(b) (Rogers–Ramanujan identities)

$$\text{(I)}\qquad \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}=\prod_{m=1}^{\infty}\frac{1}{(1-q^{5m+1})(1-q^{5m+4})};$$

$$\text{(II)}\qquad \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n}=\prod_{m=1}^{\infty}\frac{1}{(1-q^{5m+2})(1-q^{5m+3})}.$$

(c) (Gordon identities) *Let* $0\le l\le k$ *be integers; then*

$$\sum_{N_1\ge\cdots\ge N_k\ge 0}\frac{q^{N_1^2+\cdots+N_k^2+N_{k-l+1}+\cdots+N_k}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-1}-N_k}(q)_{N_k}}=\prod_{m>0\,;\ m\not\equiv 0,\pm(k-l+1)\bmod 2k+3}\frac{1}{1-q^m}.$$

Part (a) corresponds to the weight $\boldsymbol{\lambda}=0$, part (b) to $\boldsymbol{\lambda}=(0,0,1)$ and $(0,1,1)$, and part (c) to $\boldsymbol{\lambda}=(0,l,k)$.

The product decomposition of the right-hand side of the Gordon identities

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{(2k+3)n^2+(2l+1)n}{2}}=\prod_{m\equiv 0,\pm(k-l+1)\bmod 2k+3}(1-q^m)$$

is a special case of the Jacobi triple product

$$\sum_{n\in\mathbb{Z}}(-1)^n u^{\frac{n(n+1)}{2}}v^{-n}=(1-v)\prod_{m=1}^{\infty}(1-u^m v^{-1})(1-u^m)(1-u^m v)$$

for $u=q^{2k+3}$ and $v=q^{k-l+1}$.

Theorem 3.3.2(c), together with Proposition 2.6.1′, implies some classical results of the theory of partitions ([17, Theorem 7.5]).

3.4. The decomposition $M=\bigsqcup_{\omega\in W_{\mathrm{aff}}\cap M}Y_\omega$ from Theorem 3.2.1 is not a stratification in the strict sense: the boundary of the stratum Y_{S_n} is not contained in the union of strata of greater codimension. To get a real stratification, one must divide some of the strata into smaller ones, for example, decompose Y_{T_1} into the union $(Y_{T_1}\setminus Y'_{T_1})\cup Y'_{T_1}$ of two strata of codimension 1 and 2, etc. As usual, to the corrected stratification of M there corresponds a resolution of the $\hat{\mathfrak{n}}$-module $H^0(M,L_{\boldsymbol{\lambda}})$ (the Cousin resolution), which consists of spaces of distributions supported on strata. The character of the resolution is given by the Lefschetz formula (3.3.1), which shows that this resolution is rather complicated. Nevertheless, its initial terms, corresponding to the strata of codimension 0 and 1, admit an explicit description, and this leads to the proof of Theorem 2.2.1′.

Let $U=Y_1=\widehat{N}\cdot\mathbf{1}$ be the open dense stratum of M, let $U_1=U_{S_1}\cap M$ and $U_2=(U_{T_1}\cap M)\setminus\overline{Y}_{S_1}$ be open neighborhoods of the (codimension 1) strata Y_{S_1} and $Y_{T_1}\setminus Y'_{T_1}$, respectively, and let $U_1^*=U_1\setminus Y_{S_1}$ and $U_2^*=U_2\setminus Y_{T_1}$. One has $U_1^*=U_1\cap U$ and $U_2^*=U_2\cap U$ (see the proof of Theorem 3.2.1).

From now on, let us fix a bundle $L_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda}=(0,l,k)$, and write $H^0(Z)$ instead of $H^0(Z,L_{\boldsymbol{\lambda}})$. By Hartogs' theorem, the sequence of restriction maps

$$0\to H^0(M)\to H^0(U)\to H^0(U_1^*)/H^0(U_1)\oplus H^0(U_2^*)/H^0(U_2)$$

is exact. The dual sequence has the form

$$0\leftarrow W\overset{\pi}{\leftarrow}\mathbb{C}[e_{-1},e_{-2},\dots]\xleftarrow{(\varphi_1,\varphi_2)}M_1\oplus M_2,$$

where π is the natural projection and $M_i=[H^0(U_i^*)/H^0(U_i)]^*$.

Theorem 2.2.1′ follows from the next lemma.

Lemma 3.4.1. (i) *M_1 is a free $\mathbb{C}[e_{-1}, e_{-2}, \ldots]$-module of rank 1 with the generator τ, $\varphi_1(\tau) = e_{-1}^{k-l+1}$;*

(ii) *The $\mathbb{C}[e_i]$-module M_2 is generated by elements σ_i, $i \in \mathbb{Z}$, $\varphi_2(\sigma_i) = S_i^{(k+1)}$.*

The proof is a straightforward calculation. (Cf. [5, Lemma 14.5.5]; calculations for (ii) are based on the coordinate transformation formula (3.2.3).)

3.5. In conclusion, we will say a few words about the Lefschetz formula for singular varieties in order to justify some statements of Sect. 3.1. The subject of this subsection will be also useful in §4.

Let X be a compact complex algebraic variety (possibly with singularities) with an invertible sheaf L, and let $\mathbf{T}$ be a torus acting by holomorphic transformations of the pair (X, L) with isolated fixed points. A natural analog of the Lefschetz formula is the following:

$$\sum(-1)^i \operatorname{ch}(\mathbf{T}, H^i(X, L)) = \sum_{\mathbf{T}x=x} \operatorname{ch}(\mathbf{T}, \mathcal{O}_x(L)) \tag{3.5.1}$$

($\mathcal{O}_x(L)$ is the space of germs of sections of L at the point x) [18].

For example, let X be contained in a nonsingular variety Y, let $\widetilde{L}$ be a holomorphic line bundle on Y, let $L = \widetilde{L}|_X$, and let the torus $\mathbf{T}$ act on $(Y, \widetilde{L})$, preserving the variety X. Let $x \in X$ be a fixed point with respect to $\mathbf{T}$, let $(z_1, \ldots, z_n)$ be local coordinates on Y with origin at the point x such that the action of $\mathbf{T}$ on the cotangent space T_x^*Y diagonalizes with weights $\lambda_1, \ldots, \lambda_n$ in these coordinates, and let X be a locally full intersection of hypersurfaces $f_j(z_1, \ldots, z_n) = 0$, $j = 1, \ldots, m$, where f_j is homogeneous with respect to the action of $\mathbf{T}$ with weight μ_j (and the elements f_j form a regular sequence in the local ring $\mathcal{O}_x(Y)$). Then in formula (3.5.1) the local summand has the form

$$\operatorname{ch}(\mathbf{T}, \mathcal{O}_x(L)) = \frac{e^{i\nu}\prod_{j=1}^m (1 - e^{i\mu_j})}{\prod_{k=1}^n (1 - e^{i\lambda_k})} \tag{3.5.2}$$

(ν is the weight of $\mathbf{T}$ on the fiber L_x); this follows from the weight decomposition of the Koszul complex.

Consider the situation of 3.1: $X = M_n$, $Y = F$, $\widetilde{L} = L_\lambda$, $\mathbf{T}$ is the maximal torus of the current group. It is known that in this case formula (3.5.1) coincides with the Demazure formula [16, 19] for the element T_n of the affine Weyl group. Decompose T_n into the product $T_n = S_{2n}S_{2n-1}\ldots S_2S_1$ of reflections corresponding to the roots $\alpha > 0$ such that $(T_n)^{-1}\alpha < 0$. (Conjugating S_2 by S_1, S_3 by S_2S_1 etc. in this decomposition, we obtain the reduced simple decomposition of T_n.) The Demazure formula is the result of the application of the sequence of operators $\Sigma_{S_{2n}}\Sigma_{S_{2n-1}}\cdots\Sigma_{S_2}\Sigma_{S_1}$ on the group algebra of the weight lattice of the torus $\mathbf{T}$ to the element $e^{-i\lambda}$, where

$$\Sigma_{S_\alpha}(\chi) = \frac{\chi}{1 - e^{i\alpha}} + \frac{S_\alpha \cdot \chi}{1 - e^{-i\alpha}}$$

(S_α is the reflection corresponding to the root $\alpha > 0$).

By induction on n we get

$$\begin{aligned}
\operatorname{ch} H^0(M_n, L_\lambda)^* &= \operatorname{ch}(\mathbb{C}[e_{-1}, e_{-2}, \ldots, e_{-2n}]v) \\
&= \sum_{m=0}^{2n-1} \frac{e^{iT_m\cdot\lambda}\cdot\binom{2n-1}{m}_q}{(1-(q^{m+1}z)^{-1})\ldots(1-(q^{2m}z)^{-1})(1-q^{2m+1}z)\ldots(1-q^{m+2n}z)} \\
&\quad + \sum_{m=1}^{2n} \frac{e^{iS_m\cdot\lambda}\cdot\binom{2n-1}{m-1}_q}{(1-(q^m z)^{-1})\ldots(1-(q^{2m-1}z)^{-1})(1-q^{2m}z)\ldots(1-q^{m+2n-1}z)}
\end{aligned} \tag{3.5.3}$$

($\binom{2n-1}{j}_q = \frac{(q)_{2n-1}}{(q)_j(q)_{2n-1-j}}$ is a q-binomial coefficient.)

The presence of the numerator $(1-q^{2n-j})\ldots(1-q^{2n-1})$ in the local terms of formula (3.5.3) means that the varieties M_n are singular.

As $n \to \infty$, formula (3.5.3) tends formally to (3.3.1). In this sense, it is natural to consider that formula (3.3.1) coincides with the Demazure formula for the "infinite element"

$$\omega_0 = \lim_{n\to\infty} T_n = \ldots S_4S_3S_2S_1 = S_1S_0S_1S_0\ldots$$

of the affine Weyl group.

§4. The Case $\mathfrak{g}=\mathfrak{sl}_3$: the Lefschetz Formula and Relationships with $\mathfrak{sl}_2$[†]

4.1. Let us denote the simple roots of the Lie algebra $\mathfrak{sl}_3$ by α and β, and the highest root by $\gamma=\alpha+\beta$. Let the corresponding root vectors be e^α, e^β, and $e^\gamma=[e^\alpha, e^\beta]$. Let the opposite root vectors be f^α, f^β, and f^γ. Denote the coroots by h^α, h^β, and $h^\gamma=h^\alpha+h^\beta$. The corresponding basis in $\widehat{\mathfrak{sl}}_3$ consists of $e_i^\alpha=e^\alpha\otimes t^i$, $e_i^\beta=e^\beta\otimes t^i$ etc., and of the central element K.

We will try to preserve the notation of §§2, 3 for similar objects. Thus, $\boldsymbol{\lambda}=(m,\lambda,k)$ is a weight of the Lie algebra $\widehat{\mathfrak{sl}}_3$ (m is the energy, λ is a weight of $\mathfrak{sl}_3$, and $k=\boldsymbol{\lambda}(K)$), V is an irreducible representation of $\widehat{\mathfrak{sl}}_3$ with highest weight $\boldsymbol{\lambda}$, $\widehat{G}=\widetilde{SL}(3,\mathbb{C}[t,t^{-1}]])$, $L_{\boldsymbol{\lambda}}=\widehat{G}\times_{\mathbf{B}_+}\mathbb{C}_{(-\boldsymbol{\lambda})}$ is the Borel–Weil line bundle on $F=\widehat{G}/\mathbf{B}_+$, and $V\simeq H^0(F,L_{\boldsymbol{\lambda}})^*$.

The affine Weyl group $W_{\text{aff}}=W\ltimes\check{T}=S_3\ltimes\mathbb{Z}^2$ contains the lattice $\check{T}=\operatorname{Hom}(\mathbb{T},T)\subset\mathfrak{h}_{\mathbb{R}}$, where T is the maximal torus of $SU(3)$ with Lie algebra $\mathfrak{h}_{\mathbb{R}}$ (the element $mh^\alpha+nh^\beta\in\check{T}$ will be denoted by $T_{m\alpha+n\beta}$), and also contains the reflections with respect to the roots $(i,\alpha,0)$, $(i,\beta,0)$, and $(i,\gamma,0)$ (denoted, respectively, by S_{-i}^α, S_{-i}^β, and S_{-i}^γ). One has $S_0^\alpha=S^\alpha\in W$ and $S_n^\alpha=T_{n\alpha}\circ S^\alpha$, and similarly for β and γ. The reflections S^α, S^β, and S_1^γ correspond to the simple roots α, β, and $(1,-\gamma,0)$ of $\widehat{\mathfrak{sl}}_3$. Elements $\xi\in\check{T}$ and $\omega\in W$ act on weights by the formulas

$$\xi\cdot(m,\lambda,k)=(m-\lambda(\xi)-k\langle\xi,\xi\rangle/2,\ \lambda+k\xi^*,\ k),\qquad \omega\cdot(m,\lambda,k)=(m,\omega\cdot\lambda,k).\tag{4.1.1}$$

Here ξ^* is the image of ξ under the isomorphism $\mathfrak{h}_{\mathbb{R}}\to\mathfrak{h}_{\mathbb{R}}^*$ induced by the canonical inner product $\langle\ ,\ \rangle$ on $\mathfrak{h}_{\mathbb{R}}$.

4.2. Let V be the basic representation of $\widehat{\mathfrak{sl}}_3$, v the vacuum vector, $\hat{\mathfrak{n}}=\mathfrak{n}_+\otimes\mathbb{C}[[t,t^{-1}]$, and $W=U(\hat{\mathfrak{n}})v\subset V$. As in 2.2, we are interested in the left ideal I in $U(\hat{\mathfrak{n}}^{\text{out}})$ annihilating the vector v. We have $f_0^\alpha v=f_0^\beta v=(e_{-1}^\gamma)^2v=0$ (these are the singular vectors in the Verma module $M_{\boldsymbol{\lambda}_0}$); hence, the following elements belong to I:

$$\begin{gathered}(e_{-1}^\gamma)^2,\quad \operatorname{ad}f_0^\alpha(e_{-1}^\gamma)^2=\pm2e_{-1}^\beta e_{-1}^\gamma,\quad \operatorname{ad}f_0^\beta(e_{-1}^\gamma)^2=\pm2e_{-1}^\alpha e_{-1}^\gamma,\\ \operatorname{ad}f_0^\alpha(e_{-1}^\beta e_{-1}^\gamma)=\pm(e_{-1}^\beta)^2,\quad \operatorname{ad}f_0^\beta(e_{-1}^\alpha e_{-1}^\gamma)=\pm(e_{-1}^\alpha)^2.\end{gathered}$$

Commuting these five expressions with the operator $L_{-1}\in\text{Vir}$ and using the relation $L_{-1}v=0$, we obtain five series of elements of the ideal I, which can be written in the short form in the notation of Remark 2.2.2 as follows:

$$e^\alpha(z)^2=e^\alpha(z)e^\gamma(z)=e^\gamma(z)^2=e^\gamma(z)e^\beta(z)=e^\beta(z)^2=0.\tag{4.2.1}$$

Theorem 4.2.2. *The left ideal I is generated by the coefficients of the power series* (4.2.1), *i.e., by the expressions $R_m=\sum_{i+j=m}e_i^\alpha e_j^\alpha$, $S_m=\sum_{i+j=m}e_i^\alpha e_j^\gamma$, etc.*

Remark 4.2.3. In fact (as in Remark 2.2.2), the infinite expressions R_m, S_m, etc. have the zero action on any vector of the space V.

By analogy with (2.3.3) and (2.6.2), it is natural to suppose that the character of the space W is given by the formula

$$\operatorname{ch}W=\sum_{a,b\geq0}\frac{q^{a^2-ab+b^2}z_1^az_2^b}{(q)_a(q)_b}\tag{4.2.4}$$

(here the variables z_1 and z_2 correspond to the two simple roots of $\mathfrak{sl}_3$; cf. (4.1.1)), or $\operatorname{ch}W(q,1,1)=\Psi_{\frac12A_2}(q)$ (see Remark 2.7.3).

Proof of formula (4.2.4).

[†] This section is written in a rather concise manner. The proofs of the most part of assertions are omitted.

Proposition 4.2.5. *Let $\tilde{I} \subset S(\hat{\mathfrak{n}}^{\text{out}})$ be the associated graded quotient of the Poincaré–Birkhoff–Witt filtration on the ideal $I \subset U(\hat{\mathfrak{n}}^{\text{out}})$. Then the ideal $\tilde{I}$ is generated by the same relations* (4.2.1).

Now the same line of reasoning as in §2.3 applied to the dual space of $\widetilde{W} = S(\hat{\mathfrak{n}}^{\text{out}})/\tilde{I}$ gives

$$\operatorname{ch} \widetilde{W}^* = \sum_{r,s,t \geq 0} \frac{q^{r^2+s^2+t^2+rs+st} z_1^{r+s} z_2^{s+t}}{(q)_r (q)_s (q)_t}.$$

Using the technique of q-binomial coefficients, it is not difficult to reduce this formula to the form (4.2.4). □

The above results can be generalized to the case of the representation with highest weight $\boldsymbol{\lambda} = (0,0,k)$: there are $2k+3$ series of relations of type $e^{\alpha}(z)^i e^{\gamma}(z)^{k+1-i} = 0$, etc. The character formula for W has the form

$$\operatorname{ch} W(q,1,1) = \Psi_{\frac{1}{2}A_2 \otimes \widetilde{B}_k^{-1}}(q). \tag{4.2.6}$$

In the general case $\boldsymbol{\lambda} = (0, \lambda, k)$, in the character formula for W extra linear terms are added to the quadratic form $\frac{1}{2}A_2 \otimes \widetilde{B}_k^{-1}$ at the exponent of q.

4.3. The variety M and the Lefschetz formula. In order to simplify our calculations, we restrict ourselves to the representations V with highest weight $\boldsymbol{\lambda} = (0,0,k)$, where k is a natural number. In this case the bundle $L_{\boldsymbol{\lambda}}$ is trivial along fibers of the projection $\pi : F \to P$ onto the Grassmannian $P = \widehat{G}/\widehat{G}^{\text{in}}$. We also denote the bundle $\pi_* L_{\boldsymbol{\lambda}} \simeq \widehat{G} \times_{\widehat{G}^{\text{in}}} \mathbb{C}_{(-\boldsymbol{\lambda})}$ by $L_{\boldsymbol{\lambda}}$.

For $\xi \in \check{T}$ and $\omega \in W \subset W_{\text{aff}} \subset F$ we have $\pi(\xi \cdot \omega) = \pi(\xi)$. Hence, the inclusion $W_{\text{aff}} \hookrightarrow F$ induces the inclusion $W_{\text{aff}}/W \simeq \check{T} \hookrightarrow P$.

We can introduce, as in §3, the subvariety $M' = \overline{\widehat{N}_+ \cdot \mathbf{1}} \subset F$ and prove that $W^* \simeq H^0(M', L_{\boldsymbol{\lambda}})$. But it will be more convenient for our purposes to consider the variety $M = \overline{\widehat{N}_- \cdot \mathbf{1}} \subset F$, where $\widehat{N}_-$ is the group of currents into the lower triangular subgroup $N_- \subset SL(3,\mathbb{C})$ with the Lie algebra $\langle f^{\alpha}, f^{\beta}, f^{\gamma} \rangle$. The fact is that the variety M, unlike M', is a union of fibers of the projection π, and, in the Lefschetz formula, after projecting to P, we can sum over the part $\check{T} \cap \pi(M)$ of the lattice $\check{T}$, instead of summing over $\omega \in W_{\text{aff}} \cap M$. (Respectively, the principal subspace $U(\hat{\mathfrak{n}}_+)v \subset V$ is replaced by $U(\hat{\mathfrak{n}}_-)v$; but the characters of the spaces $U(\hat{\mathfrak{n}}_+)v$ and $U(\hat{\mathfrak{n}}_-)v$ differ only by the replacement of z_1 by z_1^{-1} and of z_2 by z_2^{-1}, because V is symmetric with respect to the replacement of e by f.) Let us denote $\pi(M)$ by the same letter M.

Theorem 4.3.1. (1) $\check{T} \cap M = \{T_{m\alpha+n\beta} : m, n \leq 0\}$.

(2) *M is nonsingular at the points $T_{n\alpha}$ and $T_{n\beta}$ and is singular at other points $\xi \in \check{T} \cap M$.*

(3) *In the Lefschetz formula for the pair $(M, L_{\boldsymbol{\lambda}})$ the local term at the point $T_{-n\alpha}$, $n \geq 0$, is equal to*

$$\Delta_{-n\alpha} = \frac{e^{iT_{-n\alpha}\cdot\boldsymbol{\lambda}}}{\prod\limits_{\substack{\delta \text{ is a root } \widehat{\mathfrak{sl}}_3;\\ S_\delta(-n\alpha)=k\alpha+l\beta\neq-n\alpha,\,k,l\leq 0;\\ T_{n\alpha}(\delta)>0}} (1-e^{i\delta})}$$

$$= \frac{e^{iT_{-n\alpha}\cdot\boldsymbol{\lambda}}}{(1-(q^n a)^{-1})(1-(q^{n+1}a)^{-1})\ldots(1-(q^{2n-1}a)^{-1})(1-q^{2n+1}a)(1-q^{2n+2}a)\ldots}$$
$$\times \frac{1}{(1-q^{-n+1}b)\ldots(1-q^{-1}b)(1-b)(1-qb)\ldots(1-q^{n+1}c)(1-q^{n+2}c)\ldots}$$

(here S_δ is the reflection with respect to δ, $a = z_1$, $b = z_2$, and $c = z_1 z_2$); the local term $\Delta_{-n\beta}$ is obtained from $\Delta_{-n\alpha}$ by the replacement $a \leftrightarrow b$ and $\alpha \leftrightarrow \beta$.

(4) *The local term* $\Delta_{-n\gamma}$ *is equal to*

$$\frac{e^{iT_{-n\gamma}\cdot\boldsymbol{\lambda}}}{\prod\limits_{\substack{\delta \text{ is a root;}\\ S_\delta(-n\gamma)=k\alpha+l\beta\neq -n\gamma,k,l\leq 0;\\ T_{n\gamma}(\delta)>0}} (1-e^{i\delta})}\cdot\frac{(1-c^{-1})(1-(qc)^{-1})\dots(1-(q^{n-1}c)^{-1})}{(1-q)(1-q^2)\dots(1-q^n)}$$

$$= \frac{e^{iT_{-n\gamma}\cdot\boldsymbol{\lambda}}}{(1-a^{-1})(1-(qa)^{-1})\dots(1-(q^{n-1}a)^{-1})(1-q^{n+1}a)(1-q^{n+2}a)\dots}$$
$$\times\frac{1}{(1-b^{-1})(1-(qb)^{-1})\dots(1-(q^{n-1}b)^{-1})(1-q^{n+1}b)(1-q^{n+2}b)\dots}$$
$$\times\frac{(1-c^{-1})(1-(qc)^{-1})\dots(1-(q^{n-1}c)^{-1})}{(1-q)\dots(1-q^n)(1-(q^nc)^{-1})\dots(1-(q^{2n-1}c)^{-1})(1-q^{2n+1}c)(1-q^{2n+2}c)\dots}.$$

The theorem is verified by direct computation in local coordinates in a neighborhood of a point $\xi\in\check{T}$ (similarly to the proof of Theorem 3.2.1) and next by applying formula (3.5.2).

We have not succeeded in evaluating the local terms corresponding to the points $T_{-m\alpha-n\beta}$ for $m>0$, $n>0$, $m\neq n$. At these points the variety has rather complicated singularities. It seems likely that they are not even locally full intersections, thus, formula (3.5.2) cannot be used for them. As for the Demazure character formula for the "infinite element"

$$\omega_0=\lim_{n\to\infty}T_{-n\gamma}=S^\alpha S^\beta S^\alpha S_1^\gamma S^\alpha S^\beta S^\alpha S_1^\gamma\dots=\dots S_{-3}^\gamma S_{-1}^\beta S_{-2}^\gamma S_{-1}^\alpha S_{-1}^\gamma S_0^\beta S_0^\gamma S_0^\alpha,$$

this formula converges rather slowly, and the complexity of the calculations grows exponentially.

Nevertheless, we can state the following conjecture.

Conjecture 4.3.2. *For* $z_1=z_2=1$, *the contribution to the Lefschetz formula of local terms* Δ_ξ, *corresponding to the points* $\xi\in\check{T}\cap M$ *different from* $T_{-n\alpha}$, $T_{-n\beta}$, *and* $T_{-n\gamma}$, *is equal to zero.*

Seemingly, each of these terms contains the factor $(1-a)$, $(1-b)$, or $(1-c)$ in the numerator, originated from the local equation of M in a neighborhood of ξ, which is homogeneous with respect to the torus $\mathbf{T}$ with weight a, b, or c.

Proposition 4.3.3. $(\Delta_{-n\alpha}+\Delta_{-n\beta}+\Delta_{-n\gamma})|_{z_1=z_2=1}$ *is equal to*
(a) $((6n+1)q^{3n^2+n}-(6n-1)q^{3n^2-n})/(q)_\infty^3$, *if* V *is the basic representation*;
(b) $(((2k+4)n+1)q^{(k+2)n^2+n}-((2k+4)n-1)q^{(k+2)n^2-n})/(q)_\infty^3$, *if* V *is the representation with highest weight* $\boldsymbol{\lambda}=(0,0,k)$, $k\geq 0$.

Taking the sum over n and equating to (4.2.6), we obtain the series of identities:

Theorem 4.3.4 (modulo Conjecture 4.3.2).
(a) (Gauss' theorem) $(q)_\infty^3=1-3q+5q^3-7q^6+9q^{10}-11q^{15}+\dots$.
(b) (Analog of the Rogers–Ramanujan identities)

$$\sum_{a,b\geq 0}\frac{q^{a^2-ab+b^2}}{(q)_a(q)_b}=\frac{1-5q^2+7q^4-11q^{10}+13q^{14}-\dots}{(q)_\infty^3}=\frac{\sum_{n\in\mathbb{Z}}(6n+1)q^{3n^2+n}}{(q)_\infty^3}.$$

(c) (Analog of the Gordon identities)

$$\Psi_{\frac{1}{2}A_2\otimes\widetilde{B}_k^{-1}}(q)=\frac{1}{(q)_\infty^3}\sum_{n\in\mathbb{Z}}((2k+4)n+1)q^{(k+2)n^2+n}.$$

(Part (a) corresponds to $k=0$; for the notation of part (c), see 2.7.3.)

4.4. For us, a rather unexpected observation was that the right-hand side of formula 4.3.4(b) coincided with the Kac character formula for the basic representation of Lie algebra $\widehat{\mathfrak{sl}}_2$ (see, for example, [5, (14.3.5)]), and, more generally, the right-hand side of 4.3.4(c) coincided with the Kac character formula for the representation of $\widehat{\mathfrak{sl}}_2$ with highest weight $(0, 0, k)$.

Using this observation, one can simplify identities (4.3.4), replacing their right-hand side by the "boson" character formula (2.6.2) for $k = 1$ and by the "parafermionic" formula (2.6.2′) for a general k. A. E. Postnikov has noticed that after such a replacement identity 4.3.4(b) becomes obvious. (The proof concerns the Durfee square.)

We will give here an explanation for the coincidence of the characters of the space W and of the space of a representation of the Lie algebra $\widehat{\mathfrak{sl}}_2$. For example, let V be the basic representation of $\widehat{\mathfrak{sl}}_2$ (we hope that there will be no confusion in the notation). It is a quotient space of the algebra $U(\widehat{\mathfrak{sl}}_2^{\mathrm{out}})$, where $\widehat{\mathfrak{sl}}_2^{\mathrm{out}} = \langle e_i, f_i, h_i : i < 0\rangle$, by some left ideal J. Since $e_{-1}^2 v = f_0 v = L_{-1} v = 0$, the following elements of the form $(\operatorname{ad} L_{-1})^n (\operatorname{ad} f_0)^m (e_{-1}^2)$ belong to J:

$$e_{-1}^2, \quad h_{-1}e_{-1} + e_{-1}h_{-1}, \quad f_{-1}e_{-1} + e_{-1}f_{-1} - h_{-1}^2, \quad h_{-1}f_{-1} + f_{-1}h_{-1}, \quad f_{-1}^2, \tag{4.4.1}$$

$$\begin{gathered} \sum_{i+j=-n} e_i e_j, \quad \sum_{i+j=-n} (h_i e_j + e_j h_i), \quad \sum_{i+j=-n} (f_i e_j + e_i f_j - h_i h_j), \\ \sum_{i+j=-n} (h_i f_j + f_i h_j), \quad \sum_{i+j=-n} f_i f_j . \end{gathered} \tag{4.4.2}$$

Proposition 4.4.3. (a) *The five relations* (4.4.1) *generate the ideal* J.

(b) *The five series of relations* (4.4.2) *generate the ideal* $\tilde{J} \subset S(\widehat{\mathfrak{sl}}_2^{\mathrm{out}})$, *which is the associated graded quotient of the* PBW-*filtration on* $J \subset U(\widehat{\mathfrak{sl}}_2^{\mathrm{out}})$.

It remains to compare the statements 4.4.3(b) and 4.2.5, and to see that the quotient spaces $S(\hat{\mathfrak{n}}^{\mathrm{out}})/\tilde{I}$ and $S(\widehat{\mathfrak{sl}}_2^{\mathrm{out}})/\tilde{J}$ are almost the same spaces: the only difference between them is the extra sum $\sum_{i+j=m}(f_i e_j + e_i f_j)$ in the third series of the quadratic relations (4.4.2). Therefore, it is likely that the characters of the two spaces coincide.

This argument can be easily generalized to the case of arbitrary k.

References

1. V. Schechtman and A. Varchenko, "Arrangements of hyperplanes and Lie algebra homology," Invent. Math., **106**, 139–194 (1991).
2. A. A. Beilinson and V. A. Ginzburg, "Infinitesimal structure of moduli spaces of G-bundles," Duke Math. J. (IMRN), No. 4, 63–74 (1992).
3. J. Lepowsky and M. Primc, "Structure of the standard modules for the affine Lie algebra $A_1^{(1)}$," Contemporary Mathematics, Vol. 46, AMS, Providence (1985).
4. V. G. Kac, Infinite-Dimensional Lie Algebras, Cambridge University Press, Cambridge (1990).
5. A. Pressley and G. Segal, Loop Groups, Clarendon Press, Oxford (1988).
6. M. Kashiwara, "The flag manifold of Kac–Moody Lie algebra," In: Algebraic Analysis, Geometry, and Number Theory, edited by J.-I. Igusa.
7. B. L. Feigin and E. V. Frenkel, "Coinvariants of nilpotent subalgebras of the Virasoro algebra and partition identities," Adv. in Soviet Math., Volume in Honor of I. M. Gel'fand (1993).
8. B. L. Feigin and E. V. Frenkel, "Affine Kac–Moody algebras and semi-infinite flag manifolds," Commun. Math. Phys., **128**, 161–189 (1990).
9. V. G. Drinfeld, "A new realization of Yangian and of quantum affine algebras," Dokl. Akad. Nauk SSSR, **296**, No. 1, 13–17 (1987).
10. B. L. Feigin and A. V. Stoyanovsky, Quasi-particle models for the representations of Lie algebras and the geometry of the flag manifold, Preprint, RIMS-942, Kyoto, September (1993).
11. A. Kuniba, T. Nakanishi, and J. Suzuki, Characters in conformal field theories from thermodynamic Bethe Ansatz, HUTP-92/A069 preprint, December (1992).

12. M. Terhoeven, Lift of Dilogarithm to Partition Identities, Preprint, Bonn (1992).
13. R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer, Fermionic quasi-particle representations for characters of $G_1^{(1)} \times G_1^{(1)}/G_2^{(1)}$, ITP preprint (1992).
14. S. Dasmahapatra, R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer, Quasi-particles, conformal field theory, and q-series, Preprint ITP-SB-93-12, RU-93-07.
15. V. B. Mehta and A. Ramanathan, "Frobenius splitting and cohomology vanishing for a Schubert variety," Ann. Math., **122**, 27–40 (1985).
16. H. H. Andersen, "Schubert varieties and Demazure character formula," Invent. Math., **79**, 611–618 (1985).
17. G. Andrews, The Theory of Partitions, Addison-Wesley (1976).
18. A. Grothendieck et al., SGA 2, North-Holland, Amsterdam (1968).
19. S. Kumar, "Demazure character formula in arbitrary Kac–Moody setting," Invent. Math., **89**, 395–423 (1987).
20. V. Lakshmibai and C. S. Seshadri, "Thèorie monomiale standard pour $\widehat{\mathfrak{sl}}_2$," C. R. Acad. Sci. Paris Ser. I, **305**, 183–185 (1987).

Translated by A. V. Stoyanovsky