THEOREM. (i) T^{ad} splits up into the direct sum of representations $\rho \in \mathscr{E}$ (with multiplicities).

(ii) The multiplicity of the occurrence of $\rho \in \mathscr{E}$ in T^{ad} is equal to the maximal multiplicity of weight in ρ.

(iii) We may choose $\mathfrak{g}_0$-invariant subspaces $V_\rho, \rho \in \mathscr{E}$ in A such that:

a) the restriction of T^{ad} to V_ρ is equivalent to ρ;

b) for any $\rho_1, \rho_2 \in \mathscr{E}$, the restriction of T^{ad} to $V_{\rho_1}V_{\rho_2}$ (the space generated by products ab, $a \in V_{\rho_1}$, $b \in V_{\rho_2}$) is equivalent to $\rho_1 \otimes \rho_2$.

5. We comment on the last statement of the theorem. Let V and V' be two finite-dimensional $\mathfrak{g}_0$-invariant subspaces in A, and let τ and τ' be the restrictions of T^{ad} to V and V'. Since $T^{ad}(X)$ for $X \in \mathfrak{g}_0$ is a differentiation in the algebra A, VV' is a $\mathfrak{g}_0$-invariant subspace, and the restriction of T^{ad} to VV' is the factor-representation $\tau \bigcirc \tau'$. Statement (iii) of the theorem means that the subspaces V_ρ may be chosen such that the restriction of T^{ad} to $V_{\rho_1}V_{\rho_2}$ is equivalent to $\rho_1 \otimes \rho_2$, i.e., there is no cancellation. Statement (ii) means (in view of Steinberg's formula on the decomposition of the tensor product [4]), that each representation $\rho \in \mathscr{E}$ appears in T^{ad} with the minimal multiplicity for this.

6. For an arbitrary semisimple Lie algebra $\tilde{\mathfrak{g}}$, we naturally call its representation $\tilde{T}$ of differentiations in some associative algebra $\tilde{A}$ a "multiplicative model," if the analogs of statements (i)-(iii) of the theorem are satisfied. Such a model was constructed by Biederharn and Louck [5] for $\tilde{\mathfrak{g}} = \mathfrak{sl}_2(\mathbf{R})$. It clearly exists for any simple Lie algebra, and in general, is not unique.

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BERNSIDE'S PROBLEM ON PERIODIC GROUPS

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Despite the progress which has been made in the solution of Bernside's problem on periodic groups [1], there is an undoubted interest in simple examples of infinite finitely generated groups, each element of which has finite order. This interest has recently grown. It turns out that such groups are connected with the solution of a whole series of problems in functional analysis, amongst which we note the problem arising to von Neuman, on invariant means [2].

In this article we shall construct examples of (and essentially state a method of constructing) groups in which each element has finite order of the form p^N, where p is a prime number. We note that the group G (with which we shall be concerned later) is defined as a group of transformations of the interval [0, 1], preserving the Lebesgue measure. Therefore, we shall define all the transformations to within a set of measure zero. Henceforward, it will be convenient to use the following notation. The letter I over an interval Δ of the numerical axis will denote the identity transformation of the interval Δ, and the letter P over the interval $\Delta = [\alpha, \beta]$ will denote "permutation" of the halves of this interval, i.e., $Px = x + (\beta - \alpha)/2$ if $\alpha < x < (\alpha + \beta)/2$, $Px = x - (\beta - \alpha)/2$ if $(\alpha + \beta)/2 < x < \beta$. Denote by G the group generated by the transformations a, b, c, and d, whose definitions are given in Fig. 1 (over the second, third, and fourth copies of the interval [0, 1] there are "written" the infinite sequences PPIPPI. . ., PIPPIP. . ., and IPPIPP. . .). We note that the generators a, b, c, and d satisfy the following relations:

$$a^2 = b^2 = c^2 = d^2 = e; \quad bc = cb, \quad bd = db, \quad cd = dc; \quad bc = d, \quad cd = b, \quad ad = c, \tag{1}$$

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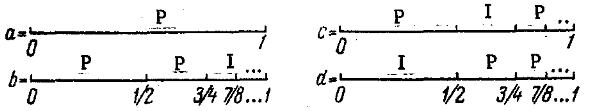

Fig. 1

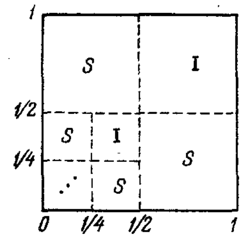

Fig. 2

where e is the unit element of the group G.

THEOREM. The group G has the following properties: it is 1) infinite, 2) periodic, and 3) cannot be given by a finite set of defining relations.

Proof. Let $W = f_1 \ldots f_n$ be a word in the symbols a, b, c, d. Denote by $\partial_a(W)$, $\partial_b(W)$, $\partial_c(W)$, $\partial_d(W)$ the number of times each symbol appears in the word W, and by $\partial(W)$ we denote the length of W. If the word W represents the element g in G and is the shortest among such words, then we denote by $\partial(g)$ the length of this shortest word. In those places when we are talking about a concrete representative W of the element g, we shall assume that $\partial_a(g) = \partial_a(W)$, $\partial_b(g) = \partial_b(W)$, etc.

1) Denote by H the subgroup of G consisting of those transformations which leave the halves of the interval [0, 1] invariant. The word $W = f_1 \ldots f_n$ defines an element of H if and only if $\partial_a(W) = 0 \pmod 2$. Therefore, the elements b, c, d, aba, aca, and ada generate H. Denote by a_Δ, b_Δ, c_Δ, d_Δ the transformations of the interval Δ which are defined analogously to the transformations a, b, c, d for the interval [0, 1]. It is easily seen that

$$\begin{array}{llll} b|_{[0,1/2]} = a_{[0,1/2]}, & b|_{[1/2,1]} = c_{[1/2,1]}; & aba|_{[0,1/2]} = c_{[0,1/2]}, & aba|_{[1/2,1]} = a_{[1/2,1]}, \\ c|_{[0,1/2]} = a_{[0,1/2]}, & c|_{[1/2,1]} = d_{[1/2,1]}; & aca|_{[0,1/2]} = d_{[0,1/2]}, & aca|_{[1/2,1]} = a_{[1/2,1]}, \\ d|_{[0,1/2]} = T_{[0,1/2]}, & d|_{[1/2,1]} = b_{[1/2,1]}; & ada|_{[0,1/2]} = b_{[0,1/2]}, & ada|_{[1/2,1]} = T_{[1/2,1]}, \end{array} \qquad (2)$$

where $S|_\Delta$ denotes the restriction of the transformation S to the interval Δ. It follows from relations (2) that the group H is isomorphic to the subgroup of the group G × G generated by the elements (a, c), (a, d), (e, b), (c, a), (d, a), (b, e), and therefore admits a homomorphism to the group G for which $(a, c) \to a$, $(a, d) \to a$, $(e, b) \to e$, $(d, a) \to d$, $(b, e) \to b$, $(c, a) \to c$. Thus, we have proved that G is infinite.

2) Denote by φ the isomorphism of H and the subgroup of G × G, generated by the elements (a, c), (a, d), (e, b), (c, a), (d, a), (b, e), where $\varphi(b) = (a, c)$, $\varphi(c) = (a, d)$, $\varphi(d) = (e, b)$, $\varphi(aba) = (c, a)$, $\varphi(aca) = (d, a)$, $\varphi(ada) = (b, e)$. By relations (1), an arbitrary element $g \in G$ can be represented by a word in the symbols a, b, c, d, in which there are no two symbols standing next to each other in the set {b, c, d}. We call a word satisfying this condition, reduced. We prove that for any $g \in G$ there exists a natural number N such that $g^{2^N} = e$. The proof is by induction on $\partial(g)$. If $\partial(g) = 1$, then from relations (1) we have $g^2 = e$. Suppose that periodicity is proved for elements $g \in G$, whose length is not greater than k. We prove that elements of length k + 1 also have bounded order. Let $\partial(g) = k + 1$ and $g = f_1 \ldots f_{k+1}$, where $f_i \in \{a, b, c, d\}$, $i = 1-(k+1)$. The word $f_1 \ldots f_{k+1}$ is reduced, and we may assume that at least one of the symbols f_1, f_{k+1} is different from a, and also that at least one of the symbols f_1, f_{k+1} does not belong to the set {b, c, d}, for otherwise g would be conjugate to an element of length less than k + 1. If $g \in H$ and $\varphi(g) = (g_l, g_r) \in G \times G$, then $\partial(g_l) \leq 1/2(k+1)$, $\partial(g_r) \leq 1/2(k+1)$. By the hypothesis, the elements g_l and g_r have finite orders. Therefore, the element g also has finite order.

Suppose now that $\partial_a(g) = 1 \pmod 2$ and $\partial_d(g) \neq 0$. In this case, $g^2 \in H$. Since $\varphi(d) = (e, b)$, $\varphi(ada) = (b, e)$ and the symbol d appears in the word $f_1 \ldots f_{k+1}$, representing the element g, then $\partial(g_l) \leq k$, $\partial(g_r) \leq k$, where $(g_l, g_r) = \varphi(g^2)$, and we may use the induction hypothesis.

Let $\partial_a(g) = 1 \pmod 2$, $\partial_d(g) = 0$, $\partial_c(g) \neq 0$, $\varphi(g^2) = (g_l, g_r)$. We have the following alternative: either $\partial(g_l) = k + 1$, $\partial_d(g_l) \neq 0$ (see [2]), or $\partial(g_l) \leq k$. An analogous statement holds for g_r. Thus we either arrive at the previous case, or we may use the induction hypothesis.

Let $\partial_a(g) = 1 \pmod 2$, $\partial_d(g) = 0$, $\partial_c(g) = 0$, $\partial_b(g) \neq 0$, $\varphi(g^2) = (g_l, g_r)$. We have the following alternative either $\partial(g_l) = k + 1$, $\partial_c(g_l) \neq 0$ (see [2]), or $\partial(g_l) \leq k$. An analogous statement holds for g_r, and we either arrive at the previous case, or we may use the induction hypothesis. Thus, we have proved that G is periodic.

3) Suppose that the group G has a representation $G = \langle a, b, c, d | R_i = e, i = 1\text{-}n \rangle$ with a finite set of defining relations. We call the quantity $\Sigma\partial(R_i)$ the length of the corresponding representation. Among all such representations, we choose one with minimal length. The representation of H (as a subgroup), constructed using the minimal representation of G, has lesser length than the original representation of G. Therefore, the factor-representation of G, obtained from the representation of H by applying the homomorphism described in 1), has lesser length than the original minimal representation of G. This contradiction shows that the group G cannot be defined by a finite set of defining relations. The theorem is proved.

Despite the fact that the group G was defined as a group of transformations on a space with a measure, it may be defined in purely algebraic terms. In particular, there exists a simple algorithm, allowing us to answer the following question for any word W in the generators of the group G: does W represent the unit element of G, or not? We also note that G is finitely approximable and has exponential growth. We give another example. Let ξ be the transformation of the square $[0, 1] \times [0, 1]$, consisting of the cyclic permutation of its quadrants, and let the transformation η be described as in Fig. 2 (S denotes the cyclic permutation of the quadrants of the square over which it is written). Then $\xi^4 = \eta^4 = e$ and the group generated by the transformations ξ and η is an infinite periodic group.

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VARIATIONAL PRINCIPLE FOR EQUATIONS INTEGRABLE BY THE INVERSE PROBLEM METHOD

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UDC 517.43

We consider (cf. [1]) the system of nonlinear equations representing the conditions for compatibility of two linear differential equations for a square N × N nonsingular matrix function $\Psi(\xi, \eta, \lambda)$

$$\Psi_\xi = U(\xi, \eta, \lambda)\Psi, \quad \Psi_\eta = V(\xi, \eta, \lambda)\Psi. \tag{1}$$

Here U and V are rational functions of the parameter λ with distinct simple poles:

$$U = U_0 + \sum_{n=1}^{N_1} \frac{U_n(\xi, \eta)}{\lambda - a_n}, \qquad V = V_0 + \sum_{n=1}^{N_2} \frac{V_n(\xi, \eta)}{\lambda - b_n}. \tag{2}$$

The compatibility conditions for Eqs. (1) have the form

$$U_{0\eta} - V_{0\xi} = [U_0, V_0], \tag{3}$$

$$U_{n\eta} = \left[U_n, V_0 + \sum_{k=1}^{N_2} \frac{V_k}{a_n - b_k}\right], \qquad V_{n\xi} = \left[V_n, U_0 + \sum_{k=1}^{N_1} \frac{U_k}{b_n - a_k}\right]. \tag{4}$$

It follows from (3) that there exists a nonsingular matrix $g(\xi, \eta)$ such that

$$U_0 = g_\xi g^{-1}, \quad V_0 = g_\eta g^{-1}. \tag{5}$$

We introduce the notation

$$\nabla_\eta = \frac{\partial}{\partial\eta} - V_0, \qquad \nabla_\xi = \frac{\partial}{\partial\xi} - U_0.$$

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