The Limit Shape of Convex Lattice Polygons and Related Topics

A. M. Vershik

UDC 514.172.45

To I. M. Gel'fand on his 80th birthday

§1. Introduction

Consider all kinds of lattice polytopes (i.e., polytopes with vertices belonging to the lattice $\mathbb{Z}^d$) contained in the d-dimensional cube with sides $[0, n]$. After the scale transformation $x_i \mapsto \frac{1}{n} x_i$, $i = 1, \dots, d$, we get the unit cube $[0,1]^d$ and the set CLP_n^d of all polytopes contained in this cube and having vertices on the lattice $L_n \equiv (\frac{1}{n}\mathbb{Z})^d$. This set may also be treated as a subspace in the space of all convex closed subsets of the unit cube. This subspace is equipped with the Hausdorff metric. Let μ_n be the uniform measure on CLP_n^d.

Problem 1 (on the limit shape). Does the sequence μ_n converge in the weak topology and is it true that its limit is a δ-measure, i.e., a measure concentrated on a single convex set?

If the answer to the second question is positive, then almost all lattice polytopes are concentrated near a single convex set after the scale transformation. This set is called their limit shape.

One of the results of the present paper is that for $d = 2$ the answer to Problem 1 is positive and the limit shape is presented explicitly. Similar problems arise in statistical physics, representation theory, the combinatorics of Young's diagrams, mathematical biology (the Richardson model, "animal growth"), and other fields. Technically, they are related to the classical asymptotic expansions of generating functions, the saddle-point method, and probabilistic considerations. Let us set up one more question.

Problem 2. Let $y = f(x)$ be a strictly convex function whose graph is contained in $I = [0,1]^2$, $f(0) = 0$, $f(1) = 1$. What is the asymptotic behavior of the number of convex polygonal lines lying in I, having all vertices on the lattice $L_n = (\frac{1}{n}\mathbb{Z})^2$, and contained in the ε-neighborhood of the graph f as $n \to \infty$, $\varepsilon \to 0$?

This problem is related to the first one. It is easy to state the multidimensional analog of Problem 2. The problem stated above happens to be closely connected with the geometry of numbers and affine differential geometry. It is solved in the present paper. In the multidimensional case it remains unsolved as yet.

In 1979, in connection with the study of certain Newton's diagrams, V. I. Arnol'd posed a question on the number of lattice polytopes of a given volume (to within the group of lattice automorphisms) and presented two-sided estimates for the two-dimensional case [1]. Soon these estimates were generalized to the multidimensional case. They are based on the estimates of the number of lattice polytope vertices via its volume (note that the latter estimates have been obtained earlier in [2]). Sharp-order estimates for the logarithms have recently been obtained by I. Bárány and the author [3]. They are based on different ideas closely connected with those of this paper. The author posed Problem 1 just after Arnol'd's work, following the pattern of similar problems for Young diagrams, partitions (see [4]), etc. The solution to Problem 1 for $d = 2$ has been suggested by I. Bárány and the author simultaneously; however, the proofs are rather different and hence Bárány's proof will be published separately [17]. The theorem on a convex lattice approximation of a convex function (Problem 2) proved to be in close connection with the notion of affine length according to Blaschke and with variational problems of a statphysical nature (the Wulff method). Yet another equally surprising connection with Diophantine analysis and with estimates of the number of points on a convex curve (Jarnik's theorem) has been revealed. Functions of partitions and

St. Petersburg Division of the Mathematical Institute of the Russian Academy of Sciences. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 28, No. 1, pp. 16–25, January–March, 1994. Original article submitted December 15, 1993.

0016–2663/94/2801–0013 $12.50

vector partitions and their extensions are the main technical tool in our analysis; however, this analysis appears to be important by itself. In the present paper we indicate the main steps of the proofs.

A series of reports on the topic of this paper generated useful commentaries and stimulated further investigations. Ya. G. Sinai suggested a statphysical interpretation of the limit shape theorem and proved the central limit theorem for fluctuations. His paper will be published in the next issue of this journal [5]. The close relation between the limit shape problem and statistical physics noticed by R. L. Dobrushin reveals itself in the analogy between the surface tension method (the Wulff method) and our theorem on curvature (Problem 2). M. M. Skriganov has drawn my attention to Jarnik's work [6], and O. M. Fomenko has told me about its up-to-date extensions [7, 8]. The author expresses his gratitude to all these people and also to I. Bárány in discussions with whom the solution of the limit shape problem has appeared and who has his own variant of the proof. The author wishes to express his thanks to V. I. Arnol'd with whom the problem was discussed in the early 80's in the context of the work [1].

Finally, the remarkable opportunity of working at IHES (France) and the contacts the author maintained during his stay at this institute promoted his work on the subject extremely.

The author dedicates this work to the outstanding mathematician Israel Gel'fand, with whom he had the luck to work for a long time and from whom the author has always been learning.

§2. Basic Notions and Connections

Notation. Let $L_n = (\frac{1}{n}\mathbb{Z})^2 = \{(x_1, x_2) : (nx_1, nx_2) \in \mathbb{Z}^2 \subset \mathbb{R}^2\}$ and let $A \subset \mathbb{R}^2$ be a convex closed set. By $\mathrm{CLP}_n(A)$ we denote the set of convex polygons with vertices at nodes of the lattice L_n lying in A and by $\mathrm{CLP}_n(A;\, x, y)$ the set of convex polygonal lines lying in A and having their ends at the points x and y and vertices at nodes of L_n (note that the convexity of a nonclosed piecewise smooth curve implies that for a chosen orientation of the plane the moving n-hedron (i.e., the tangent and normal vectors at a given point) is everywhere nondegenerate and has the same orientation). Fix the standard Euclidean metric r on $\mathbb{R}^2$ and denote by d the Hausdorff metric on the collection of all compact sets in $\mathbb{R}^2$, i.e.,

$$d(A, B) = \max_{x\in A}\min_{y\in B} r(x, y) + \max_{y\in B}\min_{x\in A} r(x, y).$$

Let $I = [0, 1]^2$, $J = [-1, 1]^2$, $e_1 = (1, 0)$, and $e_2 = (0, 1)$.

Statement of basic theorems. It is convenient to solve Problem 1 first for the case of nonclosed polygonal lines.

Theorem 2.1. *For every* $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{CLP}_n(I;\, e_1, e_2),\ d(\gamma, \Gamma) < \varepsilon\}}{\#\,\mathrm{CLP}_n(I;\, e_1, e_2)} = 1,$$

where $\Gamma = \{(x_1, x_2) : \sqrt{x_1} + \sqrt{x_2} = 1,\ x_1, x_2 \geq 0\}$.

In other words, for sufficiently large n almost all convex polygonal lines in $\mathrm{CLP}_n(I;\, e_1, e_2)$ lie in an arbitrarily small neighborhood of the parabola Γ.

Let us now proceed to the case of convex polygonal lines lying in the square J.

Theorem 2.2. *For every* $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{CLP}_n(J),\ d(\gamma, \Delta) < \varepsilon\}}{\#\,\mathrm{CLP}_n(J)} = 1,$$

where $\Delta = \{(x_1, x_2) : \sqrt{1-|x_1|} + \sqrt{1-|x_2|} = 1\}$.

The curve Δ consists of four pieces of a parabola and belongs to C^2 at the points of transition from one piece to another.

Thus, the limit shape in the sense of §1 for appropriately scaled convex lattice polygons is Δ. Namely, almost all normalized convex lattice polygons are concentrated near Δ for n sufficiently large. In other words, the uniform distributions μ_n on $\mathrm{CLP}_n(J)$ converge to the δ-measure δ_Δ.

This theorem solves Problem 1. Let us proceed to Problem 2. Let γ_f be the graph of a strictly convex function $f \in C^2([0, 1])$, $f(0) = 0$, $f(1) = 1$, and let $\varepsilon > 0$. Let $V_\varepsilon(\gamma_f)$ be a uniform neighborhood of γ_f.

Theorem 2.3. *Let $f''(t) > 0$, $t \in (0,1)$. Then*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} n^{-2/3}\ln \#\{\gamma \in \mathrm{CLP}_n(I;\,\mathbf{0},\mathbf{1}),\ \gamma \subset V_\varepsilon(\gamma_f)\} = 2^{-2/3}\varkappa \int_\gamma k(s)^{1/3}ds = 2^{-2/3}\varkappa \int_0^1 [f''(t)]^{1/3}dt,$$

where $k(\,\cdot\,)$ is the curvature of γ, $\varkappa = 3\sqrt[3]{\zeta(3)/\zeta(2)}$ (ζ is the Riemann zeta function), $\mathbf{0} = (0,0)$, and $\mathbf{1} = (1,1)$.

Consider the following variational problem: Find

$$\max_{\gamma\in C^2}\int_\gamma k(s)^{1/3}ds$$

over all strictly convex curves joining $e_1 = (1,0)$ with $e_2 = (0,1)$ and such that the tangent vector is vertical at e_2 and horizontal at e_1. The curve $\Gamma = \{(x_1, x_2) : \sqrt{x_1} + \sqrt{x_2} = 1\}$ proves to be the unique solution to this problem.

Let us now proceed to the case of closed curves.

Theorem 2.4. *Let γ_0 be a closed strictly convex planar curve and $k(\,\cdot\,)$ be its curvature. Then*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} n^{-2/3}\ln \#\{\gamma \in \mathrm{CLP}_n(\mathbb{R}^2),\ \gamma \subset V_\varepsilon(\gamma_0)\} = 2^{-2/3}\varkappa \int_{\gamma_0} k(s)^{1/3}ds\,.$$

The quantity $\int_\gamma k(s)^{1/3}ds$ is well known in differential geometry as the affine curvature. It proves to be not only a Euclidean but also an affine invariant of a curve and occurs in various problems (see [12, 13]). The curve Γ is a geodesic in affine geometry. A possible analog of Theorem 2.4 in the d-dimensional case deals with the integral $\int_\Omega k^{(d-1)/(d+1)}(s)\,d\omega(s)$, where $k(\,\cdot\,)$ is the Gaussian curvature of a convex hypersurface Ω. This quantity has numerous applications (e.g., see [14]).

The connection between Theorem 2.1 (respectively, 2.2) and Theorem 2.3 (respectively, 2.4) is not so straightforward. After proving Theorem 2.3 (respectively, 2.4) and solving the cited variational problem, we apparently obtain Theorem 2.1 (respectively, 2.2). However, both theorems are, in fact, proved simultaneously, and an independent proof of Theorems 2.3 and 2.4 is as yet unknown. It is also mysterious that the affine curvature arises in Theorems 2.3 and 2.4, since in the applications known so far integrality and affine curvature are by no means related.

From the standpoint of problems of statistical physics, Theorems 2.3 and 2.4 imply that the cubic root of the curvature plays a role similar to that of the surface tension (see [16]) in related problems. However, the appearance of the curvature is undoubtedly connected with the a priori convexity of a curve.

There exists a very interesting connection with number-theoretic problems. In 1926, Jarnik [6] proved the existence of a strictly convex curve of unit length whose smoothness is only C^1 and on which there exist $cn^{2/3}$ rational points of the form $(p_1/n, p_2/n)$ for an infinite set of denominators n. This order is precise; the number of such points does not exceed $cn^{2/3}$ on an arbitrary strictly convex curve. Note (see Theorem 3.2) that the number of vertices for a typical convex polygonal line in CLP_n is also equal to $cn^{2/3}$. On the other hand, if one requires larger differentiability, then the number of rational points will be of smaller order [7, 8]. For C^∞-curves this order does not exceed $n^{1/2+\varepsilon}$, whereas $\sqrt{n}$ is realized on parabolas [7]. Thus, the curve Γ is the only one in a neighborhood of which convex rational polygonal lines are concentrated, and it contains a substantially smaller number of rational points, i.e., some kind of degeneration (smoothing of the limit curve) takes place. This curve is, however, likely to remain the best in C^∞.

§3. Theorems on Strict Partitions

By a *partition* of a nonnegative integer n we mean its decomposition into an unordered sum of nonnegative integers. We write $p(n)$ for the number of these partitions. The function $p(n)$ is called the *partition* *function* (partitio numerorum). This function has been studied since Euler, who has found its generating function

$$\sum_{n=0}^{\infty} p(n) z^n = \prod_{k=1}^{\infty} \frac{1}{1-z^k} = F_1(z).$$

A partition of a vector $\mathbf{n} \in \mathbb{Z}_+^d$ into an unordered sum of vectors with nonnegative integer coordinates is called a *vector partition*. We will denote the number of such partitions by $p_d(\mathbf{n})$. The function $p_d(\mathbf{n})$ is called the *vector partition function*. It is not difficult to see that

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} p_d(\mathbf{n}) z^{\mathbf{n}} = \prod_{\mathbf{k}\in\mathbb{Z}_+^d,\, \mathbf{k}\neq 0} \frac{1}{1-z^{\mathbf{k}}} = F_d(z),$$

where $z^{\mathbf{k}} = z_1^{k_1} \dots z_d^{k_d}$; see [9].

We need a somewhat different notion of a partition. We refer to a partition of a vector $\mathbf{n} \in \mathbb{Z}_+^d$ without proportional summands as its *strict partition.* We will denote the number of such partitions $p_d^*(\mathbf{n})$, $d \geq 2$. The function $p_d^*(\mathbf{n})$ does not have a nontrivial analog for $d = 1$ and, thus, along with $p_d(n)$, may be treated as the multidimensional generalization of $p(\mathbf{n})$. Its generating function is

$$\widetilde{F}_d(z) = \prod{}' \frac{1}{1-z^{\mathbf{k}}}.$$

Here $z^{\mathbf{k}} = z_1^{k_1} \dots z_d^{k_d}$, and the product $\prod'$ is taken over all tuples $\mathbf{k} = (k_1, \dots, k_d)$ such that g. c. d.$(k_1, \dots, k_d) = 1$. In particular,

$$\widetilde{F}_2(z_1, z_2) = \sum_{n,m=0}^{\infty} p_2^*(n,m) z_1^n z_2^m = \prod_{\text{g.c.d.}(k_1,k_2)=1} \frac{1}{1-z_1^{k_1} z_2^{k_2}}.$$

The asymptotic behavior of $p(n)$ is well known (Hardy–Ramanujan and Rademacher, e.g., see [9]). In particular,

$$\ln p(n) = (2\pi/\sqrt{\sigma})\sqrt{n}(1+o(1)).$$

In the multidimensional case, the asymptotic behavior of $p_d(\mathbf{n})$ was under investigation only in a few works (see [10, 11]). It has been discovered that the asymptotic behavior varies substantially depending on the relations between the coefficients. Specifically, for $d = 2$ the asymptotic behavior inside and outside the zone $c_1\sqrt{n} < m < c_2 n^2$ is quite different. In this zone the asymptotics is as follows:

$$\ln p_2(n,m) = 3\sqrt[3]{\zeta(3)}\,(nm)^{1/3}(1+o(1)).$$

We will be interested in the asymptotic behavior of the function $p_2^*(n,m)$. Our technique requires a knowledge of the logarithmic asymptotics for this function in the "linear zone" $0 < \alpha_0 \leq m/n \leq \alpha_1 < \infty$.

Theorem 3.1. *Let* $\varkappa = 3\sqrt[3]{\zeta(3)/\zeta(2)}$; *then*

$$(1)\quad \ln p_2^*(n,m) = \varkappa (nm)^{1/3}(1+o(1)), \quad m = \alpha n,\ 0<\alpha<\infty,\ n\to\infty;$$

$$(2)\quad \lim_{n\to\infty} \sup_{\alpha_0\leq\alpha\leq\alpha_1} |n^{-2/3}\ln p_2^*(n,\alpha n) - \varkappa\sqrt[3]{\alpha}| = 0, \quad \alpha_0 > 0,\ \alpha_1 < \infty.$$

The factor $\zeta(2)$ appears in the denominator owing to the coprimity condition. Item (2) claims that the convergence to the limit in (1) is uniform with respect to all rays separate from the coordinate axes. The proof deals with the multidimensional saddle-point method for the Cauchy integral. The saddle-point torus depends on the parameter (n), but in the linear (and even quadratic) zone no difficulties are encountered, the more so that we deal with logarithmic asymptotics.

Main lemma (on the saddle-point contour). *As* $z_1, z_2 \to 1$ *we have*

$$(1-z_1)(1-z_2)\ln \widetilde{F}_2(z_1, z_2) = (\zeta(3)/\zeta(2))(1 + o((z_1-1), (z_2-1))).$$

Moreover, the saddle-point torus $|z_1| = r_1$, $|z_2| = r_2$ *is as follows*:

$$r_1 = 1 - \varkappa m^{1/3} n^{-2/3} + o(z_1 - 1) = 1 - \varkappa \alpha^{1/3} n^{-1/3} + o(z_1 - 1),$$

$$r_2 = 1 - \varkappa n^{1/3} m^{-2/3} + o(z_2 - 1) = 1 - \varkappa \alpha^{-2/3} n^{-1/3} + o(z_2 - 1).$$

Thus, we have determined the minimax point and, in order to obtain the logarithmic asymptotics (including the proof of uniformity with respect to a parameter), one has only to verify that the Hessian is nondegenerate in a neighborhood of the point $(1,1)$.

Let $\Lambda_{n,m}$ denote the set of all strict partitions of the vector (n, m); $\#\Lambda_{n,m} = p_2^*(n,m)$. If $\lambda \in \Lambda_{n,m}$, i.e., $\lambda = ((n_i, m_i))_{i=1}^k$, $\sum n_i = n$, $\sum m_i = m$, then the summands may be put in ascending order of the corresponding ratios m_i/n_i: $m_1/n_1 \le m_2/n_2 \le \dots \le m_k/n_k$ (note that at most one summand can have the first or the second coordinate equal to zero). Put $\theta_i = \operatorname{arctg}(m_i/n_i)$, $i = 1, \dots, k$, and $\theta(\lambda) = (\theta_1, \dots, \theta_k) \in [0, \pi/2]^k$, and let $|\lambda|$ be the number of summands in the partition λ.

The following theorem describes two important asymptotic properties of a typical strict partition; these properties pertain to the number of summands and to the distribution of angles.

Theorem 3.2. 1. *There exists an absolute constant* c *such that for every* $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{\#\{\lambda \in \Lambda_{n,n} : |n^{-2/3}|\lambda| - c| < \varepsilon\}}{\#\Lambda_{n,n}} = 1.$$

2. *For every* $\theta \in (0, \pi/2)$ *there exists a function* $\gamma(\delta) > 0$, $\gamma(\delta) \searrow 0$ *as* $\delta \searrow 0$, *such that*

$$\lim_{n\to\infty} \varkappa^{-1} n^{-2/3} \ln \# \Big\{ \lambda \in \Lambda_{n,n} : \frac{1}{n} \sum_{i\,:\,|\theta_i - \theta| < \delta} n_i > \gamma(\delta) \Big\} < 1.$$

Item 2 claims that the fraction of those summands (n_i, m_i) for which the angle $\theta_i = \operatorname{arctg}(m_i/n_i)$ is sufficiently close to a given θ can be too large only for an exponentially small number of partitions. In particular, the limit distribution of angles for a typical partition is continuous.

These theorems may also be proved for full partitions.

Let us now connect the spaces $\Lambda_{n,n}$ and $\mathrm{CLP}_n(I; \mathbf{0}, \mathbf{1})$.

Lemma. *The mapping*

$$\Lambda_{n,n} \ni \lambda = ((n_i, m_i))_{i=1}^{|\lambda|} \mapsto ((\tfrac{1}{n} n_i, \tfrac{1}{n} m_i))_{i=1}^{|\lambda|}$$

establishes the bijection between the set of strict partitions of the vector (n, n) *and the set of convex polygonal lines in* $I = [0,1]^2$ *with vertices at nodes of the lattice* $L_n = (\frac{1}{n}\mathbb{Z})^2$ *and ends at the points* $\mathbf{0} = (0,0)$ *and* $\mathbf{1} = (1,1)$.

This lemma allows us to interpret the theorems of this section as assertions on convex lattice polygonal lines.

Corollary 1. 1. $\ln \# \mathrm{CLP}_n(I; \mathbf{0}, \mathbf{1}) = \varkappa n^{2/3}(1 + o(1))$.
2. $\ln \# \mathrm{CLP}_n(J) = 4\varkappa n^{2/3}(1 + o(1))$.

From now on we deal with geometric statements.

Let Π be the parallelogram with diagonal vertices $(0,0)$ and $(1,1)$.

Corollary 2. 1. $\ln \# \mathrm{CLP}_n(\Pi; \mathbf{0}, \mathbf{1}) = \varkappa (\mathrm{Area}\, \Pi)^{1/3} n^{2/3}(1 + o(1))$, *where* $\mathrm{Area}\,\Pi$ *is the area of* Π.
2. $\lim\limits_{n\to\infty} \sup\limits_{0 < s_0 \le s \le 1} |n^{-2/3} \ln \# \mathrm{CLP}_n(\Pi; \mathbf{0}, \mathbf{1}) - \varkappa s^{1/3}| = 0$, *where* $\mathrm{Area}\,\Pi = s$.

Both corollaries can easily be derived from Theorem 3.1. In order to prove Corollary 1, it suffices to perform a linear change taking Π into I, and to prove Corollary 2 we use the uniformity in Theorem 3.1.

The assumption that the ends of the diagonal belong to the lattice can be removed, since one can deal with an arbitrary parallelogram Π and the set $\mathrm{CLP}_n(\Pi; a, b)$, where a and b are the lattice points of L_n nearest to the vertices of Π. Corollary 2 remains valid in the same formulation.

§4. Sketch of Proof of the Theorems

One can regard convex polygonal lines in $\mathrm{CLP}_n(I;\, e_1, e_2)$ as graphs of piecewise linear functions, i.e., as elements of $C([0,1])$ with the boundary conditions $x(0) = 1$ and $x(1) = 0$. The set of all convex functions on $[0,1]$ with these boundary conditions forms a compact set M in the topology of $C([0,1])$. This topology coincides with the Hausdorff topology on the polygonal lines considered as closed subsets of I. The set of probability measures on M, in turn, forms a compact set $V(M)$ in the weak measure topology. Let μ_n denote the uniform distribution on $\mathrm{CLP}_n(I;\, e_1, e_2)$. By virtue of compactness, there exists at least one limit measure $\overline{\mu} = \text{w-lim}_{k\to\infty} \mu_{n_k}$; its support $\operatorname{supp}\overline{\mu}$ is the closed set of all convex curves with ends e_1 and e_2.

Lemma 1. *If $\gamma_0 \in \operatorname{supp}\overline{\mu}$, then for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \varkappa^{-1} n^{-2/3} \ln \#\{\gamma \in \mathrm{CLP}_{n_k}(I;\, e_1, e_2),\ \gamma \subset V_\varepsilon(\gamma_0)\} = 1, \qquad (*)$$

where $V_\varepsilon(\gamma_0)$ is the ε-neighborhood of γ_0.

This relationship follows from the formula

$$0 < \overline{\mu}(V_\varepsilon(\gamma_0)) = \lim_{k\to\infty} \mu_{n_k}(V_\varepsilon(\gamma_0)) = \lim_{k\to\infty} \frac{\#\{\gamma \in \mathrm{CLP}_{n_k}(I;\, e_1, e_2), \gamma \subset V_\varepsilon(\gamma_0)\}}{\#\,\mathrm{CLP}_{n_k}(I;\, e_1, e_2)},$$

and from the estimate in Theorem 3.1.

Hence it follows that once we prove that there is a unique convex curve (namely, $\Gamma = \{(x_1, x_2) : \sqrt{x_1} + \sqrt{x_2} = 1\}$) with the property mentioned in Lemma 1, it will follow that any limit measure for the sequence μ_n is δ_Γ. This is exactly the assertion of Theorem 2.1. Note that Theorem 2.2 can be proved with the help of exactly the same approach, since only one convex curve in J, namely, $\Delta = \{(x_1, x_2) : \sqrt{1 - |x_1|} + \sqrt{1 - |x_2|} = 1\}$, has the required asymptotic behavior.

Hence, it suffices to show that the only curve for which condition $(*)$ is satisfied is the curve Γ. We shall prove a stronger assertion. Let $y = (\bar{x}_1, \bar{x}_2)$ be a rational point, $y \notin \Gamma$, $|\bar{x}_2 - (1 - \sqrt{\bar{x}_1})^2| > \varepsilon$. Let the set $A(y, \varepsilon) = A$ consist of all polygonal lines in $\mathrm{CLP}_n(I;\, e_1, e_2)$ that intersect the straight line $\{(\bar{x}_1, \lambda) : \lambda \in \mathbb{R}\}$ at some point of the interval $\bar{x}_2 - \varepsilon \le \lambda \le \bar{x}_2 + \varepsilon$.

Lemma 2. $\lim_{n\to\infty} \varkappa^{-1} n^{-2/3} \ln \#A < 1$.

Proof. The set A splits into a finite number of sets $A_{\lambda,\rho}$ according to at what rational point λ a polygonal line intersects the segment $[\bar{x}_2 - \varepsilon, \bar{x}_2 + \varepsilon]$ and what is the angle ρ of the slope at this point to the x_1-axis (from the left, for instance). Since the number of sets $A_{\lambda,\rho}$ depends on n polynomially ($\le n^4$), it suffices to obtain an estimate for $\#A_{\lambda,\rho}$ uniform with respect to all λ and ρ. Each polygonal line in $A_{\lambda,\rho}$ lies in the union $\Pi_1 \cup \Pi_2$ of two parallelograms, namely, the parallelogram Π_1 with the x_1-axis and the straight line $P = \{(x_1, x_2) : x_1/a_1 + x_2/a_2 = 1\}$ as directrices and the diagonal vertices e_1 and y and the parallelogram Π_2 with the x_2-axis and the straight line P as directrices and the diagonal vertices e_2 and y. By Corollary 2 in §3, we have

$$\lim_{n\to\infty} \varkappa^{-1} n^{-2/3} \ln \#\{\gamma \in \mathrm{CLP}_n(I;\, e_1, e_2),\ \gamma \subset \Pi_1 \cup \Pi_2\} = (\operatorname{Area}\Pi_1)^{1/3} + (\operatorname{Area}\Pi_2)^{1/3},$$

and also if the areas of the parallelograms Π_1 and Π_2 are bounded away from zero, then the convergence is uniform with respect to ρ. For small areas of the parallelograms Π_1, Π_2 we may use Item 2 of Theorem 3.1 asserting the exponential smallness of the number of partitions (= polygonal lines) with a large common length of links that have a slope close enough to that of one of the two straight lines going either through e_1 and y or through e_2 and y each. Thus, we reduce the problem for the remaining areas to the following geometric problem, whose solution has been suggested by I. Bárány (see Fig. 1).

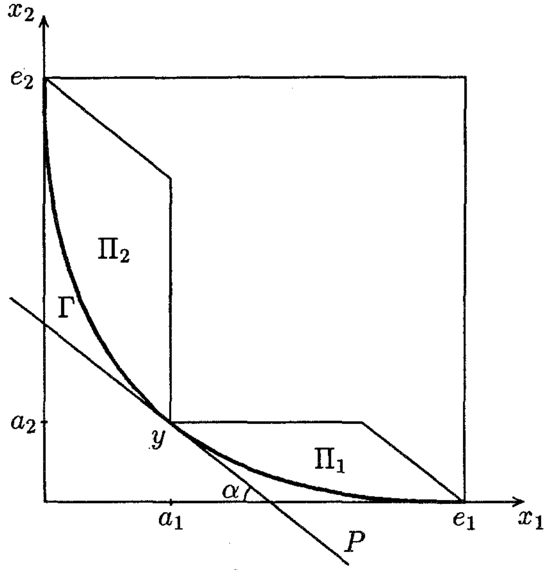

Fig. 1

Lemma 3. *Among all straight lines of the form* $P_{a_1,a_2} = \{(x_1, x_2) : x_1/a_1 + x_2/a_2 = 1\}$, $0 < a_1, a_2 < 1$ *and points* $y = (x_1, x_2)$ *on* P_{a_1,a_2}, *the sum*

$$(\text{Area}\,\Pi_1)^{1/3} + (\text{Area}\,\Pi_2)^{1/3}$$

is maximal and is equal to $1 = \text{Area}\,I$ *only for the pairs* (a_1, a_2) *and points* y *that satisfy* $a_1 + a_2 = 1$, $x_1 = a_1^2$, *and* $x_2 = a_2^2$.

Thus, points of the form $\{(x_1, x_2) : \sqrt{x_1} + \sqrt{x_2} = 1\}$ prove to be the only ones that provide the maximum value for the expression $\lim_{n\to\infty} \varkappa^{-1} n^{-2/3} \ln \# A$. Hence, Theorems 2.1 and 2.2 are proved.

Let us proceed to Theorems 2.3 and 2.4. Let $y = f(x)$, $f \in C^2([0, 1])$, $f(0) = 1$, $f(1) = 0$, $f''(t) > 0$ for $t \in (0, 1)$. If $t_1, t_2 \in (0, 1)$, then the parallelogram constructed on tangents to the graph of f at the points t_1 and t_2 with vertices at the points $(t_1, f(t_1))$, $(t_2, f(t_2))$ will be called *admissible*. Taking a partition of the interval $[0, 1]$ by the points $0 = t_0 < t_1 < \cdots < t_k = 1$, $t_i - t_{i-1} = \delta$, $i = 1, \ldots, k$, and constructing admissible parallelograms $\Pi_i = \Pi_{t_i, t_{i+1}}$ with respect to the pairs (t_i, t_{i+1}), we obtain for $\delta = \delta(\varepsilon)$ sufficiently small, the inclusion $\bigcup_i \Pi_i \subset V_\varepsilon(\gamma_f)$, where γ_f is the graph of f.

Lemma 4. *The area of an admissible parallelogram is*

$$\text{Area}\,\Pi_{t,t+\delta} = \tfrac{1}{4} f''(t)\delta^3 + o(\delta^3).$$

This elementary lemma allows us to get a lower bound for $\varkappa^{-1} n^{-2/3} \ln \#\{\gamma \in \text{CLP}_n,\ \gamma \subset V_\varepsilon(\gamma_f)\}$ by means of the integral sum for the integral $\int_0^1 f''(t)^{1/3}\,dt$.

On the other hand, given an $\varepsilon > 0$ small enough, we can construct a family of admissible parallelograms covering $V_\varepsilon(\gamma_f)$ and such that the area of their mutual overlappings is small in comparison with the total area of $V_\varepsilon(\gamma_f)$. This argument gives the desired upper bound.

Concluding remarks. 1. Convex capacity. The foregoing arguments suggest that the following notion is useful. Let A be a convex set. The quantity

$$c(A) = \lim_{n\to\infty} \varkappa^{-1} n^{-2/3} \ln \# \text{CLP}_n(A)$$

is called its *convex capacity* (if it exists at all). Here $\text{CLP}_n(A)$ is the set of all closed polygonal lines having vertices on L_n and lying in A. If we do not require closedness, then we can also deal with nonconvex A. In this case, the convex capacity can be defined as follows:

$$c(A;\, a, b) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \varkappa^{-1} n^{-2/3} \ln \#\{\gamma \in \text{CLP}_n(A;\, a, b),\ \gamma \subset V_\varepsilon(A)\},$$

$a, b \in \partial A$. In fact, we have been evaluating the convex capacity of certain sets and graphs. It would be of interest to continue these calculations.

2. On the other hand, Theorem 3.1 may be treated as the evaluation of the entropy for the uniform distribution. From this viewpoint, it is appropriate to consider some other distributions on CLP_n. Our results are possibly stable with respect to the choice of distributions.

3. Theorem 3.1 shows how the number of vertices of a typical polygonal line grows. However, one can consider some other fixed growth, say, $\sqrt{n}$, and look for the limit shapes for uniform distributions connected with this growth. This problem is similar to that on large deviations.

4. In the multidimensional case there exist two forms of generalizations of two-dimensional problems, namely, the transfer to polyhedra and the transfer to convex hulls of polygonal lines ("zonotopes"). Both cases are of interest; the second one may be investigated by means of the tools of the present paper.

References

1. V. I. Arnol'd, "The statistics of convex lattice polygons," Funkts. Anal. Prilozhen., **14**, No 1, 1–3 (1980).
2. G. E. Andrews, "A lower bound for the volume of strictly convex bodies with many boundary points," Trans. Am. Math. Soc., **106**, 270–279 (1963).
3. I. Bárány and A. M. Vershik, "On the number of convex lattice polytopes," Geometric Funct. Anal., **2**, No. 4, 381–393 (1992).
4. A. M. Vershik and S. V. Kerov, "The asymptotic behavior of the maximal and the typical dimension for irreducible representations of the symmetric group," Funkts. Anal. Prilozhen., **19**, 1, 27–36 (1985).
5. Ya. G. Sinai, "The probabilistic approach to the analysis of statistics for convex polygonal lines," to appear in Funkts. Anal. Prilozhen., **28**, 2 (1994).
6. V. Jarnik, "Über die Gitterpunkte auf konvexen Curven," Math. Z., **24**, 500–518 (1926).
7. E. Bombieri and J. Pila, "The number of integral points on arcs and ovals," Duke. Math. J., **59**, No. 2, 337–357 (1989).
8. W. Schmidt, "Integer points on curves and surfaces," Monatsh. Math., **99**, No. 1, 45–72 (1985).
9. G. E. Andrews, Number Theory: The Theory of Partitions, Addison-Wesley (1976).
10. E. Wright, "The number of partitions of a large bi-partite number," Proc. London Math. Soc., **7**, No. 5, 150–160 (1957).
11. G. Meinardus, "Zur additiven Zahlentheorie in mehreren Dimensionen," Math. Ann., **132**, 333–346 (1956).
12. W. Blaschke, Affine Differentialgeometrie, Berlin (1923).
13. P. A. Shirokov and A. P. Shirokov, Affine Differential Geometry [in Russian], Fizmatgiz, Moscow (1959).
14. I. Bárány and D. G. Larman, "Convex bodies, economic cap coverings, random polytopes," Mathematika, **35**, 274–291 (1988).
15. P. V. Sporyshev and A. M. Vershik, "Asymptotic behavior of the number of faces of random polyhedra and the neighborliness problem," Selecta Math. Soviet., **12**, No. 1 (1992).
16. R. Dobrushin, R. Kotecky, and S. Shlosman, Wulff Construction, a Global Shape from Local Interaction, AMS Mathematical Monographs, Vol. 104 (1992).
17. I. Bárány, "The limit shape theorem for convex lattice polygons," to appear.

Translated by N. A. Sidorov