Weil Representation and Norms of Gaussian Operators

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To I. M. Gel'fand on the occasion of his 80th birthday

Let G be a group, let T be a unitary representation of G in a Hilbert space H, and let $\Gamma(H)$ be the semigroup of all contractions (i.e., operators with norm ≤ 1) in H, equipped with the weak operator topology. Then, closing the image $T(G)$ of the group G in $\Gamma(H)$, we obtain a compact semigroup $\Gamma \subseteq \Gamma(H)$ with separately continuous multiplication; in particular, Γ is a compactification of the group G.

This simple construction is of special interest for infinite-dimensional groups, because it proves to be an important tool for studying representations of such groups and also because it furnishes us with interesting examples of compactifications. See the author's papers [17–19, 21], Neretin's papers [12, 14, 15], and the forthcoming paper by R. S. Ismagilov "On the irreducibility of representations of groups of measurable currents."

The present paper is devoted to a single model example: for G we take the infinite-dimensional real metaplectic group $Mp(\infty, \mathbb{R}) := \varinjlim Mp(n, \mathbb{R})$, and for T we take the Weil representation W of G in a boson Fock space with countably many degrees of freedom. The interest in the Weil representation W is related to the fact that this representation is one of the main "building blocks" for constructing representations of infinite-dimensional groups; see Olshanskii [20] and Neretin [13].

However, the main result of the paper (Theorem 3.5) deals with the Fock space $F(\mathbb{C}^N)$ with finitely many degrees of freedom and consists in computing the norm of the so-called Gaussian operators in $F(\mathbb{C}^n)$. These are integral operators with kernel of Gaussian type: the exponent of a quadratic form.

The formula for the norm of a Gaussian operator (see (3.3) and (3.10)) turns out to be not very simple; under supplementary restrictions on the operator it admits somewhat simpler versions (see (3.8), (3.13), and (3.14)). The knowledge of the norm allows us to describe the semigroup $\Gamma(n)$ of all Gaussian contractions in the Fock space $F(\mathbb{C}^n)$, $n = 1, 2, \ldots$, completely; finally, the semigroup Γ itself, as a set, is a projective limit: $\Gamma = \varprojlim \Gamma(n)$; see §4.

However, the constructive description of the semigroup Γ remains an open problem, and in §4 we state four related problems. It seems to me that these are interesting problems of operator theory. In Neretin's paper [12], for the semigroup Γ, two estimates "from below" are obtained. There also exists a p-adic version of the semigroup Γ, which compactifies the infinite-dimensional metaplectic group over a field of p-adic type. Nazarov [10] succeeded in obtaining a precise description of this semigroup. The situation over the field $\mathbb{R}$ is more complicated, and this seems to be caused by the fact that, over a p-adic field, there is no difference between operators of trace class and Hilbert–Schmidt operators.

Gaussian operators and their relationships with analysis are thoroughly examined in Howe's work [9]. However, this work has another orientation: the problems concerning norms and the compactification of the metaplectic group are not considered there. The paper [9] also deals with another realization of Gaussian operators, which is related to the "real" model of the Fock space but not to the "complex" one. The connection between the two realizations is discussed in Folland [5] and Hilgert [7]. Regarding Gaussian operators in L^p spaces, see Epperson [4] and the references therein.

The present publication, together with the cited works [10, 12] by M. L. Nazarov and Yu. A. Neretin, gives a detailed exposition of our joint note [11]. I am deeply grateful to my co-authors for numerous conversations and collaboration. The results of this paper, in their first version, were obtained by the author in the autumn of 1984.

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§1. Preliminaries

The proofs of the facts stated in this section can be found in the following sources: Berezin [2], Berezin and Shubin [3], Folland [5], and Neretin [12].

Denote by μ_n, $n = 1, 2, \dots$, the Gaussian measure on $\mathbb{C}^n$ with density $\pi^{-n}\exp(-z^* z)$ with respect to the Lebesgue measure. Here $z \in \mathbb{C}^n$ is regarded as a column vector and $z^* = \overline{z}^t$ as a row vector (the symbol $(\,\cdot\,)^t$ stands for transposition). The space $F(\mathbb{C}^n)$, formed by the entire functions on $\mathbb{C}^n$ contained in $L^2(\mathbb{C}^n, \mu_n)$, is closed in $L^2(\mathbb{C}^n, \mu_n)$ and so is itself a Hilbert space. It is called the *Bargmann–Segal space* and is a convenient model of the boson Fock space with n degrees of freedom. Note that $F(\mathbb{C}^n)$ is canonically isomorphic to $F(\mathbb{C}^1)^{\otimes n}$.

For any $z \in \mathbb{C}^n$, the function $f_z(w) := \exp(z^* w)$ lies in $F(\mathbb{C}^n)$ and has the property that $(f, f_z) = f(z)$ for all $f \in F(\mathbb{C}^n)$. Any bounded operator A in $F(\mathbb{C}^n)$ is uniquely determined by its *symbol* $K(z, w) := (Af_w, f_z)$ and can be written as an integral operator,

$$(Af)(z) = \int K(z, w) f(w) \mu_n(dw). \tag{1.1}$$

Moreover, the concept of a symbol and the representation (1.1) hold for a wide class of unbounded operators in $F(\mathbb{C}^n)$.

Let Ω be a complex symmetric matrix of size $2n \times 2n$ written in block form $\Omega = [\Omega_{ij}]$, where the indices i and j take the values 1 and 2 and each of the four blocks Ω_{ij} is of size $n \times n$. Let us assign to Ω the kernel $K(\Omega) = K(\Omega\,|\,z, w)$, where

$$K(\Omega\,|\,z, w) := \exp\left(\tfrac{1}{2}(z \oplus \overline{w})^t \Omega (z \oplus \overline{w})\right) = \exp\left(\tfrac{1}{2} z^t \Omega_{11} z + \tfrac{1}{2} w^* \Omega_{22} \overline{w} + z^t \Omega_{12} \overline{w}\right), \qquad z, w \in \mathbb{C}^n. \tag{1.2}$$

We are interested in operators in $F(\mathbb{C}^n)$ with symbol of the form $\mathrm{const} \cdot K(\Omega)$, which we will call *Gaussian operators* (a precise definition is given below).

Example 1.1. Let C be a matrix of size $n \times n$ with $\|C\| \le 1$, and let $\Omega(C) := \begin{bmatrix} 0 & C \\ C^t & 0 \end{bmatrix}$. The operator A_C of the change of a variable

$$(A_C f)(z) := f(C^t z), \qquad f \in F(\mathbb{C}^n),\ z \in \mathbb{C}^n, \tag{1.3}$$

is a bounded Gaussian operator with symbol $K(\Omega(C))$. Moreover, A_C is a contraction, and if the matrix C is unitary, then the operator A_C is also unitary.

Example 1.2. If $\|\Omega\| < 1$, then a bounded Gaussian operator $A(\Omega)$ with kernel (1.2) exists and is a Hilbert–Schmidt operator, since the kernel is square integrable. If $\widetilde{\Omega}$ is another matrix with $\|\widetilde{\Omega}\| < 1$, then

$$\operatorname{tr}(A(\Omega)^* A(\widetilde{\Omega})) = \det((1 - \Omega^* \widetilde{\Omega})^{-1/2}). \tag{1.4}$$

Proposition 1.3. *For the existence of a bounded Gaussian operator with symbol* (1.2), *the following conditions on the matrix* Ω *are necessary*:

$$\|\Omega\| \le 1, \quad \|\Omega_{11}\| < 1, \quad \|\Omega_{22}\| < 1. \tag{1.5}$$

Proof. Suppose that A is a bounded operator with symbol (1.2). Then the function $(Af_0)(z) = \exp(\frac{1}{2} z^t \Omega_{11} z)$ lies in $F(\mathbb{C}^n)$, whence $\|\Omega_{11}\| < 1$. Replacing A by A^*, we obtain $\|\Omega_{22}\| < 1$. Further, for any $x = z \oplus \overline{w} \in \mathbb{C}^{2n}$

$$|\exp(\tfrac{1}{2} x^t \Omega x)| = |(Af_w, f_z)| \le \|A\| \, \|f_w\| \, \|f_z\| = \|A\| \exp(\tfrac{1}{2} x^* x). \tag{1.6}$$

Reducing the matrix Ω to the diagonal form by the transformation $\Omega \mapsto U^t \Omega U$ with an appropriate unitary matrix U, we conclude from (1.6) that $\|\Omega\| \le 1$. □

In fact, conditions (1.5) are also sufficient. This will be proved in two ways, see Corollary 2.9 and Theorem 3.5. Neretin [12] proposed another approach, based on the fixed-point method.

Following Neretin [12], we introduce the dense subspace $F_0(\mathbb{C}^n)$ in $F(\mathbb{C}^n)$ spanned by the functions of the form

$$f_{\omega,a}(z) = \exp\left(\tfrac{1}{2}z^t\omega z + a^t z\right), \qquad z \in \mathbb{C}^n, \tag{1.7}$$

where $a \in \mathbb{C}^n$ and ω is a complex symmetric matrix of size $n \times n$ with $\|\omega\| < 1$. Denote by $S(n)$ the space of all complex symmetric matrices Ω of size $2n \times 2n$ that satisfy conditions (1.5).

Proposition 1.4 (Neretin [12]). *For any $\Omega \in S(n)$, there is an operator $A(\Omega)\colon F_0(\mathbb{C}^n) \to F_0(\mathbb{C}^n)$ well defined by the integral* (1.1) *with kernel $K(z,w) := K(\Omega|z,w)$.* □

It will be convenient for us to take $A(\Omega)$ as an initial definition of the Gaussian operator with kernel $K(\Omega|z,w)$, $\Omega \in S(n)$.

Set $V := \mathbb{C}^n \oplus \mathbb{C}^n$. We equip V with an indefinite inner product $\Phi(x,y)$ and a bilinear skew-symmetric form $B(x,y)$:

$$\Phi(x,y) := y^*Jx, \quad \text{where } J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B(x,y) := y^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x. \tag{1.8}$$

The forms Φ and B are related by the formula $\Phi(x,y) = iB(x, \operatorname{conj}(y))$, where conj stands for the antilinear involution on V sending a vector $y = y_1 \oplus y_2$ (where $y_1, y_2 \in \mathbb{C}^n$) into the vector $i(\overline{y}_2 \oplus \overline{y}_1)$. Denote by $V_{\mathbb{R}}$ the real form of the space V corresponding to the involution conj. Then the quadruple $(V, \Phi, B, \operatorname{conj})$ is uniquely determined by the real symplectic space $(V_{\mathbb{R}}, B)$. The group $\operatorname{Aut}(V_{\mathbb{R}}, B)$ is the real symplectic group $Sp(n, \mathbb{R})$ of rank n. On the other hand, the same group, viewed as the automorphism group of the triple (V, Φ, B), can be realized as $U(n,n) \cap Sp(n, \mathbb{C})$, where $U(n,n) := \operatorname{Aut}(V, \Phi)$ and $Sp(n, \mathbb{C}) := \operatorname{Aut}(V, B)$. Then we obtain the well-known *complex realization* of the group $Sp(n, \mathbb{R})$ by block matrices of type $g_{\alpha\beta} := \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix}$.

Proposition 1.5 (Berezin [2, Chap. II, (4.26)], Vergne [26]). *The Gaussian operators*

$$W_n(g_{\alpha\beta}) = \pm(\det\alpha)^{1/2} A(\Omega(\alpha,\beta)), \quad \textit{where } \Omega(\alpha,\beta) := \begin{bmatrix} \overline{\beta}\alpha^{-1} & (\alpha^{-1})^t \\ \alpha^{-1} & -\alpha^{-1}\beta \end{bmatrix}, \tag{1.9}$$

define a two-valued projective unitary representation of the group $Sp(n, \mathbb{R})$ in the space $F(\mathbb{C}^n)$. □

The representation W_n is called the *Weil representation* of the group $Sp(n, \mathbb{R})$. The function $g_{\alpha\beta} \mapsto \pm(\det\alpha)^{1/2}$ admits the choice of a single-valued branch when lifted to the two-fold covering $Mp(n, \mathbb{R}) \to Sp(n, \mathbb{R})$, which is called the *metaplectic group.* This allows us to interpret W_n as an ordinary unitary representation of the group $Mp(n, \mathbb{R})$ as well.

§2. Linear Relations and Orbits

As in §1, we put $V = \mathbb{C}^n \oplus \mathbb{C}^n$, where n takes the values $1, 2, \dots$. Vectors of V will be denoted by $x = x_1 \oplus x_2$ or $y_1 \oplus y_2$, where $x_i, y_i \in \mathbb{C}^n$. By definition, a *linear relation* $L\colon V \to V$ is an arbitrary linear subspace in $V \oplus V$. We will write vectors of $V \oplus V$ in the form

$$\xi = x_1 \oplus x_2 \oplus y_1 \oplus y_2, \quad \text{where } x_1, x_2, y_1, y_2 \in \mathbb{C}^n. \tag{2.1}$$

The permutation $x_1 \leftrightarrow y_1$ of the components x_1 and y_1 in (2.1) defines an involutive transformation $\mathcal{P}$ on the set of linear relations, which we will call the *Potapov–Ginzburg transformation*, cf. Azizov and Iokhvidov [1, Chap. V, §1]. The transformation $\mathcal{P}$ has a number of remarkable properties; in particular, see Proposition 2.1.

Any operator $T\colon V \to V$ can be interpreted as a linear relation if T is replaced by its *graph*

$$\operatorname{graph}(T) := \{x \oplus Tx \mid x \in V\} \subset V \oplus V. \tag{2.2}$$

Denote by $\mathrm{Contr}(V,J)$ the set of all maximal J-contracting linear relations $L\colon V \rightarrow V$. In other words, $L \in \mathrm{Contr}(V,J)$ if $\dim L = 2n$ and the Hermitian form $\Phi_- := \Phi \oplus (-\Phi)$ on $V \oplus V$ turns out to be nonnegative on L. In particular, if $T\colon V \to V$ is a J-contracting operator (i.e., $\Phi(Tx, Tx) \le \Phi(x,x)$ for all $x \in V$ or, in matrix terms, $J - T^*JT \ge 0$), then $\mathrm{graph}(T) \in \mathrm{Contr}(V,J)$.

By $\mathrm{Contr}(V)$ we denote the space of all contracting operators $T\colon V \to V$, i.e., operators with $\|T\| \le 1$. We will not distinguish between T and $\mathrm{graph}(T)$.

Proposition 2.1. *The Potapov–Ginzburg transformation establishes a bijection $\mathcal{P}\colon L \to T$ between* $\mathrm{Contr}(V,J)$ *and* $\mathrm{Contr}(V)$.

Proof. Let $L \in \mathrm{Contr}(V,J)$. Let us show that $\mathcal{P}(L)$ coincides with $\mathrm{graph}(T)$ for some operator T. For $\xi \in L$ we have

$$0 \le \Phi_-(\xi,\xi) = -\|x_1\|^2 + \|x_2\|^2 + \|y_1\|^2 - \|y_2\|^2 = (\|y_1\|^2 + \|x_2\|^2) - (\|x_1\|^2 + \|y_2\|^2). \tag{2.3}$$

Thus, $y_1 \oplus x_2 = 0$ implies $x_1 \oplus y_2 = 0$. Since $\dim L = 2n$, this means that the vector $(y_1 \oplus x_2) \oplus (x_1 \oplus y_2)$ runs over the graph of an operator $T\colon V \to V$ if ξ runs over L. The equivalence of the conditions $L \in \mathrm{Contr}(V,J)$ and $T \in \mathrm{Contr}(V)$ is obvious (due to (2.3)). □

Let us denote by V' and V'' the first and the second copies of the space V in the direct sum $V \oplus V$; sometimes we will identify them with V itself. For $L \in \mathrm{Contr}(V,J)$ we put $\mathrm{Ker}\, L := L \cap V'$ and $\mathrm{Ind}\, L := L \cap V''$. By the condition $\Phi_-|L \ge 0$ we have $\Phi|\,\mathrm{Ker}\, L \ge 0$ and $\Phi|\,\mathrm{Ind}\, L \le 0$. We will write the contracting operator $T = \mathcal{P}(L)$ in the block form $T = [T_{ij}]$, where $i, j = 1, 2$. Let us introduce another Hermitian form $\Phi_+ := \Phi \oplus \Phi$ on $V \oplus V$.

Proposition 2.2. *Let $L \in \mathrm{Contr}(V,J)$ and let $T = [T_{ij}]$ be the corresponding operator from* $\mathrm{Contr}(V)$. *The following conditions are equivalent*:

(i) $\Phi|\,\mathrm{Ker}\, L > 0$ *and* $\Phi|\,\mathrm{Ind}\, L < 0$;
(ii) $\|T_{12}\| < 1$ *and* $\|T_{21}\| < 1$;
(iii) *the form $\Phi_+|L$ is nondegenerate.*

Proof. (i) $\Longrightarrow$ (ii). Let us show that $\|T_{12}\| < 1$. Since $\|T\| \le 1$, we have $T_{12}^*T_{12} + T_{22}^*T_{22} \le 1$. Therefore, it suffices to verify that $\|T_{12}a\| < \|a\|$ for any nonzero $a \in \mathbb{C}^n$ such that $T_{22}a = 0$. Let us assume that the vector ξ, written in the form (2.1), ranges over L. Then T transforms $y_1 \oplus x_2$ into $x_1 \oplus y_2$. Put $y_1 = 0$ and $x_2 = a$. Then $y_2 = T_{21}y_1 + T_{22}x_2 = 0$. Therefore, $\xi \in \mathrm{Ker}\, L$ and $\xi \ne 0$. Due to (i), $\Phi(\xi,\xi) > 0$, i.e., $\|x_1\| < \|x_2\|$, and this means $\|T_{12}a\| < \|a\|$. In the same way, setting $x_2 = 0$ and $y_1 = b$, where $T_{11}b = 0$, $b \ne 0$, we check that $\|T_{21}\| < 1$.

(ii) $\Longrightarrow$ (iii). Consider the n-dimensional subspaces L^- and L^+ in L that are distinguished by the conditions $x_2 = 0$ and $y_1 = 0$, respectively. Now we will show that L^- is strictly negative and L^+ is strictly positive with respect to the form Φ_+. Since $\dim L = 2n$, this will immediately imply that L is nondegenerate.

Let $\xi \in L^-$ and $\xi \ne 0$. Then $x_2 = 0$, $y_1 \ne 0$, and, by (ii), $\|y_2\| = \|T_{21}y_1\| < \|y_1\|$. Together with the definition of the form Φ_+, this implies $\Phi_+(\xi,\xi) < 0$. Similarly it is checked that $\Phi_+(\xi,\xi) > 0$ for all nonzero $\xi \in L^+$.

(iii) $\Longrightarrow$ (i). Since $\Phi|\,\mathrm{Ker}\, L = \Phi_-|\,\mathrm{Ker}\, L \ge 0$, in order to prove that $\Phi|\,\mathrm{Ker}\, L > 0$, it suffices to check that $\xi \in \mathrm{Ker}\, L$ and $\Phi_-(\xi,\xi) = 0$ imply $\xi = 0$. Now note that $\Phi_-(\xi,\eta) = 0$ for all $\eta \in L$, because $\Phi_-|L \ge 0$. But, since ξ has the form $x \oplus 0$, this also means that $\Phi_+(\xi,\eta) = 0$ for all $\eta \in L$. By (iii), we conclude that $\xi = 0$. The property $\Phi|\,\mathrm{Ind}\, L < 0$ is checked similarly. □

Denote by $\Lambda(n)$ the set of all linear relations $L \in \mathrm{Contr}(V,J)$ that satisfy the equivalent conditions (i)–(iii) of Proposition 2.2 and are also Lagrangian subspaces in $V \oplus V$ with respect to the form $B_- := B \oplus (-B)$.

Proposition 2.3. *$\Lambda(n)$ is a semigroup with respect to the multiplication of linear relations.*

Proof. Recall that the product $N = LM$ of the linear relations L and M consists of those vectors $x \oplus z \in V \oplus V$ for which there exists a vector $y \in V$ such that $x \oplus y \in M$ and $y \oplus z \in L$; this operation generalizes the multiplication of linear operators. Now let $L, M \in \Lambda(n)$. Then it is easily verified that N is a nonnegative subspace with respect to the form Φ_-, isotropic with respect to the form B_-, and satisfies condition (i) of Proposition 2.2. The only nonevident property is $\dim N = 2n$. We can derive this from the fact that L and M are Lagrangian relations; see Guillemin and Sternberg [6, (9.6)]. Another reasoning is as follows. Let us put $S = \mathcal{P}(L)$ and $T = \mathcal{P}(M)$, and let us show that $\mathcal{P}(N) = \operatorname{graph}(U)$, where $U\colon V \to V$ is a linear operator. To do this, it suffices to check that, for any fixed $z_1, x_2 \in \mathbb{C}^n$, the following system of linear equations on the vector $y = y_1 \oplus y_2 \in V$ is solvable:

$$T_{21}y_1 + T_{22}x_2 = y_2, \qquad S_{11}z_1 + S_{12}y_2 = y_1. \tag{2.4}$$

But this fact is evident if one takes into account that $\|T_{21}\| < 1$ and $\|S_{12}\| < 1$. $\square$

Proposition 2.4. *Let $\mathcal{P}'$ stand for the transformation that assigns the matrix* $\Omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} T$, *where* $T := \mathcal{P}(L)$, *to a linear relation* $L \in \operatorname{Contr}(V, J)$. *Then*

(i) *$\mathcal{P}'$ establishes a bijection $\Lambda(n) \to S(n)$;*

(ii) *the transfer of the semigroup structure from $\Lambda(n)$ to $S(n)$ by means of $\mathcal{P}'$ leads to the following multiplication in $S(n)$:*

$$\Omega * \widetilde{\Omega} = \begin{bmatrix} \Omega_{11} + \Omega_{12}\widetilde{\Omega}_{11}(1 - \Omega_{22}\widetilde{\Omega}_{11})^{-1}\Omega_{21} & \Omega_{12}(1 - \widetilde{\Omega}_{11}\Omega_{22})^{-1}\widetilde{\Omega}_{12} \\ \widetilde{\Omega}_{21}(1 - \Omega_{22}\widetilde{\Omega}_{11})^{-1}\Omega_{21} & \widetilde{\Omega}_{21}\Omega_{22}(1 - \widetilde{\Omega}_{11}\Omega_{22})^{-1}\Omega_{12} + \Omega_{22} \end{bmatrix}. \tag{2.5}$$

Proof. (i) Conditions (1.5) characterizing the matrices $\Omega \in S(n)$ rephrase the condition $\|T\| \le 1$ from Proposition 2.1 and condition (ii) of Proposition 2.2. Finally, the symmetry condition $\Omega^t = \Omega$ is equivalent to the condition $B_- | L = 0$.

(ii) This is verified by direct computation. $\square$

Proposition 2.5. *For $\Omega, \widetilde{\Omega} \in S(n)$, we have*

$$A(\Omega)A(\widetilde{\Omega}) = \det((1 - \Omega_{22}\widetilde{\Omega}_{11})^{-1/2})A(\Omega * \widetilde{\Omega}). \tag{2.6}$$

Proof. A direct calculation based on a formula for the Gaussian integral (Berezin [2, Chap. I, (2.16)]) shows that the application of both sides of formula (2.6) to the vector $f_{\omega,a} \in F_0(\mathbb{C}^n)$ gives the same result. $\square$

Thus, the multiplication (2.5) in $S(n)$ has two natural interpretations: on the one hand, it corresponds to the multiplication of linear relations, and on the other hand, it corresponds to that of Gaussian operators. This fact also means that the mapping $L \mapsto A(\mathcal{P}'(L))$ defines a projective representation of the semigroup $\Lambda(n)$ by Gaussian operators.

Let us consider the group $Sp(n, \mathbb{R})$ in its complex realization (see §1).

Proposition 2.6. (i) *The mapping $g \mapsto \operatorname{graph}(g)$ specifies an isomorphism of the group $Sp(n, \mathbb{R})$ onto the group of invertible operators of the semigroup $\Lambda(n)$.*

(ii) *The image of the group $Sp(n, \mathbb{R})$ in $S(n)$ under the mapping $\mathcal{P}' \circ \operatorname{graph}$ consists of all unitary symmetric matrices Ω with $\det \Omega_{12} = \det \Omega_{21} \neq 0$.*

(iii) *The operator $W_n(g)$, where $g \in Sp(n, \mathbb{R})$, is proportional to the Gaussian operator $A(\mathcal{P}'(\operatorname{graph}(g)))$.*

Proof. (i) This is trivial.

(ii) This is trivial, too. Note that the condition $\det\Omega_{12} = \det\Omega_{21} \neq 0$, which means that Ω comes from the graph of an operator, implies the conditions $\|\Omega_{11}\| < 1$ and $\|\Omega_{22}\| < 1$, because Ω is unitary.

(iii) It suffices to check that $\Omega(\alpha,\beta) = \mathcal{P}'(\mathrm{graph}(g_{\alpha\beta}))$, in the notation of (1.9). □

By Proposition 2.6, the group $Sp(n,\mathbb{R})$ acts by two-sided translations on the semigroup $\Lambda(n) \cong S(n)$. Now we will deal with classifying orbits of this two-sided action. The heart of this problem (if one digresses from the isotropy condition $B_-|L = 0$) consists in reducing to a canonical form a couple (Φ_-, Φ_+) of Hermitian forms, defined on a finite-dimensional linear space L, such that Φ_- is nonnegative and Φ_+ is nondegenerate. Precisely these conditions play a crucial role in simplifying the classification problem.

A linear relation $L \in \Lambda(n)$ will be called *nondegenerate* if $\operatorname{Ker} L = 0$ and $\operatorname{Ind} L = 0$. Nondegenerate elements $L \in \Lambda(n)$ are the graphs of the operators $g \in Sp(n,\mathbb{C})$ that are J-contractions (i.e., $J - g^*Jg \geq 0$). They form a subsemigroup in $Sp(n,\mathbb{C})$, which will be denoted by $Sp_{\leq}(n,\mathbb{C})$. For more detail about semigroups of this kind, see Olshanskii [16] and Hilgert and Neeb [8]. Note that the image of the semigroup $Sp_{\leq}(n,\mathbb{C})$ in $S(n)$ consists of all symmetric matrices Ω of size $2n \times 2n$ such that $\|\Omega\| \leq 1$ and $\det\Omega_{12} = \det\Omega_{21} \neq 0$.

A linear relation $L \in \Lambda(n)$ will be called *totally degenerate* if $L = \operatorname{Ker} L \oplus \operatorname{Ind} L$. In other words, L is specified by a couple of transversal B-isotropic n-dimensional subspaces in V, one of which is strictly positive and another of which is strictly negative with respect to the form Φ. In terms of matrices Ω, this means that $\Omega_{12} = \Omega_{21} = 0$, $\|\Omega_{11}\| < 1$, and $\|\Omega_{22}\| < 1$. The totally degenerate elements $L \in \Lambda(n)$ form a single orbit under the action of the group $Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R})$; any such element can be reduced to the following canonical form: $\operatorname{Ker} L = \{0 \oplus x_2\}$ and $\operatorname{Ind} L = \{y_1 \oplus 0\}$, where x_2 and y_1 range over $\mathbb{C}^n$.

For arbitrary elements $K \in \Lambda(k)$ and $M \in \Lambda(m)$, where $k, m = 1, 2, \dots$, the direct sum $L = K \oplus M$, which is an element of the semigroup $\Lambda(k+m)$, is defined in an obvious way. In terms of block matrices Ω, this operation $\oplus$ means that any of the four blocks $(\mathcal{P}'(L))_{ij}$, where $i, j = 1, 2$, equals the direct sum $(\mathcal{P}'(K))_{ij} \oplus (\mathcal{P}'(M))_{ij}$.

Proposition 2.7. *Under the action of the group $Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R})$, an arbitrary element $L \in \Lambda(n)$ is reduced to the form $K \oplus M$, where K is a totally degenerate element of $\Lambda(k)$, M is a nondegenerate element of $\Lambda(m)$, and $k + m = n$.*

(Of course, it is possible that K or M equals $\{0\}$, i.e., L itself is already a nondegenerate or totally degenerate element.)

Proof. *Step 1.* Recall that V' and V'' stand for the first and the second copies of the space V in $V \oplus V$. For $L \in \Lambda(n)$ let us denote by $\operatorname{Dom} L$ and $\operatorname{Ran} L$ its projections to V' and V'', respectively, and by L^* its orthogonal complement in $V \oplus V$ with respect to the form Φ_-. Then $\operatorname{Dom} L$ coincides with the orthogonal complement of $\operatorname{Ker} L^* := L^* \cap V'$ with respect to the form Φ on $V' = V$, and $\operatorname{Ran} L$ coincides with the orthogonal complement of $\operatorname{Ind} L^* := L^* \cap V''$ with respect to the form Φ on $V'' = V$.

Indeed, e.g., let us check the first claim. It is evident that $\operatorname{Dom} L$ is orthogonal to $\operatorname{Ker} L^*$. Conversely, let us assume that a vector $x \in V'$ is orthogonal to $\operatorname{Ker} L^*$. Then there exists a linear functional on $L^* + V'$ that vanishes on L^* and coincides with the functional $\Phi(\,\cdot\,, x)$ on V'. Let us extend it to a linear functional on the whole space $V' \oplus V''$ and write it in the form $\Phi_-(\,\cdot\,, \xi)$. Then ξ lies in L and has the form $x \oplus y$ with some $y \in V''$, that is, $x \in \operatorname{Dom} L$. The second claim is verified in exactly the same way.

Step 2. Let us show that $\operatorname{Dom} L$ and $\operatorname{Ran} L$ are nondegenerate with respect to the form Φ on $V' = V'' = V$. By Step 1, it suffices to prove a similar statement for $\operatorname{Ker} L^*$ and $\operatorname{Ind} L^*$. Note that L^* is a maximal nonpositive subspace with respect to Φ_-, because L is maximal nonnegative. Further, we also note that L^* is nondegenerate with respect to Φ_+: indeed, denote by $L^\perp$ the orthogonal complement to L with respect to Φ_+; then $L^\perp$ is nondegenerate together with L; on the other hand, by the definition of the forms Φ_- and Φ_+, L^* is transformed into $L^\perp$ by the operator $x \oplus y \mapsto x \oplus (-y)$; since this operator preserves all forms, we conclude that L^* is nondegenerate with respect to Φ_+, together with $L^\perp$. Thus, we have shown that L^* is both a maximal nonnegative subspace with respect to $(-\Phi)_-$ and a nondegenerate subspace with respect to $(-\Phi)_+$. By Proposition 2.2, $\Phi|\operatorname{Ker} L^* < 0$ and $\Phi|\operatorname{Ind} L^* > 0$. In particular, $\operatorname{Ker} L^*$ and $\operatorname{Ind} L^*$ are nondegenerate.

Step 3. Since all four subspaces $\operatorname{Ker} L$, $\operatorname{Dom} L$, $\operatorname{Ind} L$, and $\operatorname{Ran} L$ are nondegenerate with respect to the form Φ, there exist Φ-orthogonal decompositions $V' = U' \oplus W'$ and $V'' = U'' \oplus W''$, where

$$\begin{aligned} U' &:= \operatorname{Ker} L + \operatorname{Ker} L^*, \qquad & W' &:= \operatorname{Dom} L \ominus \operatorname{Ker} L, \\ U'' &:= \operatorname{Ind} L + \operatorname{Ind} L^*, \qquad & W'' &:= \operatorname{Ran} L \ominus \operatorname{Ind} L. \end{aligned} \tag{2.7}$$

Further, due to the relation between Φ, B, and conj and due to the fact that L is Lagrangian, the space L^* coincides with $(\mathrm{conj} \oplus \mathrm{conj})(L)$. In particular, $\operatorname{Ker} L^* = \mathrm{conj}(\operatorname{Ker} L)$ and $\operatorname{Ind} L^* = \mathrm{conj}(\operatorname{Ind} L)$. Thus, U' and U'' are stable with respect to the involution conj, and then their orthogonal complements W' and W'' also have the same property.

We put $K := \operatorname{Ker} L \oplus \operatorname{Ind} L$ and $M := L \cap (W' \oplus W'')$. Then L coincides with $K \oplus M$, because K is contained in L and is a nondegenerate subspace with respect to $\Phi_\pm$.

Step 4. It is clear that $M \cap W' = \{0\}$ and $M \cap W'' = \{0\}$. On the other hand, $\dim M = \frac{1}{2} \dim(W' \oplus W'')$. It follows that $\dim W' = \dim W''$ and, hence, $\dim U' = \dim U''$. Acting by the group $Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R})$, we can match the space $U' \oplus U''$ with the subspace $(\mathbb{C}^k \oplus \mathbb{C}^k) \oplus (\mathbb{C}^k \oplus \mathbb{C}^k)$ in $V' \oplus V'' = (\mathbb{C}^n \oplus \mathbb{C}^n) \oplus (\mathbb{C}^n \oplus \mathbb{C}^n)$, where $k = \dim U' = \dim U''$. This means that $L = K \oplus M$ is contained in the image of the semigroup $\Lambda(k) \times \Lambda(n-k)$ under its canonical embedding into $\Lambda(n)$. It is clear that K is totally degenerate and M is nondegenerate. □

Theorem 2.8. *Under the action of the group $Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R})$, any linear relation $L \in \Lambda(n)$ is reduced to a canonical form, which is the direct sum of linear relations from $\Lambda(1)$ of the following four types:*

1) $L^{(1)} = \{0 \oplus x_2 \oplus y_1 \oplus 0\} \subset \mathbb{C}^4$; $\mathcal{P}'(L^{(1)}) = 0$;
2) $L_s^{(2)}$ *is the graph of the operator* $\mathbb{C}^2 \to \mathbb{C}^2$ *given by the matrix* $\begin{bmatrix} s^{-1} & 0 \\ 0 & s \end{bmatrix}$, *where* $0 < s < 1$; $\mathcal{P}'(L_s^{(2)}) = \begin{bmatrix} 0 & s \\ s & 0 \end{bmatrix}$;
3) $L_a^{(3)}$ *is the graph of the operator* $\mathbb{C}^2 \to \mathbb{C}^2$ *given by the matrix* $1 + (a^{-1} - 1)\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, *where* a *is any fixed number from* $(0,1)$; $\mathcal{P}'(L_a^{(3)}) = \begin{bmatrix} a-1 & a \\ a & a-1 \end{bmatrix}$;
4) $L^{(4)}$ *is the graph of the identity operator* $\mathbb{C}^2 \to \mathbb{C}^2$; $\mathcal{P}'(L^{(4)}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Proof. By Proposition 2.7, we can assume that L is nondegenerate, i.e., L is the graph of some operator $T \in Sp_{\le}(n,\mathbb{C})$. Then we must prove that L can be reduced to the direct sum of elements of type $L_s^{(2)}$, $L_a^{(3)}$, and $L^{(4)}$. It is known (Potapov [23, Chap. II]) that any J-contracting invertible operator T can be uniquely written in the form $T = g \exp H$, where $g \in U(n,n)$ and H is a J-self-adjoint J-nonpositive operator, i.e., $JH = (JH)^* \le 0$. Now let $T \in Sp_{\le}(n,\mathbb{C})$. Then, using the involution that singles out the subgroup $Sp(n,\mathbb{C}) \subset GL(n,\mathbb{C})$, we get $g \in Sp(n,\mathbb{R})$ and $H \in i\mathfrak{sp}(n,\mathbb{R})$, where $\mathfrak{sp}(n,\mathbb{R})$ stands for the Lie algebra of the Lie group $Sp(n,\mathbb{R})$. Moreover, H belongs to the convex $Sp(n,\mathbb{R})$-invariant cone $C_n \subset i\mathfrak{sp}(n,\mathbb{R})$ formed by the J-nonpositive operators in $i\mathfrak{sp}(n,\mathbb{R})$. Thus, our problem is reduced to the well-known problem of classifying $Sp(n,\mathbb{R})$-orbits in the cone f_n, see Paneitz [22, Lemma 7.1]. (This classification can be also obtained by Potapov's methods [23].) □

Corollary 2.9. *Conditions* (1.5) *from Proposition* 1.3 *are not only necessary but also sufficient for a Gaussian operator* $A(\Omega)$ *to be bounded.*

Proof. By Proposition 1.5 and Theorem 2.8, one may assume that Ω has the canonical form. Further, using the isomorphism $F(\mathbb{C}^n) = F(\mathbb{C}^1)^{\otimes n}$, we reduce our statement to the case $\Omega = \mathcal{P}(L)$, where $L \in \Lambda(1)$ is one of the elements listed in Theorem 2.8. For elements of the first, second, and fourth types, the verification of the boundedness of the operator $A(\Omega)$ is trivial. For $L = L_a^{(3)}$ this is slightly more difficult, and here one can use, e.g., a construction which is given below in the proof of Theorem 4.2, Step 2.

§3. Computing Norms of Gaussian Operators

Consider the subsemigroup $\Lambda_0(n) := \{L \in \Lambda(n) \,|\, \Phi_-|L > 0\}$ in $\Lambda(n)$. It is invariant with respect to the action of the group $Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R})$, and in the canonical representation of the elements $L \in \Lambda_0(n)$ only components of type $L^{(1)}$ or $L_s^{(2)}$ can occur. The transformation $\mathcal{P}'$ maps $\Lambda_0(n)$ into the subsemigroup $S_0(n) := \{\Omega \in S(n) \mid \|\Omega\| < 1\}$. Under the action of the group $Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R})$, any matrix $\Omega \in S_0(n)$ can be transformed to the canonical form

$$\Omega(s_1,\dots,s_n) := \begin{bmatrix} 0 & \operatorname{diag}(s_1,\dots,s_n) \\ \operatorname{diag}(s_1,\dots,s_n) & 0 \end{bmatrix}, \qquad 0 \le s_1,\dots,s_n < 1, \tag{3.1}$$

where $\operatorname{diag}(s_1,\dots,s_n)$ stands for the diagonal matrix with diagonal entries $s_1,\dots,s_n$. This is a particular case of the matrices $\Omega(C)$ from Example 1.1. Note that possible zeros among the numbers $s_1,\dots,s_n$ correspond to components of type $L^{(1)}$.

Proposition 3.1. *Let us assign to an arbitrary matrix $\Omega \in S(n)$ the pencil of Hermitian matrices $(1-\Omega^*\Omega) - \lambda(J - \Omega^* J \Omega)$. Then the roots of the characteristic equation*

$$\det((1-\Omega^*\Omega) - \lambda(J - \Omega^* J\Omega)) = 0 \tag{3.2}$$

are constant on the orbits of the two-sided action of the group $Sp(n,\mathbb{R})$. If $\Omega = \Omega(s_1,\dots,s_n)$, see (3.1), then the roots have the form $\lambda = \pm(1-s_i^2)(1+s_i^2)^{-1}$, $1 \le i \le n$.

Proof. Consider the pencil of Hermitian forms $(\Phi_-|L) - \lambda(\Phi_+|L)$ on the space $L := (\mathcal{P}')^{-1}(\Omega)$. Equip L with a basis by making use of the projection $\xi \mapsto y_1 \oplus x_2$, which gives an isomorphism $L \to \mathbb{C}^{2n}$ (see the proof of Proposition 2.1). In this basis the forms $\Phi_-|L$ and $\Phi_+|L$ will be represented by the matrices $1-\Omega^*\Omega$ and $J - \Omega^* J\Omega$, respectively. Since the spectrum of the pencil $(\Phi_-|L) - \lambda(\Phi_+|L)$ is clearly invariant with respect to $Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R})$, we obtain the first claim of the proposition. The second claim is checked by a simple calculation. □

By Proposition 3.1, the (unordered) collection of numbers $s_1,\dots,s_n$, which arises as the result of reducing a matrix $\Omega \in S_0(n)$ to the canonical form, is a single-valued function of Ω.

Recall that for $\Omega \in S_0(n)$, the Gaussian operator $A(\Omega)$ is a nonzero Hilbert–Schmidt operator and, hence, a bounded operator.

Proposition 3.2. *Let $\Omega \in S_0(n)$. Then*

$$\|A(\Omega)\|^{-1} = \det((1-\Omega^* J\Omega J)^{1/4}) \prod_{i=1}^{n} (1+s_i^2)^{-1/2}. \tag{3.3}$$

Proof. *Step 1.* In the notation of Example 1.1, we put $I = A_{-1}$, where 1 denotes the unit matrix of size $n \times n$. It is easily verified that $IA(\Omega) = A(\Omega)I = A(J\Omega J)$ for all $\Omega \in S(n)$. Now, let $\Omega \in S_0(n)$. Then, by (1.4),

$$\operatorname{tr}(A(\Omega)^* I A(\Omega)) = \det((1-\Omega^* J\Omega J)^{-1/2}). \tag{3.4}$$

Step 2. Let us show that the function

$$\operatorname{tr}(A(\Omega)^* I A(\Omega))\, \|A(\Omega)\|^{-2}, \qquad \Omega \in S_0(n), \tag{3.5}$$

is constant on the orbits of the group $Sp(n,\mathbb{R})\times Sp(n,\mathbb{R})$. Indeed, note that the operator I commutes with the Weil representation W_n (this operator just determines the decomposition of W_n into two irreducible components). Therefore, expression (3.5) will not change after replacing the operator $A(\Omega)$ by the operator $W_n(g_1)A(\Omega)W_n(g_2)^{-1}$, where g_1 and g_2 are arbitrary elements of $Sp(n,\mathbb{R})$. On the other hand, by Propositions 2.5 and 2.6(iii), we have

$$W_n(g_1)A(\Omega)W_n(g_2)^{-1} = \text{const} \cdot A(\widetilde{\Omega}), \tag{3.6}$$

where $\widetilde{\Omega} \subset S_0(n)$ is the result of the action on the matrix Ω by the element (g_1, g_2) of the group $Sp(n, \mathbb{R}) \times Sp(n, \mathbb{R})$. Let us substitute (3.6) into (3.5) for the operator $A(\Omega)$. Then under the trace symbol and under the norm symbol the factors $|\,\mathrm{const}\,|^2$ will appear, which will cancel each other, so that the result will come to replacing Ω by $\widetilde{\Omega}$, and this only means the invariance of the function (3.5).

Step 3. Let us show that expression (3.5) equals $\prod(1+s_i^2)^{-1}$. Indeed, by the result of Step 2, we can calculate (3.5) by substituting $\Omega = \Omega(s_1, \dots, s_n)$ into this expression. Then the operator $A(\Omega)$ will turn into the operator A_C from Example 1.1, where the matrix C equals $\mathrm{diag}(s_1, \dots, s_n)$. It is easy to verify that the norm of this operator equals 1. Furthermore, the operator $A(\Omega)^* I A(\Omega)$ turns into the operator A_{-C^2}, whose trace is easily calculated, and the result is equal to $\prod(1+s_i^2)^{-1}$.

Step 4. Consider the identical transformation

$$\|A(\Omega)\|^{-2} = \frac{\mathrm{tr}(A(\Omega)^* I A(\Omega))}{\|A\|^2} \, \frac{1}{\mathrm{tr}(A(\Omega)^* I A(\Omega))}. \tag{3.7}$$

By Step 3, the first factor equals $\prod(1+s_i^2)^{-1}$.By Step 1, the second factor is given by formula (3.4). Finally, we obtain (3.3). □

Remark 3.3. The same reasoning, but without using the operator I, leads to another formula for the norm:

$$\|A(\Omega)\|^{-1} = \det((1-\Omega^*\Omega)^{1/4}) \prod_{i=1}^{n} (1-s_i^2)^{-1/2}, \qquad \Omega \in S_0(n). \tag{3.8}$$

At first sight, (3.8) is simpler than (3.3). However, as we will see now, formula (3.3) holds for all $\Omega \in S(n)$, not only for $\Omega \in S_0(n)$, while the substitution in (3.8) of a matrix Ω, which does not lie in $S_0(n)$, leads to an uncertainty of type $0/0$.

As was mentioned in the proof of Proposition 3.1, the matrix $J - \Omega^* J \Omega$ describes the form $\Phi_+|L$ in an appropriate basis of the space L. Since the form $\Phi_+|L$ is nondegenerate for all $L \in \Lambda(n)$, the matrix $J - \Omega^* J \Omega$ is invertible for all $\Omega \in S(n)$. Now let us assign to $\Omega \in S(n)$ the Hermitian matrix

$$X(\Omega) := (1-\Omega^*\Omega)^{1/2} (J - \Omega^* J \Omega)^{-1} (1-\Omega^*\Omega)^{1/2} \tag{3.9}$$

and set $|X(\Omega)| = \sqrt{X(\Omega)^2}$.

Let us denote by $\overline{S}(n)$ the closed matrix ball $\{\Omega \mid \|\Omega\| \le 1\}$ in the space of complex symmetric matrices of size $2n \times 2n$.

Proposition 3.4. *Define a function $\varphi_n \ge 0$ on $\overline{S}(n)$ by putting $\varphi_n(\Omega) = 0$ for $\Omega \in \overline{S}(n) \setminus S(n)$ and*

$$\varphi_n(\Omega) = |\det(J - \Omega^* J \Omega)|^{1/4} \det((1 + |X(\Omega)|)/2)^{1/2} \tag{3.10}$$

for $\Omega \in S(n)$. The function φ_n is continuous on $\overline{S}(n)$ and vanishes precisely on the set $\overline{S}(n) \setminus S(n)$. Further, if $\Omega \in S_0(n)$, then $\|A(\Omega)\|^{-1} = \varphi(\Omega)$.

Proof. The first factor in (3.10) is a continuous function on $\overline{S}(n)$ vanishing precisely on the set $\overline{S}(n) \setminus S(n)$. The second factor is well defined on $S(n)$ and is a continuous, bounded, nowhere vanishing function on this set. Since $S(n)$ is open in $\overline{S}(n)$, this gives the first claim of the proposition.

Let us show that the right-hand side of (3.3) coincides with $\varphi_n(\Omega)$ for $\Omega \in S_0(n)$. It is clear from (3.3) that $\det(1 - \Omega^* J \Omega J) > 0$. Therefore, this determinant coincides with $|\det(J - \Omega^* J \Omega)|$. Further, the roots of Eq. (3.2) are precisely the eigenvalues of the matrix $X(\Omega)$. By Proposition 3.1, these eigenvalues have the form $\pm(1-s_i^2)(1+s_i^2)^{-1}$, $1 \le i \le n$. Thus, the spectrum of the matrix $(1 + |X(\Omega)|)/2$ consists of the numbers $(1+s_i^2)^{-1}$ taken with multiplicity 2. Therefore, the second factor in (3.10) equals the second factor in (3.3). □

Theorem 3.5. *For any matrix $\Omega \in S(n)$, the Gaussian operator $A(\Omega)$ is bounded and $\|A(\Omega)\|^{-1} = \varphi_n(\Omega)$, where the function φ_n is defined in Proposition* 3.4.

It should be emphasized that the reasoning given below does not use the proof of sufficiency of conditions (1.5) for the boundedness of the operator $A(\Omega)$, given in Corollary 2.9. Thus, Theorem 3.5 provides an independent proof of this fact. Combining Theorem 3.5 with Proposition 3.4, we obtain that the function $S(n) \ni \Omega \mapsto \|A(\Omega)\|^{-1}$, being extended by 0 outside $S(n)$, is continuous on the whole closed ball $\overline{S}(n)$.

Proof. For $0 < \varepsilon < 1$, we consider the operators $A_\varepsilon = A_{\varepsilon \cdot 1}$ in $F(\mathbb{C}^n)$; see Example 1.1. The operators A_ε form a one-parameter semigroup of self-adjoint contractions in the space $F(\mathbb{C}^n)$. They preserve the dense subspace $F_0(\mathbb{C}^n)$, and $A_\varepsilon A(\Omega) A_\varepsilon = A(\varepsilon^2\Omega)$ for all $\Omega \in S(n)$. Note that $\varepsilon^2\Omega \in S_0(n)$ for all $\Omega \in S(n)$. Finally, A_ε strongly converges to 1 as $\varepsilon \nearrow 1$. Using these properties, we obtain for an arbitrary vector $f \in F_0(\mathbb{C}^n)$ (as $\varepsilon \nearrow 1$)

$$\begin{aligned} \|A(\Omega) f\| &= \lim \|A_\varepsilon A(\Omega) A_\varepsilon f\| = \lim \|A(\varepsilon^2\Omega) f\| \\ &\leq \lim \varphi_n(\varepsilon^2\Omega)^{-1} \|f\| = \varphi_n(\Omega)^{-1} \|f\|, \qquad \Omega \in S(n), \end{aligned} \tag{3.11}$$

by continuity of the function φ_n. Hence, $A(\Omega)$ is bounded for any $\Omega \in S(n)$, and $\|A(\Omega)\| \leq \varphi_n(\Omega)^{-1}$.

On the other hand, since A_ε is a contraction, for any $\varepsilon \in (0,1)$ we have

$$\|A(\Omega)\| \geq \|A_\varepsilon A(\Omega) A_\varepsilon\| = \|A(\varepsilon^2\Omega)\| = \varphi_n(\varepsilon^2\Omega)^{-1}, \tag{3.12}$$

whence $\|A(\Omega)\| \geq \varphi_n(\Omega)^{-1}$. Thus, $\|A(\Omega)\| = \varphi_n(\Omega)^{-1}$. □

Remark 3.6. Let us state without proof two more formulas for the norm of the Gaussian operator $A(\Omega)$. The first formula is valid for all $\Omega \in S(n)$ corresponding to elements of the semigroup $Sp_{\leq}(n, \mathbb{C})$:

$$\|A(\Omega)\|^{-1} = |\det \Omega_{12}|^{1/2} \prod_{i=1}^{n} s_i(\Omega)^{-1/2}. \tag{3.13}$$

The second formula is valid for $n = 1$ and all $\Omega \in S(1)$:

$$\|A(\Omega)\|^{-1} = \left(\frac{(\det(1 - \Omega^*\Omega))^{1/2} + (\det(1 - \Omega^* J \Omega J))^{1/2}}{2} \right)^{1/2}. \tag{3.14}$$

§4. The Compactification of the Infinite-Dimensional Metaplectic Group in the Weil Representation

Note that the standard Gaussian measure μ_n in $\mathbb{C}^n$ (see §1) is invariant with respect to the transformations $z \mapsto Uz$, where $U \in U(n)$. Thus, the definition of the Bargmann–Segal space $F(\mathbb{C}^n)$ makes sense for an abstract finite-dimensional Hilbert space over $\mathbb{C}$.

Now, fix a countable-dimensional complex Hilbert space E and assign to it the *Bargmann–Segal space* $F(E)$ consisting of all the complex functions f on E with the following properties: a) if $E' \subset E$ is a finite-dimensional subspace, then $f|E' \in F(E')$; b) $\|f\|^2 := \sup \|f|E'\|^2 < \infty$, where the supremum is taken over all finite-dimensional $E' \subset E$. A slightly different (but equivalent) definition is given by Segal [24].

It will be convenient for us to fix an orthonormal basis $e_1, e_2, \dots$ in E and to identify $\mathbb{C}^n$ with $\mathbb{C}e_1 + \cdots + \mathbb{C}e_n$ for all $n = 1, 2, \dots$. Then $F(E)$ is identified with the Hilbertian completion of the algebraic inductive limit $\varinjlim F(\mathbb{C}^n)$. In particular, $F(E)$ is a Hilbert space. This is a convenient model of the boson Fock space with infinitely many degrees of freedom; see Berezin [2] and Segal [24].

The definition of a symbol of an operator and the definition of a Gaussian operator with symbol $K(\Omega)$ remain meaningful in the space $F(E)$. But now the matrix Ω has size $2\infty \times 2\infty$ and each of its blocks has to be interpreted as a bounded operator in E. Denote by S the set of all symmetric block matrices Ω of size $2\infty \times 2\infty$ satisfying inequalities (1.5) and such that the diagonal blocks Ω_{11}, Ω_{22} are Hilbert–Schmidt operators in E. Then an analogue of Proposition 1.1 holds: if there exists a bounded Gaussian operator $A(\Omega)$ with symbol $K(\Omega)$, then $\Omega \in S$. Conversely, as Neretin [12] has shown, for all $\Omega \in S$ one can still define Gaussian operators $A(\Omega)$ with a common dense invariant domain $F_0(E) \subset F(E)$; moreover, for these operators $A(\Omega)$, the multiplication formula (2.6), where the product $*$ is still defined by (2.5), remains valid.

Problem I. Describe the subset $S^* \subset S$ of those $\Omega \in S$ for which the Gaussian operator $A(\Omega)$ is bounded.

Neretin [12] has shown that S^* is distinct from S, and evaluated "from below" the set S^* in two different ways.

For $n = 1, 2, \ldots$, define the *truncation mapping* $\theta_n \colon S \to S(n)$ as follows: any of the blocks Ω_{ij} (which is a $\infty \times \infty$ matrix) is replaced by its upper left corner of size $n \times n$. Let P_n stand for the orthoprojection $F(E) \to F(\mathbb{C}^n)$. The common domain $F_0(E)$ of Gaussian operators $A(\Omega)$, as defined by Neretin [12], is invariant with respect to P_n; moreover, $P_n(F_0(E))$ coincides with $F_0(\mathbb{C}^n)$. This permits us to introduce, for $\Omega \in S$, the operator $P_n A(\Omega) P_n$, which may be viewed as an operator in $F_0(\mathbb{C}^n)$; the latter is simply $A(\theta_n(\Omega))$.

Introduce the function $\varphi(\Omega) \geq 0$ on S which equals $\|A(\Omega)\|^{-1}$ on $S^* \subset S$ and vanishes on $S \setminus S^*$. For $n = 1, 2, \ldots$ and any $\Omega \in S$, we have $\|P_n A(\Omega) P_n\|^{-1} = \varphi_n(\theta_n(\Omega))$, where the function φ_n was defined in Proposition 3.4. It follows that the functions $\varphi_n \circ \theta_n$ on S form a monotone nonincreasing sequence, which pointwise converges to φ.

Problem II. Calculate the function φ.

Denote by $\overline{\Gamma}(n)$ the set of all Gaussian contractions in $F(\mathbb{C}^n)$. The operators from $\overline{\Gamma}(n)$ have the form $aA(\Omega)$, where $\Omega \in S(n)$, $a \in \mathbb{C}$, $0 \leq |a| \leq \varphi_n(\Omega)$. The set $\overline{\Gamma}(n)$ is a semigroup with the multiplication law

$$(aA(\Omega))(bA(\widetilde{\Omega})) = ab \det\left((1 - \Omega_{22}\widetilde{\Omega}_{11})^{-1/2}\right) A(\Omega * \widetilde{\Omega}), \tag{4.1}$$

as follows from (2.6). Let us equip $\overline{\Gamma}(n)$ with the weak operator topology: on the set of nonzero operators this topology coincides with the topology of convergence of the parameters (a, Ω), and the weak convergence to the zero operator is equivalent to the convergence of the parameter a to 0. Note that $\overline{\Gamma}(n)$ is compact.

Denote by $\overline{\Gamma}$ the set of all Gaussian contractions in $F(E)$. The operators from $\overline{\Gamma}$ have the form $aA(\Omega)$, where $\Omega \in S^*$ and $a \in \mathbb{C}$, $0 \leq |a| \leq \varphi(\Omega)$. Let us equip $\overline{\Gamma}$ with the weak operator topology: this topology has the same description in terms of parameters (a, Ω), where the convergence of the matrices Ω is taken with respect to the weak operator topology in $E \oplus E$. Then $\overline{\Gamma}$ is a compact semigroup with separately continuous multiplication; the latter is given by formula (4.1) as before.

An explicit description of the semigroup $\overline{\Gamma}$ of Gaussian contractions runs into Problems I and II. However, $\overline{\Gamma}$ can be characterized as follows: as a topological space, $\overline{\Gamma}$ is the projective limit $\varprojlim \overline{\Gamma}(n)$; the projections $\overline{\Gamma}(m) \to \overline{\Gamma}(n)$, where $m > n$, or $\overline{\Gamma} \to \overline{\Gamma}(n)$ are given by the mapping $aA(\Omega) \mapsto P_n aA(\Omega) P_n = aA(\theta_n(\Omega))$.

Consider the group $Mp(\infty, \mathbb{R}) := \varinjlim Mp(n, \mathbb{R})$ and its Weil representation $W := \varinjlim W_n$ in the space $F(E)$. We will denote by $G(n)$ (respectively, by G) the image of the group $Mp(n, \mathbb{R})$ (respectively, $Mp(\infty, \mathbb{R})$) in the group of unitary operators of the space $F(E)$. The group G consists of all Gaussian operators of form $aA(\Omega)$ such that $\Omega = \Omega^t$ defines a unitary operator in $E \oplus E$, each of the matrices Ω_{11}, Ω_{22}, $\Omega_{12} - 1$, $\Omega_{21} - 1$ has only a finite number of nonzero coefficients, and $a^2 = \det \Omega_{12} = \det \Omega_{21} \neq 0$.

Let us take the closure of the group G in the semigroup $\Gamma(F(E))$ of all contractions of the space $F(E)$, where $\Gamma(F(E))$ is equipped with the weak operator topology. Then we obtain a compact semigroup $\Gamma \supset G$ with separately continuous multiplication (for more details on semigroup compactifications of infinite-dimensional groups, see Olshanskii [21] and Neretin [14, 15]). Clearly, $\Gamma \subseteq \overline{\Gamma}$.

Proposition 4.1. $P_n \in \Gamma$, $n = 1, 2, \ldots$.

Proof. Fix n and consider the sequence (C_N) of operators in E of the following form: $C_N e_j = e_j$ except for the numbers $j = n+1, \ldots, n+4N$; $C_N e_{n+i} = e_{n+2N+i}$ and $C_N e_{n+2N+i} = e_{n+i}$ for $i = 1, \ldots, 2N$. In the notation of Example 1.1, put $A_N := A_{C_N}$. Since $\det C_N = 1$, the operator A_N is contained in G. For any $k \geq n$ we have $P_k A_N P_k = P_n$, provided $2N \geq k$. Thus, A_N weakly converges to P_n as $N \to \infty$. □

By Proposition 4.1, the set $\Gamma(n) := P_n \Gamma P_n | F(\mathbb{C}^n)$ is an operator semigroup in $F(\mathbb{C}^n)$. Clearly, $\Gamma(n) \subseteq \overline{\Gamma}(n)$.

Theorem 4.2. (i) $\Gamma(n)=\overline{\Gamma}(n)$, $n=1,2\dots$.
(ii) $\Gamma=\overline{\Gamma}=\varprojlim\Gamma(n)$.

Proof. *Step 1.* Let $g\in Mp(n,\mathbb{R})$ and $aA(\Omega)$ be the corresponding operator from $G(n)$. If g tends to infinity in the locally compact group $Mp(n,\mathbb{R})$, then $\varphi_n(\Omega)\to 0$, whence $a\to 0$, i.e., $aA(\Omega)$ weakly converges to the zero operator. Thus, $\Gamma(n)$ contains 0.

Step 2. Assume that $L\in\Lambda(n)$ has the standard form (Theorem 2.8) and $\Omega:=\mathcal{P}'(L)\in S(n)$. Then there exists a unitary matrix $\widetilde{\Omega}\in S(2n)$ such that $\theta_n(\widetilde{\Omega})=\Omega$ and $|\det\widetilde{\Omega}_{12}|=\varphi_n(\Omega)$. In fact, it suffices to check this for $n=1$ and for the elements $L\in\Lambda(1)$ enumerated in Theorem 2.8. In the case $L=L^{(1)},L^{(2)}_s,L^{(4)}$ the verification is trivial. In the case $L=L^{(3)}_a$ one can take as $\widetilde{\Omega}\in S(2)$ the following matrix:

$$\widetilde{\Omega}_{11}=\widetilde{\Omega}_{22}=\begin{bmatrix}a-1 & ib\\ ib & a-1\end{bmatrix},\quad\text{and}\quad \widetilde{\Omega}_{12}=\widetilde{\Omega}_{21}=\begin{bmatrix}a & ib\\ ib & a\end{bmatrix},\qquad b:=\sqrt{a(1-a)}. \tag{4.2}$$

Step 3. Let us show that $\overline{\Gamma}(n)\setminus\{0\}$ is contained in $P_nG(2n+1)P_n$. Indeed, both sets of operators are invariant with respect to the two-sided action of the group $G(n)$. Therefore, it suffices to check that for any matrix $\Omega\in S(n)$ in the canonical form and any a, $0<|a|\le\varphi_n(\Omega)$, the operator $aA(\Omega)$ is contained in $P_nG(2n+1)P_n$. But this fact is implied by the result of Step 2; the necessity of adding 1 to $2n$ is caused by the possibility of the strict inequality $|a|<\varphi_n(\Omega)$ (if $|a|=\varphi_n(\Omega)$, then one can deal with the group $G(2n)$).

Step 4. The results of Steps 1 and 3 show that $\Gamma(n)=\overline{\Gamma}(n)$ for all n, which in turn implies $\Gamma=\overline{\Gamma}$. □

Remark 4.3. Denote by $\overline{G}$ the weak closure of the group $G\cong Mp(\infty,\mathbb{R})$ in the group of all unitary operators of the space $F(E)$. The group $\overline{G}$ consists of all Gaussian operators $aA(\Omega)$ such that the matrix $\Omega\in S$ is unitary and $0\ne|a|^4=\det(1-\Omega_{11}^*\Omega_{11})=\det\Omega_{12}^*\Omega_{12}$. It is isomorphic to the central extension of the group of proper linear canonical transformations with infinitely many degrees of freedom by means of the circle. The latter group is formed by the infinite symplectic matrices $g_{\alpha\beta}$ such that the block α is a Hilbert–Schmidt operator (see Berezin [2, Chap. II, §4, Theorem 1], Vergne [26], and Shale [25]). On the other hand, the group $\overline{G}$ coincides with the group of invertible elements of the semigroup $\Gamma=\overline{\Gamma}$.

Problem III. Is it possible to find a canonical form of operators from the semigroup $\Gamma=\overline{\Gamma}$ with respect to the two-sided action of the group $\overline{G}$?

Problem IV. Consider the orthoprojection $P\colon E\to E'$ onto a subspace $E'\subset E$ with infinite dimension and codimension. Is it true that $P\overline{G}P$ coincides with the semigroup of all nonzero Gaussian contractions in E' ?

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