is valid for all $t \in (0,1]$.

Let Γ_φ denote the smallest possible constant C in (4).

Theorem 3. *Let* $\psi \in \Phi$. *The following properties are equivalent:*

(i) *if* $\varphi \in \Phi$ *and* $\varphi \le \psi$, *then* $\Gamma_\varphi < \infty$;

(ii) $\sup\limits_{\varphi\in\Phi,\varphi\le\psi} \Gamma_\varphi < \infty$;

(iii) *there exist constants* $a, \varepsilon > 0$ *such that* $\psi(t) \le at^\varepsilon$ *for all* $t \in [0,1]$.

In particular, if $\varphi \in \Phi$ and $\varphi(t) \le at^\varepsilon$, then $\Gamma_\varphi \le 5a/\varepsilon$.

Theorem 4. *Let* E *be an* r.i. *space and* $E \supset L_p$ *for some* $p < \infty$. *Then the equivalence* (3) *holds.*

Theorem 4 improves the main result of [3]. It shows that a criterion for the validity of (3) cannot be stated in terms of the Boyd indices of E. Note that the condition

$$\sup_{k,t} \frac{k^{1/q}\varphi_E(t^k/k!)}{\varphi_E(t)} < \infty,$$

where $\varphi_E(t) = \|1|_{[0,t]}\|_E$, is necessary for the validity of (3).

We use probabilistic estimates in the coincidence problem ([6, 4.9.B]) in the proof of Theorem 2.

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Yangians of Lie Superalgebras of Type $A(m,n)$

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1. Yangians of Lie superalgebras are related to rational solutions of the graded quantum Yang–Baxter equation [3] in just the same way as Yangians of Lie algebras are related to solutions of the quantum Yang–Baxter equation. In this note we consider Yangians of simple Lie superalgebras of type $A(m,n)$. Unlike [4], we use the definition of Yangians from [1, 8]. Starting from a system of zero-order generators, for which the relations and the comultiplication law are similar to those of the universal enveloping superalgebra, we introduce first-order generators, for which the comultiplication law is more complicated (see Subsec. 3). The remaining relations follow from the compatibility conditions for the algebra and coalgebra structures. From the finite system of generators and relations thus obtained we derive (quite naturally, in our opinion) a new system of generators and relations (Subsec. 4), which is used in the statement of the PBW theorem (Subsec. 5) and in proving the existence of the universal R-matrix (Subsec. 6). The author is grateful to S. Z. Levendorskii for useful discussions.

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2. In what follows we are interested in classical superalgebras of type $A(m,n)$, $m \neq n$ [5]. Let us recall their definition. Any classical Lie superalgebra G is contragredient and hence is characterized by its Cartan matrix $A = (a_{ij})_{i,j=1}^{r}$ and by the set $\tau \subseteq I = \{1, \dots, r\}$ of indices of odd generators. If $G = A(m,n)$, then $r = m+n-1$ and $\tau = \{m\}$. The Lie superalgebra $G = A(m,n)$ is generated by the elements x_i^+, x_i^-, and h_i, $i \in I$, which satisfy the relations $[x_i^+, x_i^-] = \delta_{ij}h_i$, $[h_i, h_j] = 0$, $[h_i, x_j^\pm] = \pm a_{ij}x_j^\pm$, and $\operatorname{ad}^2(x_i^\pm)(x_j^\pm) = 0$, $i \neq j$, where $p(x_i^\pm) = 0$ for $i \notin \tau$, $p(x_m^\pm) = 1$, and $p(h_i) = 0$ are the degrees of the generators and $(\operatorname{ad} a)(b) := [a,b] = ab - (-1)^{p(a)p(b)}ba$. Moreover, yet another relation has been noted recently [6]: $[\operatorname{ad} x_{m-1}^\pm(x_m^\pm), \operatorname{ad} x_{m+1}^\pm(x_m^\pm)] = 0$. Note that for $A(m,n)$ the nonzero elements of the Cartan matrix $A = (a_{i,j})_{i,j=1}^{m+n-1}$ are $a_{i,i} = 2$ and $a_{i,i+1} = a_{i+1,i} = -1$ for $i < m$, and $a_{i-1,i} = a_{i,i-1} = 1$ and $a_{i,i} = -2$ for $i > m$.

3. In what follows we denote the superalgebra $A(m,n)$ by G.

Definition 1. The Yangian $\overline{Y}(G)$ is the Hopf superalgebra over $\mathbb{C}$ determined by the generators $x_{i,0}^\pm$, $h_{i,0}$, $x_{i,1}^\pm$, and $h'_{i,1}$, $i \in I = \{1, \dots, m+n-1\}$, and by the relations $[h_{i,0}, h_{j,0}] = [h_{i,0}, h'_{j,1}] = [h'_{i,1}, h'_{j,1}] = 0$, $[h_{i,0}, x_{j,0}^\pm] = \pm a_{ij}x_{j,0}^\pm$, $[h'_{i,1}, x_{j,0}^\pm] = \pm a_{ij}x_{j,1}^\pm$, $[x_{i,0}^+, x_{j,0}^-] = \delta_{ij}h_{i,0}$, $[x_{i,1}^+, x_{j,0}^-] = \delta_{ij}h_{i,1} := \delta_{ij}(h'_{i,1} + \frac{1}{2}h_{i,0}^2)$, $[x_{i,1}^\pm, x_{j,0}^\pm] = [x_{i,0}^\pm, x_{j,1}^\pm] \pm (a_{ij}/2)(x_{i,0}^\pm x_{j,0}^\pm + x_{j,0}^\pm x_{i,0}^\pm)$, $[x_{m,1}^\pm, x_{m,0}^\pm] = 0$, $[x_{i,0}^\pm, [x_{i,0}^\pm, x_{j,0}^\pm]] = 0$, $i \neq j$, $[[x_{m-1,1}^\pm, x_{m,0}^\pm], [x_{m+1,0}^\pm, x_{m,0}^\pm]] = 0$, $[[h'_{i,1}, x_{i,1}^+], x_{j,1}^-] + [x_{i,1}^+, [h'_{i,1}, x_{j,1}^-]] = 0$, $i \neq m$, and $[[h'_{m-1,1}, x_{m,1}^+], x_{m,1}^-] + [x_{m,1}^+, [h'_{m-1,1}, x_{m,1}^-]] = 0$.

The comultiplication Δ is defined on $\overline{Y}(G)$ by the formulas $\Delta(h_{i,0}) = h_{i,0} \otimes 1 + 1 \otimes h_{i,0}$, $\Delta(x_{i,0}^\pm) = x_{i,0}^\pm \otimes 1 + 1 \otimes x_{i,0}^\pm$, $i \neq m$, and $\Delta(x_{m,0}^\pm) = x_{m,0}^\pm \otimes 1 - 1 \otimes x_{m,0}^\pm$, $\Delta(h'_{i,1}) = h'_{i,1} \otimes 1 + 1 \otimes h'_{i,1} + [h_{i,0} \otimes 1, \Omega'_2]$, $\Delta(x_{i,1}^\pm) = a_{i,i}^{-1}[\Delta(h'_{i,1}), \Delta(x_{i,0}^\pm)]$, $i \neq m$, and $\Delta(x_{m,1}^\pm) = a_{m-1,m}^{-1}[\Delta(h'_{m-1,1}), \Delta(x_{m,0}^\pm)]$. Here $\Omega'_2 = \sum_{\alpha \in \Delta_+}(-1)^{p(\alpha)}x_\alpha^- \otimes x_\alpha^+$, $p(\alpha) := p(x_\alpha^\pm)$.

Theorem 1. *The superalgebra structure is compatible with the cosuperalgebra structure on* $\overline{Y}(G)$.

4. Let us introduce a new system of generators for the Yangian $Y(G)$, where $G = A(m,n)$. This system is similar to the one introduced by W. G. Drinfel'd for Yangians of simple Lie algebras [1, 7]. This system of generators is especially useful in the proof of the PBW theorem and its corollaries.

Let us introduce generators $\widetilde{x}_{i,k}^\pm$ and $\widetilde{h}_{i,k} (\in \overline{Y}(G))$, $i \in I$, $k \in \mathbb{Z}_+$, by the formulas $\widetilde{x}_{i,0}^\pm = x_{i,0}^\pm$, $\widetilde{x}_{i,k+1}^\pm = \pm a_{i,i}^{-1}[h'_{i,1}, \widetilde{x}_{i,k}^\pm]$, $i \neq m$, $\widetilde{x}_{m,k+1}^\pm = \pm a_{m-1,m}^{-1}[h'_{m-1,1}, \widetilde{x}_{m,k}^\pm]$, and $\widetilde{h}_{i,k} = [\widetilde{x}_{i,k}^+, \widetilde{x}_{i,0}^-]$.

Definition 2. Let $Y(G)$ denote the Hopf superalgebra over $\mathbb{C}$ determined by the generators $x_{i,k}^\pm$ and $h_{i,k}$, $i \in I$ and $k \in \mathbb{Z}_+$, and by the relations $[h_{i,k}, h_{j,l}] = 0$, $\delta_{i,j}h_{i,k+l} = [x_{i,k}^+, x_{j,l}^-]$, $[h_{i,0}, x_{j,l}^\pm] = \pm a_{ij}x_{j,l}^\pm$, $[h_{i,k+1}, x_{j,l}^\pm] = [h_{i,k}, x_{j,l+1}^\pm] \pm (a_{ij}/2)(h_{i,k}x_{j,l}^\pm + x_{j,l}^\pm h_{i,k})$ for $i \neq m$ or $j \neq m$, $[h_{m,k+1}, x_{m,l}^\pm] = 0$, $[x_{m,k+1}^\pm, x_{m,l}^\pm] = 0$, $[x_{i,k+1}^\pm, x_{j,l}^\pm] = [x_{i,k}^\pm, x_{j,l+1}^\pm] \pm (a_{ij}/2)(x_{i,k}^\pm x_{j,l}^\pm + x_{j,l}^\pm x_{i,k}^\pm)$ for $i \neq m$ or $j \neq m$, $[x_{i,k}^\pm, [x_{i,s}^\pm, x_{j,l}^\pm]] + [x_{i,s}^\pm, [x_{i,k}^\pm, x_{j,l}^\pm]] = 0$, $i \neq j$, and $[[x_{m-1,k}^\pm, x_{m,r}^\pm], [x_{m+1,l}^\pm, x_{m,t}^\pm]] = 0$ for arbitrary integer m, r, l, and t. Note that the parity function assumes the following values on the generators: $p(x_{i,k}^\pm) = 0$ for $i \in I \setminus \tau$ $(i \neq m)$ and $k \in \mathbb{Z}_+$, $p(h_{i,k}) = 0$ for $i \in I$ and $k \in \mathbb{Z}_+$, and $p(x_{m,k}^\pm) = 1$.

Theorem 2. *The correspondence* $\widetilde{x}_{i,k}^\pm (\in \overline{Y}(G)) \to x_{i,k}^\pm (\in Y(G))$, $\widetilde{h}_{i,k} (\in \overline{Y}(G)) \to h_{i,k} (\in Y(G))$ *defines a superalgebra isomorphism* $\overline{Y}(G) \to Y(G)$.

5. Following [9], let us state the PBW theorem for $Y(A(m,n))$. The second index of the generators $x_{i,k}^\pm$ and $h_{i,k}$ will be called the *degree* of these generators. The degree of a monomial is, by definition, the sum of the degrees of its factors. For a polynomial, the maximal degree of its monomials will be called the degree of the polynomial. Let $Y(G)_k$ denote the space of elements in $Y(G)$ whose degree does not exceed k. We obtain the following filtration on $Y(G)$:

$$0 = Y(G)_{-1} \subset Y(G)_0 = U(G) \subset Y(G)_1 \subset \cdots \subset Y(G)_k \subset Y(G)_{k+1} \subset \cdots .$$

Let us construct the root vectors for $Y(G)$. Let $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_p}$ be a decomposition of a positive root into a sum of roots such that $x_\alpha^\pm = [x_{i_1}^\pm, [x_{i_2}^\pm, \dots [x_{i_{p-1}}^\pm, x_{i_p}^\pm] \dots]]$ is a nonzero root vector from $G_{\pm\alpha}$.

Let $k = k_1 + \cdots + k_p$ and $\overline{k} = (k_1, \ldots, k_p) \in \mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+ = Z_+^p$. Let us define the root vectors by the formulas

$$x^{\pm}_{\alpha,\overline{k}} = [x^{\pm}_{i_1,k_1}, [x^{\pm}_{i_2,k_2}, \ldots [x^{\pm}_{i_{p-1},k_{p-1}}, x^{\pm}_{i_p,k_p}] \ldots]],$$
$$x_{\pm\alpha,\overline{k}} = x^{\pm}_{\alpha,\overline{k}}, \quad h_{\alpha,\overline{k}} = [x^{+}_{\alpha,0}, x^{-}_{\alpha,\overline{k}}], \quad h_{i,\overline{k}} = h_{\alpha,\overline{k}}.$$

It is easy to check that if $k = k'_1 + \cdots + k'_p$ is another decomposition of the number k and $\overline{k}' = (k'_1, \ldots, k'_p)$, then $x^{\pm}_{\alpha,\overline{k}} - x^{\pm}_{\alpha,\overline{k}'} \in Y(G)_{k-1}$ and $h_{\alpha,\overline{k}} - h_{\alpha,\overline{k}'} \in Y(G)_{k-1}$.

For each number k let us fix some vector $\overline{k}$ determining a decomposition of this number. Let us introduce a linear order $<$ on the set $\{x^{+}_{\alpha,\overline{k}}, x^{-}_{\beta,\overline{l}}, h_{j,m}\}$, $\alpha, \beta \in \Delta$, $j \in I$, and $k, l, m \in \mathbb{Z}_+$. To this end, denote the set of ordered monomials in $x^{+}_{\alpha,\overline{k}}$, $x^{-}_{\beta,\overline{l}}$, and $h_{j,m}$ by $\Omega(<)$. We choose the said linear order as follows. Let $\beta(1) < \cdots < \beta(N)$ be a linear order on Δ_+. We require that $x^{+}_{\alpha,\overline{k}} < h_{j,\overline{l}} < x^{-}_{\beta,\overline{s}}$ for every α, β, j, k, l, and s; if $i < j$, then $x^{\pm}_{\beta(i),\overline{k}} < x^{\pm}_{\beta(j),\overline{l}}$ and $h_{i,\overline{k}} < h_{j,\overline{l}}$ for any k and l; if $k < l$, then $x^{\pm}_{\beta(i),\overline{k}} < x^{\pm}_{\beta(i),\overline{l}}$ and $h_{j,\overline{k}} < h_{j,\overline{l}}$ for all $i, j \in I$.

Theorem 3. $\Omega(<)$ *is a* PBW *basis of the Yangian* $Y(G)$.

6. We now prove the existence of a pseudotriangle structure on $\overline{Y}(G)$.

Let $T_\lambda \colon \overline{Y}(G) \to \overline{Y}(G)$, $\lambda \in C$, be the automorphism defined by the formulas $T_\lambda(x) = x$, $x \in U(g)$, $T_\lambda(x^{\pm}_{i,1}) = x^{\pm}_{i,1} + \lambda x^{\pm}_{i,0}$, and $T_\lambda(h'_{i,1}) = h'_{i,1} + \lambda h'_{i,0}$.

Theorem 4. *There exists a unique formal series* $R(\lambda) = 1 + \sum_{k=1}^{\infty} R_k \lambda^{-k}$, *where* $R_k \in \overline{Y}(G) \otimes \overline{Y}(G)$, *such that*

$$(T_\lambda \otimes \mathrm{id})\Delta'(a) = R(\lambda)(T_\lambda \otimes \mathrm{id})(\Delta(a))R(\lambda)^{-1}, \qquad R^{21}(\lambda)R^{12}(-\lambda) = 1,$$
$$R^{12}(\lambda_1 - \lambda_2)R^{13}(\lambda_1 - \lambda_3)R^{23}(\lambda_2 - \lambda_3) = R^{23}(\lambda_2 - \lambda_3)R^{13}(\lambda_1 - \lambda_3)R^{12}(\lambda_1 - \lambda_2),$$

where $\Delta' = \sigma \circ \Delta$, $\sigma(x \otimes y) = (-1)^{p(x)p(y)} y \otimes x$, $a \in \overline{Y}(G)$.

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