EXAMPLE OF AN ALGEBRAIC CURVE IN C^2 SUCH THAT ITS COMPLEMENT HAS FUNDAMENTAL GROUP WITH ELEMENTS OF FINITE ORDER

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In this note we answer the question posed by Arnol'd [1] of whether the fundamental group of the complement to an algebraic curve in $\mathbf{C}^2$ can have nontrivial elements of finite order.

1. Consider the curve $\Gamma \subset \mathbf{C}^2$ defined by the equation

$$\lambda(\lambda^2 - 2z^2(3+z^2)\lambda + z^2(3z^2+1)^2) = 0. \tag{1}$$

It can be shown that $\pi_1(\mathbf{C}^2 \setminus \Gamma)$ admits a corepresentation by two generators a, b with relations $ab^2 = b^2a$, $a^3b = ba^3$, $(ab)^2 = (ba)^2$. The distinguished point and the loops representing a and b lie in the intersection of $\mathbf{C}^2 \setminus \Gamma$ with the complex line $z = z_0$, where z_0 is real and sufficiently close to 0. This intersection is a complex line punctured at the points

$$\lambda_1 = z_0^2(3+z_0^2) + z_0\sqrt{(z_0^2-1)^3}, \quad \lambda_2 = 0, \\ \lambda_3 = z_0^2(3+z_0^2) - z_0\sqrt{(z_0^2-1)^3}. \tag{2}$$

The distinguished point lies on the positive half of the real axis, and a and b are represented by elementary circuits around λ_2 and λ_3, respectively.

The correspondence $a \to (1, 2, 3)$, $b \to (1, 2)$ defines an epimorphism of $\pi_1(\mathbf{C}^2 \setminus \Gamma)$ onto the permutation group S(3), whence $aba^{-1}b^{-1} \neq 1$. It is easily verified that $(aba^{-1}b^{-1})^3 = 1$, i.e., $aba^{-1}b^{-1}$ is an element of order three.

2. See [3], e.g., concerning the definition of the braid group B(n) and its action on the free group $\mathbf{F}_n$, denoted by $\beta \circ \tau$, $\beta \in \mathbf{F}_n$, $\tau \in B(n)$ below. It is known (cf. [1, 2]) that B(n) is isomorphic to the fundamental group of the region $G_n \subset \mathbf{C}^n$ consisting of the points $z_1, \ldots, z_n$ at which the polynomial $\lambda^n + z_1\lambda^{n-1} + \ldots + z_n$ does not have multiple zeros.

Let Γ be an arbitrary algebraic curve in $\mathbf{C}^2$. We may assume without loss of generality that Γ is the set of zeros of a polynomial of the form $p(z, \lambda) = \lambda^n + a_1(z)\lambda^{n-1} + \ldots + a_n(z)$ with discriminant $d(z) \not\equiv 0$, where the $a_i(z)$ are polynomials in one variable. Then p induces a homomorphism $p_*\colon \mathbf{F}_k\ (=\pi_1(\mathbf{C} \setminus \{z \mid d(z) = 0\})) \to B(n)$ $(=\pi_1(G_n))$. Let $\tau_1, \ldots, \tau_m$ be any set of generators of the group $p_*\{\mathbf{F}_k\}$.

<u>THEOREM.</u> The group $\pi_1(\mathbf{C}^2 \setminus \Gamma)$ admits the following corepresentation. Generators: $\beta_1, \ldots, \beta_n$ (n is the degree of the polynomial p with respect to λ). Relations: $\beta_j \circ \tau_i = \beta_j$, $j = 1, \ldots, n$, $i = 1, \ldots, m$.

The loops representing $\beta_1, \ldots, \beta_n$ are the elementary circuits around the punctured points of the generic fiber of the projection of $\mathbf{C}^2 \setminus \Gamma$ onto the z axis.

The difference between this theorem and Zariski's theorem is that the braids τ_i are evaluated instead of covering isotopies, which makes the computation easier. Thus for curve (1), when the generators of $\pi_1(G_3)$ are defined by the polynomials $\sigma_1 = (\lambda - \lambda_3)/(\lambda^2 - (\lambda_1 + \lambda_2)\lambda + ((\lambda_1 + \lambda_2)^2 - z(\lambda_1 - \lambda_2)^2)/4)$, $\sigma_2 = (\lambda - \lambda_1)(\lambda^2 - (\lambda_2 + \lambda_3)\lambda + (\lambda_2 + \lambda_3)^2 - z(\lambda_2 - \lambda_3)^2)/4)$, where λ_1, λ_2, λ_3 correspond to (2) and z runs over the unit circle with the positive orientation, the action of B(3) on F_3 is given by the formulas $\alpha_1 \circ \sigma_1 = \alpha_1\alpha_2\alpha_1^{-1}$, $\alpha_2 \circ \sigma_1 = \alpha_1$, $\alpha_3 \circ \sigma_1 = \alpha_3$, $\alpha_1 \circ \sigma_2 = \alpha_1$, $\alpha_2 \circ \sigma_2 = \alpha_2\alpha_3\alpha_2^{-1}$, $\alpha_3 \circ \sigma_2 = \alpha_2$. One can put $\tau_1 = (\sigma_1\sigma_2)^3$, $\tau_2 = \sigma_2\sigma_1^3\sigma_2^{-1}$, $\tau_3 = \sigma_2^{-1}\sigma_1^3\sigma_2$, $\tau_4 = \sigma_2^4$, $\tau_5 = \sigma_1^4$, so that then $a = \beta_2$, $b = \beta_3$, $\beta_1 = \beta_2^{-1}\beta_3\beta_2$.

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PROPERTIES OF BASES OF SOME KOETHE SPACES AND THEIR SUBSPACES

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A sequence (e_n) in a countably normed space E is said to be orderable if there exists a system of prenorms $(|\cdot|_s)$ defining the initial topology in E and such that for each pair of indices $n_1, n_2 \in N = \{1, 2, \ldots\}$ either $\frac{|e_{n_1}|_s}{|e_{n_1}|_{s+1}} \geqslant \frac{|e_{n_2}|_s}{|e_{n_2}|_{s+1}}$ $\forall s \in N$ or $\frac{|e_{n_1}|_s}{|e_{n_1}|_{s+1}} \leqslant \frac{|e_{n_2}|_s}{|e_{n_2}|_{s+1}}$ $\forall s \in N$ (where we take $0/0 = 0$).

This generalizes the notion of regular bases due to Dragilev (cf. [1]).

The purpose of this note is to study properties of unconditional bases in products of spaces having absolute orderable bases.

It is possible in large classes of Koethe spaces to solve the problem of quasiequivalence of unconditional bases and obtain generalizations of results of Dragilev [2], Zakharyuta [3], and the author [4] relating to classes of nuclear spaces, and of Mityagin [5], Baran and the author [6] for individual classes of nonnuclear spaces.

1. A monotone Koethe matrix (MKM) is a matrix of nonnegative numbers $[a_s(n)]_{s,n=1}^{\infty}$ such that $a_s(n) \leq a_{s+1}(n)$, n, s = 1, 2, . . . for increasing row indices.

An MKM is called regular [1] if the ratios $a_s(n)/a_{s+1}(n)$ are nonincreasing in n for each s. A regular MKM $[a_s(n)]$ corresponds to continuous functions

$$a_s(t) = a_s(n) + (t-n)[a_s(n+1) - a_s(n)], \quad n \leqslant t < n+1,\ n,\ s = 1, 2, \ldots, \tag{1}$$

such that the ratios $a_s(t)/a_{s+1}(t)$ are nonincreasing in t.

Let (E_n) be a sequence of Frechet spaces (= complete metrizable locally convex spaces) with a fixed system of prenorms $(|\cdot|_s^{(n)})_{s=1}^{\infty}$, n = 1, 2, . . . defining the topology in the E_n. We will denote by $\left(\sum\{E_n, (|\cdot|_s^{(n)})\}\right)_{l_1[a_s(n)]}$ the space of all sequences $x = (x_n)$, $x_n \in E_n$, n = 1, 2, . . . such that

$$|x|_s = \sum_n |x_n|_s^{(n)} a_s(n) < \infty, \quad s = 1, 2, \ldots, \tag{2}$$

with topology defined by the system of prenorms $(|\cdot|_s)$.

LEMMA 1. A Frechet space with an absolute orderable basis is isomorphic to a space of the form $\left(\sum_n l_1^{k(n)}\right)_{l_1[a_s(n)]}$ defined by some regular matrix $[a_s(n)]$ and set of spaces $l_1^{k(n)} = \left\{\xi = (\xi_i)_{i=1}^{k(n)} : \|\xi\|_{l_1} = \sum_{i=1}^{k(n)} |\xi_i| < \infty\right\}$ with $|\xi|_s^{(n)} = s\|\xi\|_{l_1}$, s = 1, 2, . . . , k(n) ≤ ∞, n = 1, 2, ...

THEOREM 1. Let $E = \left(\sum_n \{E_n, (|\cdot|_s^{(n)})\}\right)_{l_1[a_s(n)]}$, where the $E_n = \left(\sum_i l_1^{k(n,i)}\right)_{l_1[a_s^{(n)}(i)]}$ are determined by the regular MKM $[a_s^{(n)}(i)]$, n = 1, 2, . . . , k(n, i) ≤ ∞, n, i = 1, 2, . . . and norms in $l_1^{k(n,i)}$ of the form $|\cdot|_s^{(n,i)} = s\|\cdot\|_{l_1}$, $n, i \in N$. Assume the MKM $[a_s(n)]$ satisfies the condition $a_s(n) \leqslant \frac{1}{2n^2} a_{s+1}(n)$, $s, n \in N$ and that the prenorms in E_n and E are defined by Eqs. (2).

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