INVARIANT CONVEX CONES AND ORDERINGS IN LIE GROUPS

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In this article we find a criterion for the existence of an invariant convex cone in the space of an irreducible linear representation of a connected Lie group. The presence of an invariant convex cone in the tangent Lie algebra of the Lie group G is a necessary, but not sufficient, condition for the existence of a continuous invariant ordering in the group G (see the definition in Paragraph 3 of Sec. 1). In this article we find a criterion for the existence of a continuous invariant ordering in a connected simple Lie group.

At the end of the article we give a list of problems.

The facts used in the article about the structure of simple real Lie groups may be found, e.g., in [4, 3].

1. Formulation of the Results

1. In Theorems 1-3 the following notation is fixed: V is a finite-dimensional real vector space, $G \subset GL(V)$ is a connected irreducible semisimple linear Lie group, K is a maximal compact subgroup of G, T is a maximal connected triangular subgroup, and $P = N(T)$ is a minimal parabolic subgroup.

To avoid constant repetition, we agree to understand by a convex cone in the space V (if it is not otherwise stipulated) a closed convex cone different from $\{0\}$ and V.

THEOREM 1. There exists a convex cone in the space V, invariant with respect to G, if and only if any of the following equivalent conditions is satisfied:

1) in the space V there exists a vector different from zero which is invariant with respect to K;

2) in the space V, there exists a ray (with origin at zero) which is invariant with respect to P.

It is known that property 1) means that the space V may be equivariantly embedded in the space $\mathbf{R}[G/K]$ of polynomial functions on the symmetric space G/K. The equivalence of 1) and 2) allows us to obtain very simply (see Paragraph 6 of Sec. 2) the well-known description of "representations of class 1," which consists of the fact that these are irreducible representations, whose leading weights are real and even [5].

If conditions 1) and 2) are satisfied, then in the space V there exist a unique, to within proportion, nonzero vector v_K, which is invariant with respect to K, and a unique, to within proportion, nonzero vector v_T, which is an eigenvector for T (leading vector). In this case, the group G is absolutely irreducible as a linear group.

For any set $M \subset V$, we set

$$\mathrm{Con}\, M = \{\textstyle\sum c_i v_i : v_i \in M,\ c_i \in \mathbf{R}^+\}.$$

THEOREM 2. If the conditions of Theorem 1 are satisfied, then in the space V there exists a unique, to within multiplication by (−1), minimal invariant convex cone

$$C_{\min}(V) = \mathrm{Con}\, Gv_T = \overline{\mathrm{Con}\, Gv_K}. \quad (1)$$

Clearly, if C is an invariant convex cone in the space V, then

$$C^* = \{v' \in V' : \langle v, v' \rangle \geqslant 0\ \forall v \in C\}$$

is an invariant convex cone in the conjugate space V'.

THEOREM 3. If the conditions of Theorem 1 are satisfied, then in the space V there exists a unique, to within multiplication by (−1), maximal invariant convex cone

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$$C_{\max}(V) = (C_{\min}(V'))^*. \quad (2)$$

If $v \mapsto f_v$ is an equivariant embedding of V in $\mathbf{R}[G/K]$, then

$$C_{\max}(V) = \{v \in V: f_v(x) \geqslant 0 \ \forall x \in G/K\}. \quad (3)$$

Example. Let V be the space of real forms of degree m in n variables, and let G be the image of the group $GL_n(\mathbf{R})$ in its natural linear representation in the space V. An invariant convex cone in the space V exists if and only if m is even. Then $C_{\max}(V)$ is the set of positive semidefinite forms, and $C_{\min}(V)$ is the set of forms which may be represented as a sum of m-th degree linear forms.

2. Theorems 1-3 may be applied to the adjoint representation Ad of a connected noncommutative simple Lie group G. Denote by K the maximal subgroup of G for which the group Ad K is compact. It is known that the dimension of the center Z(K) of K is equal to 0 or 1.

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the tangent Lie algebras of the groups G and K, respectively, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the Cartan decomposition of the algebra $\mathfrak{g}$.

THEOREM 4. There exists a convex cone in the algebra $\mathfrak{g}$ which is invariant with respect to Ad G, if and only if

$$\dim Z(K) = 1. \quad (4)$$

If condition (4) is satisfied, then in the space $\mathfrak{m}$ there exists a complex structure, which is invariant with respect to Ad K. It may be defined by the formula

$$Iy = [k_0, y] \ (y \in \mathfrak{m}), \quad (5)$$

where k_0 is a suitable element of the center $\mathfrak{z}(\mathfrak{k})$ of the algebra $\mathfrak{k}$.

THEOREM 5. Let condition (4) be satisfied. The interior of the cone $C_{\max}(\mathfrak{g})$ consists of all elements of the algebra $\mathfrak{g}$, which are conjugate to elements of the set

$$\mathfrak{k}^+ = \{x \in \mathfrak{k}: I^{-1}\,\mathrm{ad}_{\mathfrak{m}} x > 0\}. \quad (6)$$

It is easily shown (see Paragraph 5 of Sec. 3) that $C_{\max}(\mathfrak{g}) = C_{\min}(\mathfrak{g})$ only in the case when the group G is locally isomorphic to $Sp_n(\mathbf{R})$. In this case, the elements of the algebra $\mathfrak{g}$ may be interpreted in a natural way as quadratic forms in $\mathbf{R}^n$, and the unique invariant cone in $\mathfrak{g}$ consists of all positive semidefinite forms.

3. The existence of an invariant cone in the Lie algebra $\mathfrak{g}$ is connected, although not in a unique way, with the existence of an invariant continuous ordering in the Lie group G.

An invariant ordering in the group G is a (partial) ordering which is invariant with respect to left and right shifts. An invariant ordering is defined uniquely by the set $P = \{g \in G: g \geqslant e\}$. The set P is a semigroup, which is invariant with respect to inner automorphisms; moreover,

$$P \cap P^{-1} = \{e\}. \quad (7)$$

Conversely, for any invariant semigroup $P \subset G$, satisfying condition (7), there exists an invariant ordering in G such that $P = \{g \in G: g \geqslant e\}$.

An ordering for which $P = \{e\}$ is called trivial. Henceforward we shall consider only nontrivial orderings.

An invariant ordering in the Lie group G is called continuous, if the semigroup P is closed and is topologically generated by any neighborhood of zero in it.

With each continuous invariant ordering in the Lie group G, there is associated an invariant strictly convex cone C(P) in the tangent Lie algebra $\mathfrak{g}$, consisting of all the vectors which are tangential to a curve lying in P. Moreover, $\exp C(P) \subset P$.

THEOREM 6. Let G be a connected simple Lie group. There exists a continuous invariant ordering in the group G if and only if its center Z(G) is infinite.

We note that by this theorem, a continuous invariant ordering may exist only in those connected simple Lie groups which do not admit an exact linear representation.

2. Proofs of Theorems 1-3

1. LEMMA. Any connected triangular group T of automorphisms of the (closed) convex cone $C \subset V$ has an eigenvector in C.

Proof. This is by induction on dim T. First consider the case dim T = 1. Let PV be a projective space associated with V, and let p be the canonical mapping from $V \setminus \{0\}$ onto PV. By the lemma in [1], any trajectory of the group T in the space PV has a limit, which is clearly a fixed point for T. Taking the trajectory of the point pv, $v \in C$, we obtain the fixed point pv_0, $v_0 \in C$. The vector v_0 is an eigenvector for T.

If dim T > 1, then we take a connected normal subgroup T_1 of T of codimension 1. By the induction hypothesis, the group T_1 has an eigenvector in C. This means that there exists a character χ: $T_1 \to \mathbf{R}^+$ such that the weighted subspace

$$V_1 = \{v \in V: gv = \chi(g)\, v \quad \forall g \in T_1\}$$

has nonzero intersection with C. The subspace V_1 is invariant with respect to T, and $V_1 \cap C$ is an invariant convex cone in it. Substituting V for V_1, we reduce the proof to the case when $V_1 = V$, i.e., all the operators in T_1 are scalar. In this case $T = T_1 \times T_2$, where T_2 is a one-parameter group, and the statement of the lemma is true, by the above-proved facts.

2. In the notation of Paragraph 1 of Sec. 1, suppose that in the space V there exists a convex cone C which is invariant with respect to G. The cone C is automatically strictly convex, i.e., $C \cap (-C) = \{0\}$. In fact, $C \cap (-C)$ is an invariant subspace different from V, and since G is irreducible, it must be zero.

Any compact group of automorphisms of a strictly convex cone has a fixed point inside it (see, e.g., [2]). Therefore, in the space V there exists a nonzero vector which is invariant with respect to K.

Moreover, by the lemma in Paragraph 1, the group T has an eigenvector in C. Let V_1 be a weighted subspace of the group T containing this vector. The subgroup P = N(T), which is connected in the Zariski topology, preserves the subspace V_1. It is known that

$$P = TL, \tag{8}$$

where L is a compact group. There exists a nonzero vector v_0 in the cone $V_1 \cap C$ which is invariant with respect to L. Clearly, the ray $\mathbf{R}^+ v_0$ is invariant with respect to the group P.

Thus, in each invariant convex cone $C \subset V$ there exists a nonzero vector, which is invariant with respect to K, and a ray which is invariant with respect to P.

3. Conversely, let v_K be a nonzero vector which is invariant with respect to K. By Iwasawa's decomposition

$$G = TK \tag{9}$$

the orbit of the vector v_K with respect to the group G coincides with its orbit with respect to T. There exists a subspace U of codimension 1, which is invariant with respect to T. Since the group T is connected, it preserves each of its (closed) semispaces U^+, U^-, bounded by the subspace U. For the sake of argument, let $v_K \in U^+$; then also $Gv_K \subset U^+$. Therefore, $C = \overline{\mathrm{Con}\, Gv_K} \neq V$. Clearly, the cone C is invariant with respect to G.

Analogously, let the ray $\mathbf{R}^+ v_P$ be invariant with respect to P. There exists an element $g \in G$, such that

$$G = \overline{(gTg^{-1})\, P}. \tag{10}$$

If U is a subspace of codimension 1, invariant with respect to T, and if $g^{-1}v_P \in U^+$, then it follows from (10) that $Gv_P \subset gU^+$, and therefore $\overline{\mathrm{Con}\, Gv_P} \neq V$.

Thus Theorem 1 is completely proved.

4. The vector v_P, which is an eigenvector for the parabolic subgroup P, is its leading vector. Since the group G is irreducible, all the leading vectors are proportional, and it follows from the results of Paragraphs 2 and 3 that the cone $\overline{\mathrm{Con}\, Gv_P}$ is the unique, to within multiplication by (−1), minimal invariant convex cone in the space V.

We note that in fact the cone $\mathrm{Con}\, Gv_P$ is closed. In fact, since G = KP, then $\mathrm{Con}\, Gv_P = \mathrm{Con}\, Kv_P$.

5. The vector v_K, which is invariant with respect to K, cannot lie in any subspace U of codimension 1, which is invariant with respect to T. In fact, conversely we would have $Gv_K = Tv_K \subset U$, which is impossible, as G is irreducible.

Hence it follows that the dimension of the space V^K of vectors which are fixed with respect to K is not greater than 1.

Suppose that $V^K \neq 0$. Then there exists a subspace of codimension 1, which is invariant with respect to K, and we prove analogously to the above that any weighted subspace of the group T is one-dimensional. Since any weighted subspace of the group T is invariant with respect to the parabolic subgroup P, a one-dimensional weighted subspace is a subspace spanned by the leading vector. Therefore, the leading vector is the unique, to within proportion, vector in the space V which is an eigenvector for the group T.

The statements of Theorem 2 follow from the results of Paragraphs 2-5.

6. Let B be a Borel subgroup of the group G_C containing T, and let v_0 be the leading vector of the space V_C with respect to B.

Reformulating the equivalence of conditions 1) and 2) of Theorem 1, bearing in mind the results of Paragraph 5, we may say that $V^K \neq 0$ if and only if the space V_C is irreducible, and its leading vector v_0 is invariant with respect to the compact group L in the decomposition (8). (In this case, the vector v_0 is proportional to a real vector, since it belongs to the unique one-dimensional weighted subspace of the group P.) We find a condition on the leading weight, under which the vector v_0 is invariant with respect to L.

Let D_0 be the Cartan subgroup of the group T. Its closure in the Zariski topology is the maximal decomposable (diagonable) torus D in the group G, where

$$D = D_0 \times F, \tag{11}$$

where $F = \{g \in D: g^2 = e\}$ is a finite Abelian group. Moreover, $P = \overline{T}L_0$, where $\overline{T} = TF$ is the closure of T in the Zariski topology, and L_0 is a connected compact subgroup, commuting with D. We may say that

$$L = FL_0. \tag{12}$$

Let S be some maximal torus in the group G, containing D. Then

$$S = DS^{im}, \tag{13}$$

where S^{im} is the maximal torus in the group L_0. Assuming that $S \subset B$, we denote by Λ the leading weight of the G_C-module V_C with respect to S_C and B. The leading vector v_0 is invariant with respect to L if and only if

$$\Lambda|_{S_{im}} = 1 \text{ and } \Lambda|_F = 1. \tag{14}$$

The first of these conditions denotes the reality of the weight Λ (more precisely, the reality of its values on S), and the second, its evenness in the group of characters of the torus D.

7. We prove Theorem 3. The proof of its first part is clearly contained in formula (2). Moreover, by Theorem 2, $C_{min}(V') = \overline{\mathrm{Con}\, Gv'_K}$, where v'_K is a nonzero vector in the space V, which is invariant with respect to K. Hence,

$$C_{max}(V) = \{v \in V: \langle v, gv'_K \rangle \geqslant 0 \ \forall g \in G\}.$$

Bearing in mind that the equivariant embedding of the space V in $\mathbf{R}[G/K]$ is defined by the formula $f_V(gK) = \langle v, gv'_K \rangle$, we obtain formula (3).

3. Proofs of Theorems 4 and 5

1. In the propositions of Paragraph 2 of Sec. 2, Ad K, as is known, has no fixed vectors in the space $\mathfrak{m}$. Therefore the set of elements of the algebra $\mathfrak{g}$, which are invariant with respect to the group Ad K, coincides with the center of the algebra $\mathfrak{k}$, and Theorem 4 follows from Theorem 1, applied to the group Ad G.

2. We will now suppose that the algebra $\mathfrak{g}$ satisfies the conditions of Theorem 4.

Any invariant convex cone $C \subset \mathfrak{g}$ contains k_0 or $-k_0$. Supposing that $C \ni k_0$, we prove that any element of the interior C^0 of the cone C is conjugate to an element of the set $\mathfrak{k}^+$ (see Theorem 5).

Let $x \in C^0$. Write x in the form $x = x_{im} + x_{re}$, where x_{im} is a semisimple element with purely imaginary eigenvalues, and x_{re} is an element commuting with it with real eigenvalues. The element x_{im} is conjugate to an element of $\mathfrak{k}$. Substituting x for the conjugate element, we see that $x_{im} \in \mathfrak{k}$.

Since the centralizer $\mathfrak{z}(x_{im})$ of the element x_{im} contains the Cartan subalgebra of the algebra $\mathfrak{k}$, which is also a Cartan subalgebra of the algebra $\mathfrak{g}$, then its center lies in $\mathfrak{k}$ and therefore consits of elements with purely imaginary eigenvalues. Therefore, the element x_{re} belongs to the commutator $\mathfrak{z}(x_{im})'$ of the algebra $\mathfrak{z}(x_{im})$.

Suppose that $x_{re} \neq 0$. If x_{re} is a semisimple element, there exists a nonzero element $y \in \mathfrak{z}(x_{im})'$, such that $[x_{re}, y] = ay$ $(a \in \mathbf{R}^*)$. Then for any $t \in \mathbf{R}$,

$$(\mathrm{Ad}\exp ty)\, x = x - aty \in C,$$

which is impossible, since the cone C is strictly convex. If x_{re} is not a semisimple element, then it may be replaced by a close semisimple element of the algebra $\mathfrak{z}(x_{im})'$, also with real eigenvalues, still satisfying the condition $x \in C^0$, and we obtain a contradiction by the above method. Thus $x_{re} = 0$ and $x = x_{im} \in \mathfrak{k}$.

Since $k_0 \in C$, $x + ak_0 \in C^0$ for any $a \geq 0$. If the operator $I^{-1}\mathrm{ad}_{\mathfrak{m}} x$ is not positive definite, then for some $a \geq 0$ the operator $I^{-1}\mathrm{ad}_{\mathfrak{m}}(x + ak_0) = I^{-1}\mathrm{ad}_{\mathfrak{m}} x + a\varepsilon$ (ε is the identity operator) is singular, i.e., there exists a nonzero element $z \in \mathfrak{m}$ such that $[x + ak_0, z] = 0$. Taking z sufficiently small, we have $x + ak_0 + z \in C^0$, which contradicts what has been proved above, since the eigenvalues of the element z are real. Thus $x \in \mathfrak{k}^+$, which is what was required.

3. By Theorem 2, the cone $C_{\min}(\mathfrak{g})$ is generated by the orbit of the leading root vector of the algebra $\mathfrak{g}$. We give a method of constructing the elements of the algebra $\mathfrak{g}$, which are conjugate to the leading root vector.

Consider the representation $k \mapsto \mathrm{Ad}_{\mathfrak{m}} k$ of the group K in the space $\mathfrak{m}$, given the complex structure by formula (5), and denote by ρ its rational continuation to the group K_C. Let z_0 be the leading vector for the representation ρ. Then for a suitable normalization of z_0, the element $z_0 + [Iz_0, z_0]$ is conjugate to the leading root vector.

To prove this, consider the **Z**-graduation of the algebra $\mathfrak{g}_C$, defined by the element k_0,

$$\mathfrak{g}_C = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \quad \mathfrak{g}_p = \{x \in \mathfrak{g}_C\colon [k_0, x] = pix\}.$$

Clearly, $\mathfrak{g}_0 = \mathfrak{k}_C$, $\mathfrak{g}_{-1} + \mathfrak{g}_1 = \mathfrak{m}_C$, where $\mathfrak{g}_{-1} = \bar{\mathfrak{g}}_1$ (the bar denotes complex conjugation in the algebra $\mathfrak{g}_C$). The mapping $x \mapsto \mathrm{Re}\, x$ establishes an isomorphism of K_C-modules $\mathfrak{g}_1$ and $\mathfrak{m}$.

Let H be the maximal torus of the group G lying in K. Let B be a Borel subgroup of the group G_C containing the Borel subgroup of the group K_C, containing H. Finally, let (e, h, f) be a standard basis of the three-dimensional root subalgebra, corresponding to the leading root of the algebra $\mathfrak{g}_C$ with respect to H_C and B. Since $\bar{h} = -h$, we may normalize e so that $f = \bar{e}$.

It is easily seen that $ih + e + f \in \mathfrak{g}$ is nilpotent, and therefore is conjugate to e.

Clearly, e is the leading vector for the representation of the group K_C in the space $\mathfrak{g}_1$ with respect to the Borel subgroup $B \cap K_C$. Therefore, $z_0 = \mathrm{Re}\, e = \frac{1}{2}(e + f)$ is the leading vector for the representation ρ of the group K_C in the space $\mathfrak{m}$.

Moreover, we have

$$Iz_0 = \mathrm{Re}\, ie = \frac{i}{2}(e - f), \quad [Iz_0, z_0] = \frac{i}{2}[e, f] = \frac{1}{2} ih,$$

so that $z_0 + [Iz_0, z_0] = \frac{1}{2}(ih + e + f)$, and hence the required statement follows.

4. Fix an invariant scalar multiplication in the algebra $\mathfrak{g}$, which is positive definite on $\mathfrak{k}$ and negative definite on $\mathfrak{m}$. Using this, we establish an isomorphism between the space $\mathfrak{g}$ and its conjugate space. By Theorem 3, we then have $C_{\max}(\mathfrak{g}) = (C_{\min}(\mathfrak{g}))^*$, where the conjugate cone is understood in the sense of scalar multiplication in $\mathfrak{g}$.

By Paragraph 3, the cone $C_{\min}(\mathfrak{g})$ is generated by the orbit of the element $e_0 = z_0 + [Iz_0, z_0]$. Since the ray $\mathbf{R}^+ e_0$ is invariant with respect to the parabolic subgroup conjugate to P, then $Ge_0 = \mathbf{R}^+ Ke_0$, and therefore

$$(C_{\max}(\mathfrak{g}))^0 = \{x \in \mathfrak{g}\colon ((\mathrm{Ad}\, k)x, e_0) > 0, \ \forall k \in K\}. \tag{15}$$

To complete the proof of Theorem 5, we need to verify that $\mathfrak{k}^+ \subset (C_{\max}(\mathfrak{g}))^0$.

Let $x \in \mathfrak{k}^+$; then

$$(x, e_0) = (x, [Iz_0, z_0]) = -([x, z_0], \ Iz_0) = -(I^{-1}[x, z_0], z_0) > 0.$$

(We recall that scalar multiplication is negative definite in $\mathfrak{m}$.) Since the set $\mathfrak{k}^+$ is invariant with respect to Ad K, then $((\mathrm{Ad}\, k)x, e_0) > 0 \ \forall k \in K$. By (15) this means that $x \in (C_{\max}(\mathfrak{g}))^0$, which is what was required.

5. Using Theorem 5, it is easy to calculate, for all possible algebras $\mathfrak{g}$, the intersection of the cone $C_{\max}(\mathfrak{g})$ with the Cartan subalgebra $\mathfrak{h}$ of the algebra $\mathfrak{k}$. Clearly,

$$C_{\min}(\mathfrak{g}) \cap \mathfrak{h} \subset (C_{\max}(\mathfrak{g}) \cap \mathfrak{h})^*,$$

and if the cone $C_{\max}(\mathfrak{g}) \cap \mathfrak{h}$ does not contain its conjugate cone in $\mathfrak{h}$, then $C_{\min}(\mathfrak{g}) \neq C_{\max}(\mathfrak{g})$. The same is true for all cases, except when $\mathfrak{g}$ is the Lie algebra of the group $Sp_n(\mathbf{R})$.

4. Proof of Theorem 6

1. For any sequence (g_n) converging to one of elements of the Lie group G, the limit

$$x = \lim n (g_n - e), \tag{16}$$

calculated by coordinates, does not depend on the choice of the system of coordinates, if it is integrated as an element of the tangent Lie algebra $\mathfrak{g}$. In particular, writing equation (16) in exponential coordinates, we obtain

$$\exp x = \lim g_n^n. \tag{17}$$

Let P be the defining semigroup of a continuous invariant ordering in the Lie group G. Denote by C(P) the set of all limits of the form (16) for sequences (g_n) lying in P. It follows from (17) that

$$\exp C(P) \subset P. \tag{18}$$

By multiplying out sequences, we prove that C(P) is an additive semigroup, and by going to subsequences of the form (g_{kn}), we prove that C(P) sustains multiplication by $\frac{1}{k}$ $(k \in Z^+)$ and thus by any positive rational numbers. It follows from (18), and from the fact that P is closed, that $\exp tx \in P$ for $x \in C(P)$ and any $t \geq 0$. Hence in turn it follows that C(P) sustains multiplication by any positive numbers, and thus is a convex cone.

These results show, in particular, that the cone C(P) may be defined as the set of vectors tangential to one-parameter semigroups lying in P. Therefore, since P is closed it follows that C(P) is closed, and from property (7) we obtain the strict convexity of C(P).

Clearly the cone C(P) is invariant with respect to Ad G.

We finally show that $C(P) \neq \{0\}$. It follows from the definition of continuous ordering that there exists a sequence in $P \setminus \{e\}$ converging to e. Choose a subsequence (g_n) of it such that the unit vector $(g_n - e)/|g_n - e|$, calculated in some exponential system of coordinates, has a limit, and such that $|g_n - e| < 1/n^2$. Then for suitable natural numbers k_n we have

$$\left| |g_n^{k_n} - e| - \frac{1}{n} \right| < \frac{1}{n^2}$$

and hence,

$$\lim n (g_n^{k_n} - e) = \lim \frac{g_n - e}{|g_n - e|} \neq 0.$$

2. We prove the "necessity" in Theorem 6. In the group G, let there exist a continuous invariant ordering, and let P be its defining semigroup. Then C(P) is an invariant convex cone in the tangent Lie algbera $\mathfrak{g}$ and by Theorem 4, $\dim Z(K) = 1$.

The connected component $Z(K)^0$ of the group Z(K) is a one-parameter group with direction vector k_0. Multiplying k_0 by -1 if necessary, we may assume that $k_0 \in C(P)$; then $\exp tk_0 \in P$ for all $t \geq 0$. Hence it follows that the one-parameter group $Z(K)^0$ is not periodic; for otherwise, $P \cap P^{-1} \neq \{e\}$. Since the group $Z(K)/Z(G) = Z(\mathrm{Ad}_{\mathfrak{g}} K)$ is compact, it follows from the noncompactness of Z(K) that Z(G) is infinite.

3. Suppose now that C is an invariant convex cone in the algebra $\mathfrak{g}$.

Moving the cone C by left shifts, we obtain a left-invariant "field of cones" $\{gC\}$ on the group G. It follows from the invariance of the cone C that this field of cones is also right-invariant.

LEMMA. Under the inversion $x \mapsto x^{-1}$ the field of cones $\{gC\}$ goes to the field of opposite cones $\{-gC\}$.

Proof. We may represent inversion as the composition of a left shift on g, inversion, and then a right shift on g. Under these transformations, the cone $g^{-1}C$ goes successively to C, $-C$ and $-gC$.

4. The pointwise-differentiable curve g(t) on the group G is called admissible, if $g'(t) \in g(t)C \ \forall t$.

Denote by P_0 the set of all elements $g \in G$, for which there exists an admissible curve joining e and g, and denote its closure by P. Clearly, P_0 and P are invariant semigroups, and P_0 is generated (and P is topologically generated) by a neighborhood of one in it.

We call the differentiable function φ on the group G increasing, if it is nonconstant and

$$d_g\varphi \geqslant 0 \text{ on } gC \quad \forall g \in G. \tag{19}$$

The set of increasing functions is invariant with respect to left and right shifts. Moreover, it follows from the lemma of Paragraph 3 that for any increasing function φ, the function $\hat{\varphi}(x) = -\varphi(x^{-1})$ is also increasing.

An increasing function increases in the normal sense of the word (possibly not strictly) along any admissible curve. Therefore, if φ is an increasing function,

$$\varphi(g) \geqslant \varphi(e) \geqslant \varphi(g^{-1}) \quad \forall g \in P. \tag{20}$$

LEMMA. If there exists an increasing function φ, then $P \cap P^{-1} = \{e\}$.

Proof. By shifting the function φ, we may obtain $d_e\varphi \neq 0$. Moreover, using inner automorphisms we may obtain increasing functions $\varphi_1, \ldots, \varphi_n$ from φ, which form a system of coordinates in some neighborhood $O(e)$ of one. We shall assume for the sake of argument that $\varphi_i(e) = 0$ for all i and that $O(e)$ in the coordinates $\varphi_1, \ldots, \varphi_n$ is a sphere of radius 1 with center at the origin of coordinates.

It follows from (20) that $\varphi_i(g) = 0 \quad \forall g \in P \cap P^{-1}$. Therefore, $(P \cap P^{-1}) \cap O(e) = \{e\}$. Let $B \subset O(e)$ be a concentric closed sphere with radius $r < 1$ (in the coordinates $\varphi_1, \ldots, \varphi_n$). Any admissible curve starting at e and finishing outside B intersects the boundary of B; therefore, at its end g satisfies the inequality $\Sigma\varphi_i(g)^2 \geq r^2$. By continuity, this inequality also holds at all points $g \in P \cap (G - B)$. Hence it follows that $(P \cap P^{-1}) \cap (G - B) = \phi$. The lemma is proved.

5. Let the group G satisfy the condition of Theorem 6, i.e., the center Z(G) is infinite.

Denote by T the maximal connected subgroup of G for which the group $\mathrm{Ad}_{\mathfrak{g}}T$ is triangular. Then

$$G = KT, \tag{21}$$

and the mapping $(k, u) \mapsto ku$ is a diffeomorphism from $K \times T$ onto G.

If the group G is simply connected, then

$$K = K' \times Z(G)^0, \tag{22}$$

where K' is a simply-connected semisimple compact Lie group, and

$$Z(K)^0 = \{\exp tk_0\} \simeq \mathbf{R}. \tag{23}$$

The center of the group G lies in K and is the direct product of a finite group lying in K' and the infinite cyclic group.

In the general case, the group G is obtained by simply connected factorization by a finite central subgroup. Therefore the decomposition (22) and the isomorphism (23) also hold, with the difference that the group K' is not necessarily simply connected.

Define a differentiable function φ on G by the condition

$$g = k(\exp \varphi(g)k_0)u \quad (k \in K',\ u \in T). \tag{24}$$

In the following paragraphs we shall prove that φ is an increasing function with respect to the cones $C = C_{\min}(\mathfrak{g})$. By the lemma in Paragraph 4, it follows that the semigroup P constructed as in Paragraph 4 defines a continuous invariant ordering on the group G. This will complete the proof of Theorem 6.

6. The function φ is right-invariant with respect to T and left-invariant with respect to K', and under a left shift by an element of $Z(K)^0$ a constant is added to it. Therefore, its differential $d\varphi$ is right-invariant with respect to T, and left-invariant with respect to K.

We need to prove that

$$d_g\varphi \geqslant 0 \quad \text{on} \quad gC_{\min}(\mathfrak{g}) \tag{25}$$

for all $g \in G$. In view of the above invariance properties of $d\varphi$, it is sufficient to verify (25) for g = e.

The space $\mathfrak{g}$ splits up into a direct sum of subspaces: $\mathfrak{g} = \mathfrak{k}' + \mathfrak{z}(\mathfrak{k}) + \mathfrak{t}$, where $\mathfrak{k}'$ and $\mathfrak{t}$ are tangent Lie algebras of the groups K' and T, respectively. Clearly, $d_e\varphi = 0$ on $\mathfrak{k}' + \mathfrak{t}$ and $d_e\varphi(k_0) = 1$.

We will prove in Paragraph 11 that the cone $C_{\min}(\mathfrak{g})$, containing k_0, lies completely on one side of the hyperplane $\mathfrak{k}' + \mathfrak{t}$. Hence it will follow that φ is an increasing function.

In Paragraphs 7-10 we will find an expression for the element k_0 in terms of the real root decomposition of the algebra $\mathfrak{g}$.

7. LEMMA. Let τ: $SL_2(\mathbf{R}) \to GL(V)$ be an irreducible (n + 1)-dimensional representation. There exists a nonzero vector v in the space V which is invariant with respect to $\tau(SO_2)$, if and only if n is even, and in this case the vector v is the sum of eigenvectors of the operator $d\tau\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$, whose eigenvalues are congruent to n modulo 4.

Proof. The representation τ is isomorphic to the natural representation τ_n of the group $SL_2(\mathbf{R})$ in the space of binary forms of degree n. The degree of the sum of the squares, which only exists for even n, is a unique (to within proportion) binary form, which is invariant with respect to SO_2. It splits up into a sum of monomials, in which both indices are even, and thus the difference in indices is congruent to n modulo 4. It remains to note that each monomial is an eigenvector of the operator $d\tau_n\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$, and the eigenvalue is equal to the difference of the indices.

8. Let $\mathfrak{d}$ be the Cartan subalgebra of the algebra $\mathfrak{t}$, which is invariant with respect to the Cartan involution σ of the algebra $\mathfrak{g}$. Denote by Δ the system of roots of the algebra $\mathfrak{g}$ with respect to $\mathfrak{d}$, and by $\mathfrak{g}_\alpha$ the root subspace corresponding to the root α. Then

$$\mathfrak{g} = \mathfrak{d} + \mathfrak{l} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha, \tag{26}$$

$$\mathfrak{t} = \mathfrak{d} + \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha, \tag{27}$$

where $\mathfrak{l}$ is the centralizer of $\mathfrak{d}$ in $\mathfrak{k}$, and Δ_+ is a system of positive roots (with respect to a suitable ordering of the space $\mathfrak{d}'$). The algebra $\mathfrak{l}$ is the tangent Lie algebra of the maximal compact subgroup L of the group P = N(T) [see formula (8) and Paragraph 6 of Sec. 1].

Since $\sigma|_{\mathfrak{d}} = -1$, then $\sigma\mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$ for any $\alpha \in \Delta$.

If γ_0 is the leading root of the system Δ, then the vectors in the space $\mathfrak{g}_{\gamma_0}$ are eigenvectors for the group T. The results of Sec. 2 (formulated in Paragraph 1 of Sec. 1) show that

1) the space $\mathfrak{g}_{\gamma_0}$ is one-dimensional, and its basis vector e_0 is the leading root vector of the algebra $\mathfrak{g}$ (with respect to the Cartan subalgebra containing $\mathfrak{d}$);

2) the vector e_0 is invariant with respect to L;

3) for suitable orientation $e_0 \in C_{\min}(\mathfrak{g})$.

9. Denote by W the Weil group of the algebra $\mathfrak{g}$ with respect to $\mathfrak{d}$, and by Γ the subsystem of the system of roots Δ, formed by the roots of maximal length. For each $\gamma \in \Gamma$, there exists an element $w \in W$, such that $\gamma = w\gamma_0$. As a transformation of the space $\mathfrak{d}$, the element w is induced by some element $\bar{w}$ of the normalizer $N_K(\mathfrak{d})$ of the algebra $\mathfrak{d}$ in the group K. Set $e_\gamma = (\mathrm{Ad}\,\bar{w})e_0$. It follows from statement 2) of Paragraph 8 that e_γ does not depend on the arbitrariness of the choice of w and $\bar{w}$, and that

$$\sigma e_\gamma = e_{-\gamma}. \tag{28}$$

There exist homomorphisms f_γ: $SL_2(\mathbf{R}) \to G$ $(\gamma \in \Gamma)$, satisfying the conditions

$$df_\gamma\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = ce_\gamma,\quad df_\gamma\begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix} = ce_{-\gamma},\quad df_\gamma\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = h_\gamma \in \mathfrak{d} \quad (c > 0). \tag{29}$$

We norm the vector e_0 so that c = 1.

LEMMA. a) Δ is a system of roots of type BC_n or C_n;

b) Γ is a system of roots of type nA_1;

c) $\dim \mathfrak{g}_\gamma = 1$ for all $\gamma \in \Gamma$;

d) $k_0 = \frac{1}{2}\sum_{\gamma\in\Gamma} e_\gamma + l_0$, where l_0 lies in the center of the algebra $\mathfrak{l}$.

Proof. In view of the equivalence of roots of maximal length, c) follows from statement 1) of Paragraph 8. Moreover, it is easily seen that from all the irreducible systems of roots, only in systems of type BC_n or C_n do the roots of maximal length form a subsystem of type nA_1, so that a) follows from b).

We prove b) and d). Let

$$k_0 = \sum_{\gamma\in\Gamma} a_\gamma e_\gamma + \sum_{\alpha\in\Delta\setminus\Gamma} e_\alpha + d_0 + l_0, \tag{30}$$

where $a_\gamma \in \mathbf{R}$, $e_\alpha \in \mathfrak{g}_\alpha$, $d_0 \in \mathfrak{d}$, $l_0 \in \mathfrak{l}$. Since $e_\gamma \in C_{\min}(\mathfrak{g})$ for any $\gamma \in \Gamma$ and $(e_\gamma, e_{-\gamma}) = \frac{1}{2}|e_\gamma + e_{-\gamma}|^2 > 0$, then

$$a_\gamma = \frac{(k_0, e_{-\gamma})}{(e_\gamma, e_{-\gamma})} > 0 \quad \forall \gamma \in \Gamma.$$

Moreover, it follows from the invariance of k_0 with respect to $N_K(\mathfrak{d})$ that all the a_γ are equal to each other. Thus $a_\gamma = a > 0$ for all $\gamma \in \Gamma$.

Since $e_\gamma + e_{-\gamma} \in \mathfrak{k}$, then $f_\gamma(SO_2) \subset K$ and therefore the element k_0 is invariant with respect to $f_\gamma(SO_2)$.

Split the space $\mathfrak{g}$ into the direct sum of irreducible invariant subspaces with respect to the representation $\tau_\gamma = \mathrm{Ad} \circ f_\gamma$ of the group $SL_2(\mathbf{R})$. This can be done so that the terms in the decomposition are invariant with respect to ad $\mathfrak{d}$; under this condition, we call them γ-components. Clearly, the projections of the element k_0 onto the γ-components are invariant with respect to $\tau_\gamma(SO_2)$.

For any $\gamma, \gamma' \in \Gamma$ the element $e_{\gamma'}$ is either the leading or the lowest vector of the γ-component of dimension $\gamma'(h_\gamma) + 1$. Since the projection of k_0 onto this component is different from zero, then by the lemma in Paragraph 7, the number $\gamma'(h_\gamma)$ is even. Thus, all the elements of the Cartan matrix of the root system Γ are even. Hence it follows that Γ is a system of type nA_1.

For any $\alpha \in \Delta \setminus \Gamma$ there exists $\gamma \in \Gamma$ such that $\alpha, \alpha + \gamma \in \Delta$, but $\alpha - \gamma, \alpha + 2\gamma \notin \Delta$. The space $\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\gamma}$ is then the sum of two-dimensional γ-components. By the lemma in Paragraph 7, the projections of the element k_0 onto these components are equal to zero. Therefore, $e_\alpha = 0$ for all $\alpha \in \Gamma \setminus \Delta$.

The elements h_γ $(\gamma \in \Gamma_+)$ form a basis of the space $\mathfrak{d}$. Set $d_0 = \sum_{\gamma\in\Gamma_+} c_\gamma h_\gamma$ $(c_\gamma \in \mathbf{R})$; then $a(e_\gamma + e_{-\gamma}) + c_\gamma h_\gamma$ is the projection of the element k_0 onto the three-dimensional γ-component $\langle e_\gamma, h_\gamma, e_{-\gamma}\rangle$. By the lemma in Paragraph 7, we obtain $c_\gamma = 0$. Thus $d_0 = 0$.

The element $l_0 = k_0 - a \sum_{\gamma\in\Gamma} e_\gamma$ lies in the center of $\mathfrak{l}$, since k_0 and e_γ $(\gamma \in \Gamma)$ commute with $\mathfrak{l}$ [see statement 2) of Paragraph 8].

Finally, bearing in mind that $[e_{\gamma'}, e_\gamma] = 0$ for $\gamma' \neq -\gamma$, from the condition $[k_0, [k_0, h_\gamma]] = I^2 h_\gamma = -h_\gamma$ we have $a = 1/2$. The lemma is proved.

10. LEMMA. $l_0 \in C_{\max}(\mathfrak{g})$.

Proof. We have

$$I^{-1}\mathrm{ad}_{\mathfrak{m}} l_0 + \frac{1}{2} I^{-1}\mathrm{ad}_{\mathfrak{m}} \sum_{\gamma\in\Gamma} e_\gamma = I^{-1}\mathrm{ad}_{\mathfrak{m}} k_0 = \varepsilon.$$

We prove that $I^{-1}\mathrm{ad}_{\mathfrak{m}} l_0$ and $\frac{1}{2} I^{-1}\mathrm{ad}_{\mathfrak{m}} \sum_{\gamma\in\Gamma} e_\gamma$ are orthogonal projectors. It will then follow by Theorem 5 that $l_0 + \frac{\lambda}{2} \sum_{\gamma\in\Gamma} e_\gamma \in C_{\max}(\mathfrak{g})$ for $\lambda > 0$, and the statement of the lemma will then be obtained by taking the limit as $\lambda \to 0$.

Set $\Gamma_+ = \Gamma \cap \Delta_+$; then

$$\sum_{\gamma\in\Gamma} e_\gamma = \sum_{\gamma\in\Gamma_+} df_\gamma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{a},$$

where $\mathfrak{a} = \mathfrak{d} + \sum_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ is a subalgebra of type nA_1. It is easily seen that the representation $\mathrm{ad}_{\mathfrak{g}}$ of the algebra $\mathfrak{a}$ has irreducible components of only the following types: 1) the adjoint representation of one of the simple terms; 2) the tensor product of the simplest (two-dimensional) representations of two simple terms; and 3) the trivial representation.

Therefore, the characteristic numbers of the operator $\mathrm{ad} \sum_{\gamma\in\Gamma} e_\gamma$ are $\pm 2i, 0$. Thus the characteristic numbers of the operator $q = \frac{1}{2} I^{-1}\mathrm{ad}_{\mathfrak{m}} \sum_{\gamma\in\Gamma} e_\gamma$ may only be ± 1 or 0; however, -1 is excluded since $\sum_{\gamma\in\Gamma} e_\gamma \in C_{\min}(\mathfrak{g})$ and by Theorem 5, the operator q is positive definite. Thus q is a projector, which is what was required.

We note that in the majority of cases the algebra $\mathfrak{l}$ is semisimple and $l_0 = 0$.

11. To complete the proof of Theorem 6, it remains to show that the cone $C_{\min}(\mathfrak{g})$ lies completely on one side of the hyperplane $\mathfrak{k}' + \mathfrak{t}$ (see Paragraph 6).

We find a vector orthogonal to $\mathfrak{k}' + \mathfrak{t}$. To do this, we represent k_0 as the sum of the vectors

$$u = \sum_{\gamma \in \Gamma_+} e_\gamma + l_0 \in [\mathfrak{t}, \mathfrak{t}] + \mathfrak{l}, \quad z = \frac{1}{2} \sum_{\gamma \in \Gamma_+} (e_{-\gamma} - e_\gamma) \in \mathfrak{m}.$$

The vector u is orthogonal to $\mathfrak{k}' + \mathfrak{t}$, and moreover $(u, k_0) = (k_0, k_0) > 0$.

By the lemma in Paragraph 10, $u \in C_{\max}(\mathfrak{g})$, since $(u, x) \geq 0$ for all $x \in C_{\min}(\mathfrak{g})$. Thus all the elements of the cone $C_{\min}(\mathfrak{g})$ lie on one side of the hyperplane $\mathfrak{k}' + \mathfrak{t}$, which is what was required.

5. Problems

1. Is it true that in the space of an irreducible representation of a connected Lie group, there may only exist a finite number of invariant convex cones?

2. Find all the connected irreducible linear Lie groups for which an invariant convex cone exists and is unique to within multiplication by −1.

3. Find a necessary and sufficient condition for the existence of an invariant strictly convex cone in the space of an irreducible representation of a connected semisimple Lie group.

4. Describe all the invariant convex cones C in a simple Lie algebra in terms of the intersection with the Cartan subalgebra $\mathfrak{h}$ of a maximal compact subalgebra. Is it true that $C^* \cap \mathfrak{h} = (C \cap \mathfrak{h})^*$?

5. Let P be the defining semigroup of an irreducible invariant ordering in the connected Lie group G, and let C(P) be the corresponding convex cone in the tangent Lie algebra (see Paragraph 1 of Sec. 4). Is it true that

a) the mapping exp: $C(P) \to P$ is open at zero;

b) the semigroup P is generated (in the algebraic sense) by any neighborhood of unity in it?

The same questions may be asked for an arbitrary closed semigroup $P \subset G$, topologically generated by any neighborhood of unity in it.

6. Is it true that for any invariant strictly convex cone C in the tangent Lie algebra of a simply connected Lie group G, there exists a continuous invariant ordering in G, whose defining semigroup P satisfies the condition C(P) = C?

7. We define in a natural way the idea of a continuous invariant ordering in the homogeneous space of a Lie group. It would be interesting to describe all the connected homogeneous spaces with irreducible connected isotropy group, which admit a continuous invariant ordering (a generalization of Theorem 6).

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