A CONSTRUCTION OF HYPER-KÄHLER METRICS

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A Riemannian metric g on a manifold M^{4k} is said to be *quaternionic* if its holonomy group is reduced to Sp (1) × Sp (k) and is said to be *hyper-Kähler* if this group is reduced to Sp (k) [1]. The last condition is equivalent to the condition that on M there exists a family of complex structures that is parametrized by the points of the projective line $\mathbf{C}P^1$ (with certain natural conditions), with respect to each of which the metric is Kähler. Therefore, it is natural to percieve the hyper-Kählerness as the quaternionic Kählerness.

For k = 1 (dim M = 4) the notion of a hyper-Kähler metric transforms into the notion of a right-planar metric: g is the autodual solution of the (vacuum) Einstein equation. The construction of explicit examples of hyper-Kähler metrics is a differential problem even for k = 1. A few examples are known for k > 1. Hitchin, Rocek, etc. have announced a general method for the construction of examples that is based on the fact that under natural restrictions the hyper-Kählerness is preserved under factorization with respect to the invariant action of a compact Lie group. Here we propose another method for the construction of hyper-Kähler metrics that generalizes the method of [2-4] for the construction of the right-planar metrics. As there, we conclude the hyper-Kählerian structures in a more general class of geometric structures and obtain hyper-Kähler metrics, restricting these more general structures to suitable submanifolds.

1. Bundles of 2-Forms and Hyper-Kähler Metrics. The description, given in [2], of the right-planar metrics in the language of the quadratic bundles of 2-forms is generalized rectilinearly to the case k > 1. As in [2], at first we "complexify" the problem and consider a nondegenerate holomorphic metric g on the complex manifold M^{4k}. The notion of the hyper-Kählerness is carried over to the complex case tautologically. With each quaternionic metric we associate a family of the holomorphic 1-forms φ_{jA}, j = 1, 2,...,2k, A = 0, 1, with respect to which the following quadratic bundle of 2-forms is constructed:

$$F(t)=\sum_{l=1}^{k} F_l(t), \quad F_l(t)=\varphi_{2l-1}(t)\wedge\varphi_{2l}(t), \quad \varphi_j(t)=\varphi_{j0}+t\varphi_{j1}, \quad t\in\mathbf{C}. \tag{1}$$

The corresponding (to the bundle) metric has the form

$$g=\sum_{l=1}^{k} g_l, \quad g_l=\varphi_{2l-1,0}\varphi_{2l,1}-\varphi_{2l,0}\varphi_{2l-1,1}, \tag{2}$$

where the forms are multiplied symmetrically. The representability of the bundle F(t) in the form (1) is equivalent to the condition that its (k + 1)-th outer power is equal to zero identically with respect to t. The holonomy group of g is realized as the gauge group of the bundle F(t). The group Sp (1) = Sl (2, **C**) corresponds to the projective change of the parameter t and Sp (k) corresponds to the linear substitutions $\varphi_j(t)$, under which the representation (1) is preserved.

In the language of the form bundles, the condition of hyper-Kählerness acquires the very simple form

$$dF(t)=0, \tag{3}$$

i.e., F(t) are closed for all t. It follows from (3) that the system

$$\varphi_j(t)=0, \quad j=1,\ldots,2l, \tag{4}$$

is completely integrable, i.e., the condition of the hyper-Kählerness is an extension of the condition for the consistency of the system of linear equations (4), containing the spectral parameter t rationally (cf. [3]).

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Following Penrose [5], we naturally call the integral surfaces (4) the α-surfaces and their manifold T the twistor manifold (cf. [1]); $\dim T = 2k + 1$. To the points M on T there corresponds a 4k-parametric family of rational curves with normal bundle $\mathcal{O}(1) \oplus \ldots \oplus \mathcal{O}(1)$ (2k terms). All the data are easily rephrased in the language of twistors.

2. Construction of a Family of Hyper-Kähler Metrics. Let us generalize one of the constructions of [4] to the case $k > 1$. Let $T = \mathbf{C}P^1 \times X^{2k}$, X^{2k} be a simplicial space (over $\mathbf{C}$) with the form $J(dx) = \sum_{l=1}^{k} dx_{2l-1} \wedge dx_{2l}$; and t be an affine parameter on $\mathbf{C}P^1$. We fix p and let N be the manifold of the rational curves in T of the form

$$x_j = u_j(t) = u_{j0} + u_{j1}t + \ldots + u_{jp}t^p, \quad j = 1, \ldots, 2k. \tag{5}$$

This is the family of the sections of the vector bundle $\mathcal{O}(p) \oplus \ldots \oplus \mathcal{O}(p)$ (2k terms) on $\mathbf{C}P^1$. We have $\dim N = 2k(p + 1)$ and let $\{u_{jl}\}$ be a coordinate system on N. Let us consider on N the bundle of the 2-forms

$$G(t) = J(du(t)) = \sum_{l=1}^{k} du_{2l-1}(t) \wedge du_{2l}(t) \tag{6}$$

of degree $2(p + 1)$ in t. It satisfies all the conditions on F(t), excluding the power in t. This is a representative of the class of the structures that generalize hyper-Kähler structures. We restrict G(t) to certain 4k-dimensional manifolds in N.

We fix $2p - 2$ values $t = t_1, \ldots, t_{2p-2}$ and $2p - 2$ Lagrangian (with respect to J) manifolds $\Gamma_1, \ldots, \Gamma_{2p-2}$ in X^{2k}. Let $M_\Gamma \subset N$ denote the submanifold of the curves (5) such that

$$u(t_j) \in \Gamma_j, \quad j = 1, \ldots, 2p - 2. \tag{7}$$

THEOREM 1. The quadratic bundle of the 2-forms

$$F(t) = G(t)|_{M_\Gamma} \Big/ \prod_{j=1}^{2p-2} (t - t_j) \tag{8}$$

on M_Γ has the structure (1), satisfies Eq. (3), and therefore induces a (complex) hyper-Kähler metric on M_Γ.

Indeed, the $(k + 1)$-th outer power of F(t), as also of G(t), is equal to zero; the closedness of F(t) follows from the closedness of G(t); and $G(t_j)|_{M_\Gamma} \equiv 0$ since Γ_j are Lagrangian. We get a family of the hyper-Kähler matrics that depend on several functions of k variables.

3. Examples. The explicit computation of F(t) according to Theorem 1 and the further computation of φ_{jA} are complicated with the growth of p on account of the difficulty in the choice of an effective parametrization on M_Γ. We give the best-possible formulas for $p = 2$, $k = 2$.

Let x_1, x_2, x_3, x_4 be coordinates on X; u_{ji}, $j = 1, 2, 3, 4$ and $i = 0, 1, 2$, be coordinates on N; and $x_j = u_j(t) = u_{j0} + u_{j1}t + u_{j2}t^2$. We fix $t_1 = 0$ and $t_2 = \infty$. Let the Lagrangian manifold Γ_1 be locally given by the conditions $x_2 = \varphi(x_1, x_3)$ and $x_4 = \psi(x_1, x_3)$, and Γ_2 be given by the conditions $x_2 = \lambda(x_1, x_3)$ and $x_4 = \mu(x_1, x_3)$. We will denote expressions of the form $\partial a(x_1, x_3)/\partial x_j$ by a_j for brevity. The conditions for the Lagrangianness mean that

$$\varphi_3' - \psi_1' = 0, \quad \lambda_3' - \mu_1' = 0. \tag{9}$$

The submanifold M_Γ is given by the equation

$$u_{20} = \varphi(u_{10}, u_{30}), \quad u_{40} = \psi(u_{10}, u_{30}), \quad u_{22} = \lambda(u_{12}, u_{32}), \quad u_{42} = \mu(u_{12}, u_{32}). \tag{10}$$

We choose u_{10}, u_{30}, u_{12}, u_{32}; u_{j1}, $j = 1, 2, 3, 4$ as coordinates on M_Γ ($\dim M_\Gamma = 8$) and restrict the bundle of the form $G(t) = du_1(t) \wedge du_2(t) + du_3(t) \wedge du_4(t)$ of degree 4 to M_Γ.

According to Theorem 1, the terms of degrees 4 and 0 in t disappear during the restriction. After division by t, we get the desired quadratic bundle of 2-forms F(t). After this, it is still necessary to reduce F(t) to the form (1), to find the 1-form φ_{jA}, and to form g by (3). We will not give the form of F(t), which is easily computed directly, but at once give the formulas for φ_{jA}. We introduce the intermediate notation

$$\omega_2 = du_{21} - \varphi_1' du_{11} - \varphi_3' du_{31}, \quad \omega_4 = du_{41} - \psi_3' du_{31} - \psi_1' du_{11},$$
$$\Delta = (\lambda_1' - \varphi_1')(\mu_3' - \psi_3') - (\lambda_3' - \varphi_3')^2,$$

where the omitted variables correspond to (10). We have

$$\varphi_{10} = \Delta du_{10}, \quad \varphi_{11} = \Delta du_{11} + (\lambda_3' - \varphi_3')\,\omega_4 - (\mu_3' - \psi_3')\,\omega_2,$$
$$\varphi_{20} = \Delta^{-1}\omega_2, \quad \varphi_{21} = \Delta^{-1}[(\lambda_1' - \varphi_1')\,du_{12} + (\lambda_3' - \varphi_3')\,du_{32}],$$
$$\varphi_{30} = \Delta du_{30}, \quad \varphi_{31} = \Delta du_{31} + (\lambda_3' - \varphi_3')\,\omega_2 - (\lambda_1' - \varphi_1')\,\omega_4,$$
$$\varphi_{40} = \Delta^{-1}\omega_4, \quad \varphi_{41} = \Delta^{-1}[(\mu_3' - \psi_3')\,du_{32} + (\lambda_3' - \varphi_3')\,du_{12}].$$

In the constructed metrics, it is easy to pass to real forms. Under the assumption that Γ_2 is obtained from Γ_1 by the involution $x \mapsto \bar{x}$, we can restrict g to the real eight-dimensional manifold $u_{j2} = \overline{u_{j0}}$, j = 1, 3; $\operatorname{Re} u_{j1} = 0$, j = 1, 2, 3, 4.

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NARASIMHAN—SESHADRI CONNECTION AND KÄHLER STRUCTURE OF THE SPACE OF MODULI OF HOLOMORPHIC VECTOR BUNDLES OVER RIEMANN SURFACES

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1. Many spaces of moduli connected with Riemann surfaces have a natural Kähler manifold structure; well-known examples are Teichmüller space, the Jacobi variety, and the space of modules of stable vector bundles (matrix analog of the Jacobian). For Teichmüller space the symplectic form of the Kähler metric mentioned (the Weyl—Petersson metric) is the "$\bar{\partial}$-derivative" of the field of projective connections defining the Fuchsian uniformization of the corresponding Riemann surfaces (cf. [1-3]). This "principle," expressing the natural symplectic structure on the space of moduli in terms of a preferred connection in the bundle, turns out to be sufficiently general. In the present note we illustrate it with the example of the space of moduli of flat stable vector bundles over a compact Riemann surface.

2. Let X be a compact Riemann surface of genus g > 1. By a theorem of Narasimhan—Seshadri [4] the space N = N(n, 0) of moduli of stable vector bundles of rank n and degree 0 over X is isomorphic with the space of classes of equivalent irreducible representations of the fundamental group $\pi_1(X)$ on the unitary group U(n). This space N is a complex manifold of dimension $n^2(g - 1) + 1$ over **C**. The tangent space $T_\rho N$ to N at the point corresponding to the (irreducible) representation $\rho\colon \pi_1(X) \to U(n)$ can be identified naturally with the Dolbeault cohomology group $H^{0,1}(X, \operatorname{End} E_\rho)$, where $\operatorname{End} E_\rho$ is the bundle of endomorphisms of the n-dimensional complex bundle E_ρ, induced by the representation ρ (cf. [5]). The cotangent space $T_\rho^* N$ at this same point is isomorphic with the group $H^{1,0}(X, \operatorname{End} E_\rho)$, and the natural pairing $T_\rho N \otimes T_\rho^* N \to \mathbf{C}$ is given by the integral $-\frac{\sqrt{-1}}{2}\int_X \operatorname{tr}(\mu \wedge q)$, where $\mu \in H^{0,1}(X, \operatorname{End} E_\rho)$, $q \in H^{1,0}(X, \operatorname{End} E_\rho)$ (elements of the Dolbeault groups are considered to be harmonic forms), and tr means the matrix trace. On N there is defined the Hermitian metric

$$\langle \mu, \nu \rangle = -\frac{\sqrt{-1}}{2}\int_X \operatorname{tr}(\mu \wedge \bar{\nu}^t), \quad \mu, \nu \in H^{0,1}(X, \operatorname{End} E_\rho),$$

where ^{t} denotes the transpose. The (1, 1)-form $\omega = -(1/2)\operatorname{Im}\langle\ ,\ \rangle$ corresponding to this metric is closed [6] and defines the Kähler structure mentioned on the space of moduli N.

3. In each stable vector bundle E of degree 0 over X there exists a unique flat unitary connection A_E, compatible with the complex structure, the Narasimhan—Seshadri connection [4; 7]. This connection plays the same role in relation to the form ω on N as the Fuchsian

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