$$\left.\frac{\partial T(\lambda, \bar{\lambda}, E_0)}{\partial \bar{\lambda}}\right|_{|\lambda|=1+0} = \frac{-4\pi}{\bar{\lambda}} T(\lambda, \bar{\lambda}, E_0) h_{-\lambda_\perp}(-\lambda, -\lambda, E_0) - (2\pi)^2\lambda^2 \frac{\partial}{\partial \lambda'} h_{-\lambda_\perp}(\lambda, \lambda', E_0)|_{\lambda'=-\lambda}. \quad (8)$$

The idea of deduction of Eq. (9) consists in that, by virtue of (5), we can find the derivative of the function

$$T(\lambda, \bar{\lambda}, E) = \iint \exp\left[\frac{i\sqrt{E}}{2}((\lambda + 1/\bar{\lambda})\bar{z} + (\bar{\lambda} + 1/\lambda) z)\right] v(z, \bar{z}) \mu'(z, \bar{z}, \lambda, E) \frac{d\bar{z} \wedge dz}{2i},$$

where μ is the solution of the problem (5), (6) (see [4]), with respect to $\bar{\lambda}$ in an explicit form. Reasoning further in the same manner, we arrive at the assertion of Proposition 2.

The author is thankful to S. P. Novikov for the formulation of the problem and guidance and to P. G. Grinevich for information about the results of [4].

LITERATURE CITED

1. S. P. Novikov, Usp. Mat. Nauk, 39, No. 4, 97-98 (1984).
2. P. G. Grinevich and R. G. Novikov, Dokl. Akad. Nauk SSSR, 286, No. 1, 19-22 (1986).
3. R. G. Novikov, Teor. Mat. Fiz., 66, No. 2, 234-240 (1986).
4. P. G. Grinevich and S. V. Manakov, Funkts. Anal. Prilozhen., 20, No. 2, 14-24 (1986).
5. L. D. Faddeev, Dokl. Akad. Nauk SSSR, 165, No. 3, 514-517 (1965); ibid., 167, No. 1, 69-72 (1966).
6. A. J. Nachman and M. J. Ablowitz, Stud. Appl. Math., 71, 243-250 (1984).

COINCIDENCE OF THE HOMOLOGICAL DIMENSIONS OF THE FRECHET ALGEBRA OF SMOOTH FUNCTIONS ON A MANIFOLD WITH THE DIMENSION OF THE MANIFOLD

O. S. Ogneva UDC 513.88

The present paper is devoted to the proof of the following assertion.

THEOREM 1. For the topological algebra $C^\infty(M)$ of smooth functions on the smooth m-dimensional real manifold M, its small homological dimension, global homological dimension, and bidimension are equal to m (all dimensions are understood in the sense of the homology of topological (locally convex) algebras [1]).

In [2] Taylor calculated the bidimension of the topological algebra $C^\infty(U)$, where U is an open set in R^m. The proof of this fact as well as the proof, relating to the pure algebraic homology, of Hilbert's syzygy theorem [3, Chap. VII, Sec. 7], are based to a considerable degree on the possibility of constructing a free resolution of special form of length m, a so-called Koszul resolution. However in the general case of the topological algebra of smooth functions on an arbitrary smooth real manifold one does not have a natural definition of a system of commuting operators (of the type of the operators of multiplication by independent variables in the case $M \subset R^m$), which would let one construct Koszul free resolutions. Nevertheless, we show that the modules over the algebra $C^\infty(M)$ always have projective (generally not free) resolutions of length m, although they are of more complicated structure than Koszul resolutions. In the proof essential use is made of the projectivity of a certain natural class of modules (cf. point 2), used in the construction of the Koszul resolution and the more complex resolutions subsequently obtained with the help of the complex of smooth Čech cochains. It is necessary to note that the basic result relates to topological ("locally convex") homology and its proof uses the specifics of the apparatus of this theory. The theorem exhibits the difference of the homology of topological algebras from the Banach situation: for example, it is known that for a functional, i.e., commutative and semisimple Banach algebra the values of the global homological dimension and the bidimension are always strictly greater than one (cf. [4]).

Institute of Information Problems, Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 20, No. 3, pp. 92-93, July-September, 1986. Original article submitted August 5, 1985.

 0016-2663/86/2003-0248$12.50

1. Definitions of topological algebras (in particular, Frechet algebras), modules over them, and their homological characteristics are given in [1, 5]. For a topological algebra A we denote by $dh_A X$ the homological dimension of the left A-module X, and by ds A, dg A, db A, respectively, the left small homological dimension, the left global dimension, and the bidimension of the algebra A. These are defined as follows: $ds\,A = \sup\{dh_A X : X$ being a left A-module, $\dim X < \infty\}$, $dg\,A = \sup\{dh_A X : X$ being a left A-module$\}$, $db\,A = dh_{A^e}A$, where $A^e = A \hat{\otimes} A^{op}$ is the enveloping algebra of the algebra A (here $\hat{\otimes}$ is the complete projective tensor product [6], and A^{op} is the algebra with the "opposite" multiplication). We recall that for a topological algebra A, $ds\,A \leqslant dg\,A \leqslant db\,A$. For a locally convex space L and a system $(T_1,\ldots,T_m)$ of commuting continuous operators given on it, we denote by Kos $(L; (T_1,\ldots,T_m))$ the Koszul complex of the pair $(L; (T_1,\ldots,T_m))$ [1, 5].

We shall consider the commutative Frechet algebra $C^\infty(M)$ of smooth functions on the smooth m-dimensional real manifold M. According to Grothendieck's theorem [6, Chap. II, Sec. 3] for smooth manifolds M_1 and M_2 the spaces $C^\infty(M_1, C^\infty(M_2)) = C^\infty(M_1 \times M_2)$ and $C^\infty(M_1) \hat{\otimes} C^\infty(M_2)$ are topologically isomorphic, where the isomorphism $C^\infty(M)^e = C^\infty(M) \hat{\otimes} C^\infty(M) \simeq C^\infty(M \times M)$ is obviously an isomorphism of algebra.

The following assertions are steps in the proof of Theorem 1.

2. For an open set $U \subset M$ the space $C^\infty(U)$ of smooth functions is a Frechet module over the algebra $C^\infty(M)$ with respect to pointwise outer multiplication $f \cdot g(s) = f(s)g(s)$, where $f \in C^\infty(M)$, $g \in C^\infty(U)$, $s \in U$.

THEOREM 2. Let U be an open set from M, lying in one chart. Then $C^\infty(U)$ is a projective $C^\infty(M)$-module.

Proof. It suffices to show that $C^\infty(U)$ is a retract of the free $C^\infty(M)$-module $C^\infty(M) \hat{\otimes} C^\infty(U)$, i.e., the canonical projection $\pi_U: C^\infty(M \times U) \to C^\infty(U), \pi_U(f) = f(s,s), s \in U$ has a left inverse morphism [1, Theorem III.1.30].

Let (W, ω) be a chart (i.e., an open set $W \subset M$ together with a fixed homeomorphism ω onto an open subset of R^m), $U \subset W$; $\psi(x) = \min(1, \operatorname{dist}(\partial\omega(U), x))$, $x \in \omega(U) \subset R^m$. We take an arbitrary smooth function $\varphi(x)$, $x \in \omega(U)$, such that $0 < \varphi(x) < \psi(x)$, and from it we define a smooth function θ on $\omega(U) \times \omega(U)$: $\theta(x, y) = \exp\{|x-y|^2/(|x-y|^2 - \varphi(y)^2)\}$, if $|x-y| \leqslant \varphi(x)$, $\theta(x, y) = 0$, if $|x-y| > \varphi(y)$. For $(s, t) \in M \times U$ we set $F(s, t) = \theta(\omega(s), \omega(t))$, if $s \in U$, and 0, if $s \notin U$. Obviously $F \subset U \times U$, $F \in C^\infty(M)$ and $F(s, s) = 1, s \in U$. Now we define the map $\rho: C^\infty(U) \to C^\infty(M \times U)$ sought as follows:

$$(\rho f)(s, t) = \begin{cases} f(s) F(s, t), & \text{if } s \in U, \\ 0, & \text{if } s \notin U. \end{cases}$$

3. For each open set $U \subset M$, lying entirely in a chart, for example (W, ω), we define the operators of multiplication by the k-th coordinate function $T_k^m(U): C^\infty(U) \to C^\infty(U)$. Namely, for $\omega(t) = (\omega^1(t), \ldots, \omega^m(t)) \in R^m$, $t \in U$, we set $T_k^m(U)f = \omega^k(t)f(t)$, $k = 1,\ldots,m$. We take any point $s_0 \in U$ and we denote by C_0 the one-dimensional $C^\infty(M)$-module C with outer multiplication $f \cdot \lambda = f(s_0)\lambda$, where $\lambda \in C$, $f \in C^\infty(M)$.

THEOREM 3. The complex over C_0 $\operatorname{Kos}(C^\infty(U); (T_1^m(U), \ldots, T_m^m(U)) \xrightarrow{\pi_0} C_0$, where $\pi_0: C^\infty(U) \to C_0$, $\pi_0(f) = f(s_0)$ is a projective resolution of the $C^\infty(M)$-module C_0.

Proposition 1. $Ext^m_{C^\infty(M)}(C_0, C_0) \simeq C$.

COROLLARY. $dh_{C^\infty(M)} C_0 = m$, $ds\,C^\infty(M) \geqslant m$.

4. The following theorem lets one "locally" estimate the homological dimension from above, namely for the $C^\infty(M)^e$-module $C^\infty(U)$ with outer multiplication $f \cdot g = f(s, s)\, g(s)$, $f \in C^\infty(M)^e$, $g \in C^\infty(U)$, $s \in U$, establish that $dh_{C^\infty(M)^e} C^\infty(U) \leqslant m$. As before we assume that U lies entirely in one of the charts.

THEOREM 4. The complex

$$\operatorname{Kos}(C^\infty(U \times U); (T_1^{2m}(U \times U) - T_{m+1}^{2m}(U \times U), \ldots, T_m^{2m}(U \times U) - T_{2m}^{2m}(U \times U)) \xrightarrow{\pi} C^\infty(U),$$

where $\pi(f) = f(s, s)$, $s \in U$, is a projective resolution of the $C^\infty(M)^e$-module $C^\infty(U)$.

Now we take $\{U_i\}$, i = 1, 2,... to be an arbitrary collection of open subsets of M, each of which lies in some one chart. Then the space $\prod_{i=1}^{\infty} C^\infty(U_i)$, being the countable Cartesian product of Frechet spaces, which are simultaneously $C^\infty(M)^e$-modules, will itself be a Frechet $C^\infty(M)^e$-module [1], in which the outer multiplication is defined by the multiplication in each of the $C^\infty(U_i)$.

Proposition 2. $dh_{C^\infty(M)^e} \prod_{i=1}^{\infty} C^\infty(U_i) \leqslant m.$

5. The final step of the proof consists of establishing an upper bound for the bidimension of the algebra $C^\infty(M)$ by means of "identifications" effected with the help of the Čech complex and local upper bounds for homological dimensions (point 4).

THEOREM 5. $dh_{C^\infty(M)^e} C^\infty(M) \leqslant m.$

Proof. We inscribe in the covering of the manifold M by a countable number of charts, a countable covering $\mathfrak{U}$ of multiplicity no higher than m. The complementary complex of smooth Čech cochains corresponding to the covering $\mathfrak{U}$ is exact and admissible. Representing it in the form of a product of m + 1 short exact admissible sequences of $C^\infty(M)^e$-modules and then applying to each of the short sequences the long exact sequence for $E \times t_{C^\infty(M)^e}(\cdot, Y)$, where Y is a $C^\infty(M)^e$-module [1, Theorem III.4.4], we get that $dh_{C^\infty(M)^e} C^\infty(M) \leqslant m$.

Combining the inequalities of Theorem 5 and the corollary to Theorem 3, we see that

$$ds C^\infty(M) = dg\, C^\infty(M) = db\, C^\infty(M) = m.$$

The author thanks A. Ya. Khelemskii for his constant interest in the work and for guidance.

LITERATURE CITED

1. A. Ya. Khelemskii, Homology in Banach and Topological Algebras [in Russian], Moscow State Univ. (1986).
2. J. L. Taylor, Adv. Math., 9, 183-252 (1972).
3. S. MacLane, Homology, Springer-Verlag (1975).
4. A. Ya. Khelemskii, Tr. Sem. im. I. G. Petrovsk., No. 3, 223-242 (1978).
5. J. L. Taylor, Adv. Math., 9, 137-182 (1972).
6. A. Grothendieck, Mem. Am. Math. Soc., 16 (1955).

BIFURCATION OF SINGULAR POINTS OF GRADIENT DYNAMICAL SYSTEMS

B. A. Khesin

UDC 517.9

1. Thom announced in [1] the orbital classification of generic four-parameter local families of gradient vector fields ("Thom's law of seven elementary catastrophes"). This classification according to Thom reduces to the classification of generic four-parameter families of potential functions with respect to smooth coordinate changes in the preimage.

For one- and two-parameter families Thom's assertion is true [2]. The classification of bifurcations of typical three-parameter families of gradient fields does not reduce to the classification of deformations of potentials, and for three parameters Thom's list is incomplete [3, 4].

In the present paper we give a complete list of local bifurcations of gradient fields on a Riemannian manifold, occurring in typical three-parameter families of potentials near singular points (seven cases instead of Thom's five). For three parameters for primitive pairs (function, metric) in general position the gradient of any versal deformation of a function is a topologically orbital versal deformation of a primitive gradient field (for four parameters this is false, cf. [4]).

The author is profoundly grateful to V. I. Arnol'd for posing the problem and helpful discussions.

M. V. Lomonosov Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 20, No. 3, pp. 94-95, July-September, 1986. Original article submitted December 26, 1985.

 0016-2663/86/2003-0250$12.50