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CONSTRUCTION OF HIGHER-DIMENSIONAL NONLINEAR INTEGRABLE SYSTEMS AND OF THEIR SOLUTIONS

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1. Introduction

As the inverse scattering method (ISM) develops, the tools for constructing nonlinear equations to which this method applies get themselves improved. Such nonlinear equations may be referred to, somewhat conventionally, as integrable (rigorous integrability was proved only for a few of them). Of particular interest are the higher-dimensional integrable systems, which contain derivatives with respect to more than two variables.

An effective procedure for simultaneous construction of integrable systems and of their solutions, known presently as the "dressing method," was developed in two distinct variants in papers [1] and [2]. In both approaches the solutions were constructed using infinite dimensional analogs of the Gauss' problem of factorizing a matrix into a product of two triangular ones. In [1] the problem was the factorization of an integral operator on the line into a product of two Volterra operators, and in [2] — that of representing a function, given on a contour in the complex plane, as a product of the functions analytic in the interior and respectively in the exterior of the contour. This second problem will be called below the local Riemann problem. The two problems are equivalent in the case where the junction contour is the real line and the kernel of the operator that undergoes the factorization depends on the difference of the arguments.

Back in paper [1] it was shown that the technique of factorization of integral operators on the line is naturally fitted for solving certain classes of integrable systems that contain functions of three variables. As it turned out, among these first higher-dimensional integrable systems there are some which are very interesting from the viewpont of applications, the most notable being the Kadomtsev—Petviashvili (KP) equation and the "problem of three-wave interaction." In the years that followed their discovery, these systems were the subject of numerous investigations (see, e.g., [3-7]). In parallel, successful attempts were made to enlarge the class of higher-dimensional systems integrable by the factorization technique [8-10]. A principal feature of all these systems is that one of the space coordinates is irrevocably singled-out (namely, the one which is the variable of the functions on which the integral operators act).

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A new step was made in paper [3], in which it was virtually shown that the problem of the factorization of an integral operator on the line is equivalent to solving on the line, in the dual space, a nonlocal Riemann problem which requires to find functions analytic on the different sides of the contour and which are connected on the contour by an integral relation. It turned out that the higher-dimensional dressing method is naturally related to the nonlocal Riemann problem. Further, it was shown in [11] that an algebraic construction is connected with the nonlocal Riemann problem, which permits to build classes of overdetermined, yet compatible systems of linear equations. Moreover, it was conjectured that, using them, one may be able to construct, in a regular way, higher dimensional integrable equations in which all variables enter on an equal status. Some examples of such equations were indeed produced. The path towards the implementation of this program was outlined in [12], where it was shown that for the unknown functions in the nonlinear equations it is natural to take "dressing data" for linear operators. Earlier, an analogous idea has been successfully realized in the theory of two-dimensional integrable systems [13].

The present work has two aims: one is to give a systematic exposition of the procedure of constructing higher dimensional equations integrable by means of the Riemann problem. We bring to completion (under reasonable constraints) the results of paper [12] and give a proof of the main conjecture given therein. We also show that among the systems integrable by means of the nonlocal problem, a natural place is occupied by the higher-dimensional systems described in [8-10] and [15]. The second aim of the paper is to extend the resources of the dressing method, applied to equations already known. One way of achieving this was to pass from the language of the factorization problem to that of the Riemann problem, and thus take advantage of the freedom allowed in the choice of the junction contour. There is however a different way, namely to use for dressing a more general problem: "the nonlocal $\bar{\partial}$-problem." The latter is a natural generalization of the nonlocal Riemann problem and represents a very convenient device for producing exact solutions of two- and higher-dimensional integrable equations. (One can show that the "direct linearization" method mentioned in papers [6, 7] reduces in the most general case to precisely this problem). Finally, we compare the resources of the various variants of the dressing method.

2. Nonlocal Riemann Problem

Consider in the complex λ plane a contour Γ and suppose that on $\Gamma \times \Gamma$ there is given a complex $N \times N$ matrix-valued function $T(\lambda, \lambda')$ of two variables. We pose the following problem: to find a function $\chi(\lambda)$ analytic everywhere off the contour Γ, whose boundary values χ_1 and χ_2 on Γ are connected by the integral relation

$$\chi_2(\lambda) = \chi_1(\lambda) + \int_\Gamma \chi_1(\lambda') T(\lambda', \lambda)\, d\lambda', \tag{2.1}$$

which will be written in the form

$$\chi_2 = \chi_1 + \chi_1 * T. \tag{2.2}$$

Relation (2.1) gives a nonlocal Riemann problem on Γ, the solution of which is plainly not unique: function χ may be multiplied at left by an arbitrary constant matrix g. To ensure uniqueness, the solution of the Riemann problem must be normalized by specifying the value of χ at some arbitrary point λ. Hereafter we shall put $\chi(\infty) = 1$.

The problem formulated above is equivalent to solving the integral equation

$$k(\lambda) = \int_\Gamma T(\lambda', \lambda)\, d\lambda' + \frac{1}{2\pi i} \iint_\Gamma \frac{k(\xi) T(\lambda', \lambda)}{\lambda' - \xi + i0}\, d\lambda'\, d\xi \qquad \lambda \in \Gamma \tag{2.3}$$

for the function $k(\lambda)$, which is the jump of $\chi(\lambda)$ on the contour: $k(\lambda) = \chi_2(\lambda) - \chi_1(\lambda)$, $\lambda \in \Gamma$. In fact, off Γ function $\chi(\lambda)$ admits the representation

$$\chi(\lambda) = 1 + \frac{1}{2\pi i} \int_\Gamma \frac{k(\lambda')}{\lambda' - \lambda}\, d\lambda'. \tag{2.4}$$

Substituting (2.4) into (2.1) we are immediately led to Eq. (2.3); the symbol "+i0" in (2.3) means that when one integrates with respect to λ' one goes around the contour on the side where the boundary value χ_1 is given.

In what follows we shall always assume that Eq. (2.3), together with problem (2.1), is uniquely solvable. A sufficient condition for unique solvability is, for example, that function $\tilde{T}(\xi,\lambda)=\frac{1}{2\pi i}\int\frac{T(\lambda',\lambda)}{\lambda'-\xi+i0}\,d\lambda'$ be small. More precisely, if the norm of $\tilde{T}$ in $L_2(\Gamma\times\Gamma)$ is not larger than one, then the operator $1-\tilde{T}$ is invertible, and Eq. (2.3) has a unique solution.

The nonlocal Riemann problem possesses an interesting class of explicit solutions which contain functional parameters. Such solutions arise when the operator T in (2.2) is degenerate, i.e., function $T(\lambda', \lambda)$ is representable as

$$T(\lambda',\lambda)=\sum_{n=1}^{N} f_n(\lambda')\,g_n(\lambda). \tag{2.5}$$

In this case Eq. (2.3) obviously reduces to a system of N linear algebraic equations for the N quantities c_n that determine $k(\lambda)$ by

$$k(\lambda)=\sum_{n=1}^{N} c_n g_n(\lambda). \tag{2.6}$$

This system has the form

$$c_n+\sum c_m A_{mn}=h_n, \tag{2.7}$$

where

$$h_n=\int_\Gamma f_n(\lambda)\,d\lambda;\quad A_{mn}=-\frac{1}{2\pi i}\int_\Gamma\int_\Gamma\frac{g_m(\lambda)\,f_n(\lambda')}{\lambda'-\lambda+i0}\,d\lambda\,d\lambda'.$$

A particular case of the nonlocal Riemann problem is, of course, the local Riemann problem, in which

$$T(\lambda,\lambda')=T(\lambda)\,\delta(\lambda-\lambda'). \tag{2.8}$$

In this case Eq. (2.3) becomes

$$k(\lambda)=\left[1+\frac{1}{2\pi i}\int_\Gamma\frac{k(\xi)}{\lambda-\xi+i0}\,d\xi\right]T(\lambda) \tag{2.9}$$

and does not admit simple solutions analogous to (2.6).

3. Ideals in the Ring of Differential Operators, Generated by a Nonlocal Riemann Problem

If the kernel T of the nonlocal Riemann problem depends on some variable x, regarded as a parameter, then the solution $\chi(\lambda)$ of this problem will also depend on x. When T depends on variables x_i ($i = 1, \ldots, n$) following a special rule, a set of linear partial differential equations for $\chi(\lambda)$ can be written. As we shall see in the next section, these linear systems lead in turn to the desired nonlinear integrable equations.

Thus, suppose that function $T(\lambda',\lambda)$ depends on n supplementary variables $x_1, \ldots, x_n$ in such a way that

$$\frac{\partial T(\lambda',\lambda)}{\partial x_i}=I_i(\lambda')\,T(\lambda',\lambda)-T(\lambda',\lambda)\,I_i(\lambda), \tag{3.1}$$

where $I_i(\lambda)$ are pairwise-commuting matrix-valued rational functions of the parameter λ. We denote by D_i the "lengthened derivatives"

$$D_i f=\frac{\partial f}{\partial x_i}+fI_i(\lambda). \tag{3.2}$$

Clearly, $[D_i, D_j] = 0$.

Consider the ring m of differential operators of the form

$$M=\sum q_k D^k. \tag{3.3}$$

Here k are multiindices ($D^k=D_1^{k_1}\ldots D_n^{k_n}$), and $q_k(x_1,\ldots,x_n)$ are matrix-valued functions which do not depend on λ.

Differentiating Eq. (2.1) with respect to x_i and using relation (3.1) we get

$$D_i\chi_2 = D_i\chi_1 + D_i\chi_1 * T. \tag{3.4}$$

By induction it follows that for every operator M in m

$$M\chi_2 = M\chi_1 + M\chi_1 * T. \tag{3.5}$$

Function $M\chi$ is not, generally speaking, a solution of the Riemann problem, for it may contain additional singularities at the poles of functions $I_i(\lambda)$. In all constructions given below, a central role is played by the following result.

THEOREM 1. If the kernel $T(\lambda', \lambda)$ satisfies conditions (3.4) and the number n of variables x_i is at least two, then the ring m contains a left ideal $\tilde{m}$ consisting of all operators $\tilde{M}$ such that

$$\tilde{M}\chi \equiv 0. \tag{3.6}$$

Proof. It suffices to examine the case n = 2. Let $M = \Sigma q_l D^l$ be an operator of the type (3.3), such that at least one component of each multiindex l is different from zero. Suppose M has order k. This operator is uniquely determined to within multiplication by an arbitrary nondegenerate matrix. Due to this circumstance the number of its independent functional coefficients equals $(k + 1)(k + 2)/2 - 2$. Suppose the total order of the divisors of the poles of functions I_1, I_2 equals s. In the general position case, function $M\chi$ has sk supplementary singularities; sk is also the number of conditions imposed to the coefficients of M in order to eliminate these singularities. It is smaller than the number of independent coefficients of the operator M if

$$\frac{(k+1)(k+2)}{2} - 2 \geqslant sk. \tag{3.7}$$

This inequality is satisfied for every value of s, starting with some $k = k_0$.

With constraints that annihilate the singularities being imposed, function $M\chi$ becomes proportional to χ, thanks to the uniqueness of the solution to Riemann's problem. Denoting the proportionality coefficient by u, we have

$$\tilde{M}\chi = M\chi - u\chi \equiv 0. \tag{3.8}$$

This completes the proof.

Let us discuss the structure of the ideal $\tilde{m}$. From (3.7) it readily follows that $k_0 = 2(s - 1)$, and the equality in (3.7) is attained only for s = 2. Let $\tilde{M}_0^i$, $i = 1, \ldots, p$, denote the collection of elements of $\tilde{m}$ having order k_0. Their number p exceeds by one the difference between the left- and right-hand sides of inequality (3.7) with $k = k_0$, so that $p = s - 1$. For s = 2 we have p = 1, so that $\tilde{m}$ contains only one operator $\tilde{M}_0$ of second order. Calculating the number of elements of $\tilde{m}$ having order $k = 2 + q$, we reach the conclusion that it is equal to $(q + 1)(q + 2)/2 - 1$, i.e., to the total number of operators of order at most q belonging to m. This means that for s = 2 the ideal $\tilde{m}$ is principal and is generated by a single operator of order two.

For s > 2 the picture is more complicated. Consider the elements of the ideal m having order $k_0 + q$. For large q the number of these elements grows like $q^2/2$. Among these operators one finds those of the form $M\tilde{M}_0^i$, $i = 1, \ldots, p$, where M is an arbitrary operator of order q. The number of such operators grows with q like $pq^2/2$, i.e., p times faster than the total number of elements of the ideal. This means that between the elements there are linear relations of the form

$$\sum_{i=1}^{p} M_{ik}\tilde{M}_0^i = 0 \tag{3.9}$$

with operator coefficients. It would be interesting to clarify whether the elements $\tilde{M}_0^i$ form a basis in the ideal $\tilde{m}$.

If the number of differentiations n > 2, the minimal order of the operators belonging to $\tilde{m}$ is considerably smaller. The case n = 3 is of special interest. Here it is readily verified that for s = 2, $\tilde{m}$ contains one first order operator as well as one minimum operator of order two which does not depend on it. For s = 3 the ideal $\tilde{m}$ contains three operators of order two. The full description of the structure of the ideal $\tilde{m}$ for arbitrary m seems a rather difficult problem; incidentally, its solution may be simplified for special choices of the rational functions $I_i(\lambda)$. To consider a simple case, we put

$$I_i(\lambda)=\frac{A_i}{\lambda-\lambda_i};\quad [A_i, A_j]=0;\quad \lambda_i\neq\lambda_j. \tag{3.10}$$

The operators $\tilde{M}$ may be sought in the form

$$\tilde{M}_{ij} = D_iD_j + u_i^jD_j + u_j^iD_i + w_{ij}. \tag{3.11}$$

The requirement that there be no poles at the points $\lambda = \lambda_i$ in the expressions $\tilde{M}_{ij}\chi = 0$ yields

$$u_j^i = -\left(\partial_j\chi_i + \frac{\chi_i A_j}{\lambda_i-\lambda_j}\right)\chi_i^{-1}. \tag{3.12}$$

Here $\chi_i = \chi\,|_{\lambda=\lambda_i}$.

The quantity w_{ij} is determined by the asymptotics of χ for $\lambda \to \infty$. For the normalization $\chi \to 1$ for $\lambda \to \infty$ considered here, we get $w_{ij} = 0$. Thus, in the example under consideration the coefficients of the operators $\tilde{M}$ are expressible through the values of the function χ at the points λ_i, i.e., the singularities of functions $I_i(\lambda)$. In the general case the coefficients of the operators $\tilde{M}$ are expressible through function χ, but with some degree of arbitrariness (though the source of this might be the fact that in $\tilde{m}$ left multiplication by arbitrary differential operators is allowed). However, in each concrete case one can redefine the operator $\tilde{M}$ arbitrarily by, say, setting some of its coefficients equal to zero, so that in the end its coefficients will be uniquely determined by function χ. This requires information on function χ and a finite number of its derivatives with respect to λ at the singularities of all matrix functions $I_i(\lambda)$.

Let $T(\lambda', \lambda) = 0$, $\chi \equiv 1$. Then $\tilde{M}$ degenerates to an operator $\tilde{M}_0$. The operator $\tilde{M}$ may be referred to as the result of dressing the operator $\tilde{M}_0$ with the aid of the Riemann problem with junction function $T(\lambda, \lambda')$. Accordingly, the formulas expressing the coefficients of the operator $\tilde{M}$ through function χ may be referred to as dressing formulas, and the collection of values of χ and of a finite number of its derivatives necessary for determining the coefficients of $\tilde{M}$, the dressing data. In the example considered above, the operator (3.11) is obtained by dressing the operator $\tilde{M}_{0ij} = D_iD_j$, and (3.12) is an example of a dressing formula. The dressing data consists of the two functions χ_i, χ_j $(\chi\,|_{\lambda=\lambda_i,\lambda_j})$.

We next turn to the case where the Riemann problem is local.

Equation (3.1) takes now the form

$$\frac{\partial T(\lambda)}{\partial x_i} = [I_i(\lambda), T(\lambda)]. \tag{3.13}$$

In the local case the structure of the ideal $\tilde{m}$ is considerably simpler. Here besides differentiating relation (2.2) we are allowed to multiply it at left by an arbitrary matrix function of the parameter λ. Therefore, we can seek the operators $\tilde{M}$ in the form

$$\tilde{M}_i\chi = D_i\chi - u_i(\lambda)\,\chi. \tag{3.14}$$

Here $u_i(\lambda)$ is a rational function of λ which has singularities at exactly the same points as $I_i(\lambda)$. Obviously, the operators $\tilde{M}_i$ of first order form a basis in the ideal $\tilde{m}$.

Finally, we notice a simple but important property of the operators $\tilde{M}$ in $\tilde{m}$. The change of variables

$$\psi = \chi \exp \sum_{k=1}^{n} I_k(\lambda)\,x_k \tag{3.15}$$

takes the "long" derivatives $D_i\chi$ into the ordinary derivatives $\partial_i\psi$. The same change of variables takes the ring m into the ring r of differential operators in variables x_i with matrix coefficients, which do not contain explicitly the parameter λ. The ideal $\tilde{m}$ is taken into the ideal $\tilde{r}$ of operators R such that

$$\tilde{R}\psi \equiv 0. \tag{3.16}$$

4. Nonlinear Equations

Our main objective is to construct nonlinear partial differential equations whose solutions can be found using a nonlocal Riemann problem. We first examine the procedure for constructing such equations on a simple example. Suppose that the number of supplementary variables is $n \geqslant 3$, and, as in the previous example, $I_i(\lambda) = A_i/(\lambda - \lambda_i)$. Now the ideal $\tilde{m}$ contains $n(n-1)/2$ operators $\tilde{M}_{ij}$ of the type (3.11). Consider the set of equations

$$\tilde{M}_{ij}\chi_k = 0, \tag{4.1}$$

which may be written in detail as

$$\partial_i\partial_j\chi_k + \partial_i\chi_k \frac{A_j}{\lambda_k - \lambda_j} + \partial_j\chi_k \frac{A_i}{\lambda_k - \lambda_j} + \chi_k \frac{A_iA_j}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} =$$
$$= \left(\partial_i\chi_j + \frac{\chi_jA_i}{\lambda_j - \lambda_i}\right)\chi_j^{-1}\left(\partial_j\chi_i + \frac{\chi_kA_i}{\lambda_k - \lambda_i}\right) + \left(\partial_j\chi_i + \frac{\chi_iA_j}{\lambda_i - \lambda_j}\right)\chi_i^{-1}\left(\partial_j\chi_k + \frac{\chi_kA_j}{\lambda_k - \lambda_j}\right). \tag{4.2}$$

In the general case we have $n^2(n-1)/2$ nonlinear equations for n matrix functions $\chi_i(x_1, \ldots, x_n)$. The only case in which the number of equations is equal to the number of unknown functions is $n = 3$.

Now consider a more general case. Suppose functions $I_i(\lambda)$ $(i = 1, \ldots, n \geqslant 3)$ have divisors q_i with no points in common, but otherwise arbitrary. The ideal $\tilde{m}$ contains $n(n-1)/2$ smaller ideals, each consisting of operators which involve differentiation with respect to only two variables, say x_i and x_j; we denote these ideals by $\tilde{m}_{ij}$. We select arbitrarily some operator $\tilde{M}_{ij}$ in $\tilde{m}_{ij}$ and subject its coefficients to the number of arbitrary relations necessary to guarantee that they be expressed uniquely through function χ. Then the expressions of the coefficients involve the values of function χ and of a finite number of its derivatives with respect to λ at the points of the divisors q_i and q_j. The set of these values at the points of all divisors forms the dressing data. These data are precisely the unknown functions for which we must write the desired nonlinear equations. To do this, consider the set of relations

$$\tilde{M}_{ij}\chi = 0, \quad i \neq j \tag{4.3}$$

in the neighborhood of each point of any of the divisors q_k $k \neq i \neq j$. Our assumption that the divisors do not intersect guarantees that the coefficients of the operators $\tilde{M}_{ij}$ at these points are finite. Each such point λ_α contributes to the dressing data the value of function χ and also the values of the n_α first-derivatives of χ with respect to λ at this point. Let us differentiate equation (4.3) n_α times with respect to λ in the vicinity of $\lambda = \lambda_\alpha$ and then set in each of the resulting equations $\lambda = \lambda_\alpha$. This yields a set of equations for the dressing data.

The number of such equations coincides with the number of unknown functions only for $n = 3$. For $n > 3$ the resulting system is strongly overdetermined; nevertheless, it admits a joint solution determined by the function of two variables $T(\lambda', \lambda)$.

The procedure described above was known earlier for the simplest case of the local Riemann problem [13]. In this case the ideal $\tilde{m}_{ij}$ contains operators of first-orders, which involve only derivatives with respect to one of the variables. They generate the equations

$$\frac{\partial\chi}{\partial x_i} = u_i(\lambda)\chi - \chi I_i(\lambda'). \tag{4.4}$$

Here $u_i(\lambda)$ is a rational matrix function of λ which has the same divisor q_i as $I_i(\lambda)$. The coefficients of $u_i(\lambda)$ may be expressed through the dressing data given on the divisor q_i. Upon restricting Eq. (4.4) to the divisor q_k and permuting the roles of the variables x_i and x_k we obtain a closed system of nonlinear equations for the two collections of dressing data attached to the divisors q_i and q_k. These collections – our unknown functions – depend on two variables, x_i and x_k. Choosing a different pair of variables we obtain a new system which formally is not connected with the first. Thus, in the case of the local Riemann problem in our scheme there naturally arise equations for functions of two variables. It is important to notice that the variational principle given in [14] applies automatically to the equations obtained by the procedure described above.

The local Riemann problem is the tool used to integrate most of the nonlinear equations to which the method of the inverse problem applies. These equations are usually derived by a different method. System (4.4) may be regarded as an overdetermined system of linear equations for function χ. The compatibility conditions for this system have the form

$$\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} + [u_i, u_j] = 0. \tag{4.5}$$

Expanding these conditions into partial fractions we obtain a nonlinear system of equations, which, upon expressing the coefficients of functions u_i in terms of the dressing data, becomes the system described above. The system of equations (4.3) can be also regarded as an overdetermined system of linear equations, for which the compatibility conditions form a set of nonlinear equations on its coefficients.

We have not succeeded in finding a simple method for writing these equations in the general case. Such a method, however, exists in one important special case. To this point we have assumed that all variables x_i have an equal status. Suppose now that one of these variables (we denote it by x without any index, and use D_x for the corresponding long derivative) is distinguished by the fact that the corresponding function $I(\lambda)$ has a particularly simple form: $I(\lambda) = i\lambda$. Suppose, in addition, that the remaining functions $I_i(\lambda)$, $i = 1, \ldots, n$, are polynomials in λ. Then, obviously, the ideal $\tilde{m}$ contains elements of the form

$$\tilde{M}_i = D_i - L_i(D_x), \tag{4.6}$$

forming a basis of this ideal. Here L_i are differential operators with respect to the variable x only.

The equations

$$\tilde{M}_i\chi = 0 \tag{4.7}$$

take after the transformation

$$\chi = \psi \exp\left[-\left(\lambda x + \sum_{i=1}^{n-1} I_i(\lambda) x_i\right)\right] \tag{4.8}$$

the form

$$R_i\psi = \frac{\partial\psi}{\partial x_i} - L_i\left(\frac{\partial}{\partial x}\right)\psi = 0, \tag{4.9}$$

which does not contain explicitly the parameter λ.

The compatibility conditions have the form $[R_i, R_j] = 0$ or, in more detail,

$$\frac{\partial L_i}{\partial x_j} - \frac{\partial L_j}{\partial x_i} + [L_i, L_j] = 0. \tag{4.10}$$

Equations for the form (4.10), with L_i differential operators in variable x, were considered earlier in paper [1]. They include most of the higher-dimensional equations which can be studied by the method of the inverse problem. A more general result is*:

THEOREM 2. Suppose that among the variables x_i one variable x is singled-out by the fact that the corresponding function $I(\lambda) = i\lambda$, whereas the others $I_k(\lambda)$, $k = 1, \ldots, n-1$, are rational functions of general form. Then in the ideal $\tilde{m}$ there are $n-1$ elements of the form

$$\tilde{M}_i = A_i(D_x) D_i - B_i(D_x), \tag{4.11}$$

where A_i and B_i are differential operators in variable x.

Proof. Let $\tilde{q}_i$ be the divisor of the poles of function $I_i(\lambda)$ lying in the finite λ plane, and let n_i be the order of $\tilde{q}_i$. Consider the collections of operators $D_x^k D_i$ ($k = 0, \ldots, n_i$). These operators may have singularities in the finite λ plane only at the points of the divisor $\tilde{q}_i$. The same property is enjoyed by their linear combination

$$A(D_x) D_i = (D_x^{n_i} + u_i^1 D_x^{n_i-1} + \ldots + u_i^{n_i}) D_i, \tag{4.12}$$

which contains exactly n_i arbitrary coefficients u_i^k, $k = 1, \ldots, n_i$. We choose these coefficients by requiring that there be no singularities at the points of the divisor $\tilde{q}_i$. The resulting expression (4.12) is a polynomial in λ; we denote its degree by m. Consider an operator of the form

$$A_i(D_x) D_i - \tilde{B}_i(D_x) = M_i. \tag{4.13}$$

Here $\tilde{B} = v_i^1 D_x^m + \ldots + v_i^m D_x$. By an appropriate choice of the coefficients v_i^k we can annihilate the coefficients of all the powers of λ. As a result,

*Theorem 2 was reported at the International Congress of Mathematicians held in Warsaw [10].

$$M_i\chi = v_i^m\chi, \quad \text{т. е.} \quad \tilde{M}_i\chi = 0$$
$$\tilde{M}_i = M_i - v_i^m. \tag{4.14}$$

Transformation (3.16) takes equations $\tilde{M}_i\chi = 0$ into the equations

$$A_i\left(\frac{\partial}{\partial x}\right)\frac{\partial\psi}{\partial x_i} - B_i\left(\frac{\partial}{\partial x}\right)\psi = 0. \tag{4.15}$$

The investigation of the compatibility conditions for system (4.15) can be carried out in the general case, though this goes beyond the scope of this paper. Here we consider only an important particular case. Take $n = 3$, so that system (4.15) reduces to two equations of the following form

$$L_1\psi = A_1\left(\frac{\partial}{\partial x}\right)\frac{\partial\psi}{\partial x_1} - B_1\left(\frac{\partial}{\partial x}\right)\psi = 0; \tag{4.16}$$

$$L_2\psi = \frac{\partial\psi}{\partial x_2} - B_2\left(\frac{\partial}{\partial x}\right)\psi = 0. \tag{4.17}$$

The commutator of the operators L_1 and L_2 vanishes identically on the space of all solutions of system (4.16), (4.17). This operator does not involve differentiation with respect to x_2, and hence it is necessarily proportional to L_1. Therefore,

$$[L_2, L_1] \equiv \frac{\partial L_1}{\partial x_2} + [L_1, B_2] = pL_1, \tag{4.18}$$

where p is a differential operator in variable x. System (4.18) is the most general example of systems admitting an "LAB-triple," known at the present time and discovered by one of the authors of this paper [15].

We conclude this section by several remarks on the classes of explicit solutions of the nonlinear systems described above. By (3.1), if the kernel $T(\lambda', \lambda, x_i)$ of the nonlocal Riemann problem is degenerate in the sense of formula (2.5) at $x_i = 0$, then it has this property for all other values of x_i. The only change is that in (2.5) functions f_n and g_n depend now on x_i according to the rules

$$f_n(\lambda', x) = \exp\left(\sum_i I_i(\lambda')x_i\right) f_n(\lambda')$$
$$g_n(\lambda', x) = g_n(\lambda')\exp\left(-\sum_i I_i(\lambda')x_i\right),$$

so that the nonlocal Riemann problem remains exactly solvable; its solution is found, as above, by solving the linear algebraic system (2.7). Thus, all systems integrable by means of a nonlocal Riemann problem admit explicit solutions that contain as parameters an arbitrary number of functions of one variable $(f_n(\lambda), g_n(\lambda), \lambda \in \Gamma)$.

Interesting particular solutions of this kind are those which are rational functions in any of the variables x_i.*

These solutions may be described as follows: Let γ_i be a collection of N points in the λ plane different from the singularities of the functions $I_i(\lambda)$. Let Γ_i be a closed contour encircling the point γ_i such that the remaining points $\gamma_i, j \neq i$, and the singularities of functions $I_k(\lambda)$ lie in the exterior of Γ_i. We shall assume that the N curves Γ_i do not intersect. Further, let S_n be N constant matrices which commute with $I_i(\lambda), [S_n, I_i(\lambda)] = 0$. Also, we denote by $\chi_i(\lambda)$ functions analytic inside Γ_i, and by $\chi(\lambda)$ a function regular off all contours Γ_i. Then the nonlocal Riemann problem which leads to rational solutions is formulated as follows: find functions χ, χ_i, such that $\chi(\infty) = 1$ and on Γ_i, $i = 1, \ldots, N$

$$\chi(\lambda) = \chi_i(\lambda) + \frac{1}{2\pi i}\int_{\Gamma_i}\frac{\chi_i(\lambda')e^{F(\lambda')}S_i e^{-F(\lambda)}}{(\lambda' - \gamma_i)(\lambda - \gamma_i)}d\lambda', \tag{4.19}$$

where

$$F(\lambda) = \sum_{i=1}^{n} I_i(\lambda)x_i. \tag{4.20}$$

*Such solutions are well known for the Kadomtsev—Petviashvili equation [17].

The proof of this assertion may be of course obtained starting with system (2.7) and computing a number of integrals. However, it is preferable to examine directly problem (4.19). Since $F(\lambda)$ is analytic inside Γ_i, the integral in (4.19) can be computed, which yields

$$\chi(\lambda)=\chi_i(\lambda)+\frac{\chi_i(\gamma_i)\,S_i e^{F(\gamma_i)-F(\lambda)}}{\lambda-\gamma_i}, \tag{4.21}$$

where $\lambda\in\Gamma_i$. From (4.21) and the normalization $\chi(\infty)=1$ it follows, first of all, that

$$\chi(\lambda)=1+\sum_{i=1}^{N}\frac{\chi_i(\gamma_i)\,S_i}{\lambda-\gamma_i} \tag{4.22}$$

in the domain of definition of $\chi(\lambda)$. Therefore, the solution of the corresponding nonlinear equation, which may be expressed through the values of $\chi(\lambda)$ at the poles of the functions $I_i(\lambda)$, is completely determined by the values of $\chi_i(\lambda)$ at the points γ_i. The equations for the latter are readily found applying Cauchy's formula to determine functions $\chi_i(\lambda)$ of (4.21):

$$\chi_i(\lambda)-1+\frac{1}{2\pi i}\chi_i(\gamma_i)S_i\int_{\Gamma_i}\frac{e^{(F(\gamma_i)-F(\lambda'))}}{(\lambda'-\lambda)(\lambda'-\gamma_i)}\,d\lambda'=\sum_{j\neq i}\frac{1}{\gamma_j-\lambda}\chi_j(\lambda_j)\,S_j.$$

Setting here $\lambda=\gamma_i$ we immediately get

$$\chi_i(\gamma_i)(1-S_iF'(\gamma_i))+\sum_{i\neq j}\frac{\chi_j(\gamma_i)\,S_j}{\gamma_i-\gamma_j}=1. \tag{4.23}$$

Relations (4.23) form a system of N equations for the N functions $\chi_i(\gamma_j)$. Functions χ_i depend on variables x_i through the coefficients $F'(\gamma_i)=\sum_n x_n\frac{\partial I_n}{\partial\lambda}\Big|_{\gamma_i}$. For this reason $\chi_i(\gamma_j)$ are rational functions in each of the variables x_k. In view of formula (4.22), so are the solutions of the corresponding nonlinear equation.

5. Nonlocal $\bar{\partial}$-Problem

We develop below a more general scheme for calculating the exact solutions described in the preceding section. Let us go back to the nonlocal Riemann problem, but now consider instead of Eq. (2.3) for function $k(\lambda)$ the equivalent equation

$$k(\lambda)=\int_\Gamma R(\lambda',\lambda)\,d\lambda'+\frac{1}{2\pi i}\iint_\Gamma\frac{k(\xi)\,R(\lambda',\lambda)}{\lambda'-\xi}\,d\lambda'\,d\xi. \tag{5.1}$$

Here the integral is understood in the sense of principal value, and function $R(\lambda',\lambda)$ is connected with $T(\lambda',\lambda)$ as follows: if we think of $T(\lambda',\lambda)$ as the kernel of the integral operator T, then $R(\lambda',\lambda)$ is the kernel of the integral operator $R=T(1+{}^1/_2T)^{-1}$ (5.1).

Suppose the contour Γ consists of l pieces Γ_m. We denote by k_m the jump of function χ on each such piece. Then

$$\chi=1+\frac{1}{2\pi i}\sum_{m=1}^{l}\int_{\Gamma_m}\frac{k_m(\lambda')}{\lambda-\lambda'}\,d\lambda'. \tag{5.2}$$

The kernel $R(\lambda',\lambda)$ becomes now a matrix $R_{nm}(\lambda',\lambda)$, and Eq. (5.1) is replaced by the system

$$k_n(\lambda)=\int\sum_{m=1}^{l}R_{mn}(\lambda',\lambda)\,d\lambda'+\frac{1}{2\pi i}\sum_{p=1}^{l}\sum_{m=1}^{l}\int_{\Gamma_p}d\xi\int_{\Gamma_m}d\lambda'\,\frac{k_p(\xi)\,R_{mn}(\lambda',\lambda)}{\lambda'-\xi}. \tag{5.3}$$

Let the contours Γ_n $(-\infty<n<\infty)$ be lines parallel to the x axis which intersect the y axis at the points $y_n=n\Delta$. We put $k_n(x)=\Delta k(x,y_n)$, $R_{nm}(x,x')=\Delta^2R(x,x',y_n,y_n)$. Letting $\Delta\to 0$, we see that relations (5.2) and (5.3) take the respective forms

$$\chi(\lambda,\bar\lambda)=1+\frac{1}{2\pi i}\int\frac{k(\lambda',\bar\lambda')}{\lambda-\lambda'}\,d\lambda'\,\overline{d\lambda}'; \tag{5.4}$$

$$k(\lambda,\bar\lambda)=\int R(\lambda',\bar\lambda',\lambda,\bar\lambda)\,d\lambda'\,d\bar\lambda'+\frac{1}{2\pi i}\int\frac{k(\lambda'',\bar\lambda'')\,R(\lambda',\bar\lambda',\lambda,\bar\lambda)}{\lambda''-\lambda'}\,d\lambda'\,d\bar\lambda'\,d\lambda''\,d\bar\lambda''.$$

In these formulas

$$\frac{1}{\lambda-\lambda'}=\lim_{\varepsilon\to 0}\frac{\bar\lambda-\bar\lambda'}{|\lambda-\lambda'|^2+\varepsilon^2}. \tag{5.5}$$

Moreover, from (5.2) it follows that

$$k(\lambda,\bar\lambda)=2i\frac{\partial\chi}{\partial\bar\lambda}. \tag{5.6}$$

so that Eq. (5.4) becomes

$$2i\frac{\partial\chi}{\partial\bar\lambda}=\int\chi(\lambda',\bar\lambda')R(\lambda',\bar\lambda',\lambda,\bar\lambda)\,d\lambda'\,d\bar\lambda'. \tag{5.7}$$

This equation describes a nonlocal $\bar\partial$-problem which is a natural generalization of the nonlocal Riemann problem.

If function $T(\lambda',\lambda)$, depending on n supplementary variables x_i, is subject to Eqs. (3.1), then these equations are satisfied by function $R(\lambda',\lambda)$ too. Using again the passage to limit described above we conclude that the limiting function $R(\lambda',\bar\lambda',\lambda,\bar\lambda)$ satisfies the system of equations

$$\frac{\partial R}{\partial x_i}=I_i(\lambda')R(\lambda',\bar\lambda',\lambda,\bar\lambda)-R(\lambda',\bar\lambda',\lambda,\bar\lambda)I_i(\lambda). \tag{5.8}$$

The algebraic relations derived in Secs. 3 and 4 do not depend on the choice of contour Γ and hence are preserved on passing to the limit. Thus, we can construct an ideal $\tilde{m}$ in the ring of differential operators (3.3) using function χ, i.e., the solution of the nonlocal $\bar\partial$-problem whose kernel satisfies Eq. (5.8). The values of this function and of its derivates with respect to λ, taken at the points of the divisors of functions $I_i(\lambda)$, form the necessary collection of dressing data; moreover, the dressing formulas remain the same as in the case of the nonlocal Riemann problem. At the same time we obtain a new method for constructing exact solutions to all nonlinear equations exhibited above.

The indicated method can be also extended to the two-dimensional equations solvable by the local Riemann problem. Thus, starting with the local problem and repeating the foregoing arguments we are led to the local $\bar\partial$-problem

$$\frac{\partial\chi}{\partial\bar\lambda}=\frac{1}{2i}\chi(\lambda)R(\lambda,\bar\lambda), \tag{5.9}$$

where function $R(\lambda,\bar\lambda)$ satisfies the system of equations

$$\frac{\partial R}{\partial x_i}=[I_i(\lambda),R(\lambda,\bar\lambda)]. \tag{5.10}$$

The solution of system (5.9) may be used to construct solutions of Eqs. (4.5) as successfully as for the local Riemann problem.

6. Connections with Marchenko's Equation

We now return to the special case described in Secs. 3 and 4, where one of the variables x is singled out by the condition $I = i\lambda$, and first of all to Eqs. (4.10). In paper [1] a dressing method based on the utilization of Marchenko's equation was used to construct solutions of these equations. In [9, 10] this method with some changes was applied to systems (4.18) as well as to more general systems which are compatibility conditions for Eqs. (4.15). Of principal interest is the question of how the methods developed in earlier works fit into the scheme described here.

Suppose that in the nonlocal Riemann problem (2.1) the contour Γ is the real axis. By assumption,

$$\frac{\partial T}{\partial x}=i(\lambda'-\lambda)T,\quad T(\lambda',\lambda,x)=T_0(\lambda',\lambda)e^{i(\lambda'-\lambda)x}. \tag{6.1}$$

We shall omit the dependence on the other variables x_i. Consider the functions

$$F(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T_0(\lambda',\lambda)e^{i(\lambda' x-\lambda z)}\,d\lambda'\,d\lambda \tag{6.2}$$

and

$$K(x, z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} k(\lambda, x)\, e^{i\lambda(x-z)}\, d\lambda. \tag{6.3}$$

Multiplying Eq. (2.3) by $e^{i\lambda(x-z)}$ and integrating with respect to λ along the real axis, we obtain the Marchenko equation

$$K(x, z) + F(x, z) + \int_{x}^{\infty} K(x, s)\, F(s, z)\, ds = 0. \tag{6.4}$$

The same equation can be obtained for a more general choice of contour. Thus, suppose that the contour consists of two pices, Γ_1 and Γ_2, lying in the upper and, respectively, the lower half plane. The nonlocal Riemann problem is now specified by a mtrix $T_{ij}(\lambda', \lambda)$, $i, j = 1, 2$. Suppose that the only nonzero entry in this matrix is $T_{21}(\lambda', \lambda, x) = T_0(\lambda', \lambda)\, e^{i(\lambda'-\lambda)x}$. Then, upon setting

$$F(x, z) = \frac{1}{2\pi}\int_{\Gamma_1} d\lambda \int_{\Gamma_2} d\lambda' T_0(\lambda', \lambda)\, e^{i(\lambda' x - \lambda z)} \tag{6.5}$$

and

$$K(x, z) = -\frac{1}{2\pi}\int_{\Gamma_1} K(\lambda, x)\, e^{i\lambda(x-z)}\, d\lambda, \tag{6.6}$$

we are again led to Eq. (6.4). Now if we choose, as in Sec. 2, the contours Γ_1 and Γ_2 as collections of lines parallel to the real axis, and then pass to the limit in the formulas (4.5) and (4.6), we get the most general expression for F(x, z) and K(x, z) which still permits us to use Eq. (6.4):

$$F(x, z) = \frac{1}{2\pi}\int_{\mathrm{Im}\,\lambda\leqslant 0} d\lambda'\, d\bar{\lambda}' \int_{\mathrm{Im}\,\lambda\geqslant 0} d\lambda\, d\bar{\lambda}\, T_0(\lambda, \bar{\lambda}, \lambda', \bar{\lambda}')\, e^{i(\lambda x - \lambda' z)} \tag{6.7}$$

and

$$K(x, z) = -\frac{1}{2\pi}\int_{\mathrm{Im}\,\lambda\leqslant 0} d\lambda\, d\bar{\lambda}\; k(\lambda, \bar{\lambda}, x)\, e^{i\lambda(x-z)}. \tag{6.3}$$

Notice that now F(x, z) grows exponentially as x, z $\to -\infty$.

We thus see that in the particular case with a singled-out variable x the methods developed here are essentially more general than those resting on Marchenko's equation.

7. Some Examples

Let us illustrate the effectiveness and flexibility of the methods developed here on some examples. One of the most important equations to which these methods apply is the Kadomtsev—Petviashvili (KP) equation. We confine our analysis to the KP — 1 equation

$$\frac{\partial}{\partial x}\left(u_t - 3uu_x - \frac{1}{4}u_{xxx}\right) + \frac{3}{4}u_{yy} = 0. \tag{7.1}$$

The integration of this equation is connected with a scalar nonlocal Riemann problem and the corresponding $\bar{\partial}$-problem. The number of supplementary variables n is three. To these variables, there correspond the first-order differential operators

$$D_1 = \frac{\partial}{\partial x} + i\lambda;\quad D_2 = \frac{\partial}{\partial y} + i\lambda^2;\quad D_3 = \frac{\partial}{\partial t} - i\lambda^3. \tag{7.2}$$

Therefore, the kernel R in problem (5.7) has the form

$$R(\lambda', \bar{\lambda}', \lambda, \bar{\lambda}) = R_0(\lambda', \bar{\lambda}', \lambda, \bar{\lambda})\, e^{i(\lambda'-\lambda)x + i(\lambda'^2-\lambda^2)y - i(\lambda'^3-\lambda^3)t}. \tag{7.3}$$

Function χ is analytic in λ in the neighborhood of the infinity and admits there the expansion

$$\chi = 1 + \frac{\chi_0}{\lambda} + 0\left(\frac{1}{\lambda^2}\right). \tag{7.4}$$

The dressing formula that expresses the unknown function u through χ is

$$u = -2i\frac{\partial}{\partial x}\chi_0. \tag{7.5}$$

The technique developed here yields immediately the N-soliton solutions of the KP − 1 equation. We take R_0 in the form

$$R_0 = \sum_{p=1}^{n} r_p \delta(\lambda' - \xi_p)\,\delta(\lambda - \eta_p). \tag{7.6}$$

(The symbol $\delta(\lambda - \eta_p)$ stands for the two-dimensional delta-function in the (λ_1, λ_2) − plane, $\lambda = \lambda_1 + i\lambda_2$.)

Here r_p are arbitrary complex constants. Using the formula $\frac{\partial}{\partial\bar{\lambda}}\frac{1}{\lambda-\lambda_0} = \pi\delta(\lambda-\lambda_0)$, we get

$$\chi = 1 + \sum_{p=1}^{n}\frac{\chi_p}{\lambda - \eta_p}, \tag{7.7}$$

in which function χ_p of the variables x, y, t satisfies the system of equations

$$\chi_p = \frac{r_p e^{i\Phi_p}}{2\pi i}\left(1 + \sum_{q=1}^{n}\frac{\chi_q}{\xi_p - \eta_q}\right), \tag{7.8}$$

where

$$\Phi_p = (\xi_p - \eta_p)\,x + (\xi_p^2 - \eta_p^2)\,y - (\xi_p^3 - \eta_p^3)\,t.$$

For function u we obtain from (7.4) and (7.5)

$$u = 2i\frac{\partial}{\partial x}\sum_{p=1}^{n}\frac{\chi_p}{p}. \tag{7.9}$$

For arbitrary complex constants ξ_p, η_p, and r_p, solution (7.9) is, generally speaking, complex and has singularities.

To eliminate such singularities and make the solution real it is necessary to subject ξ_p, η_p, and r_p to a number of constraints. Thus, for n = 2, we take $\xi_1 = i\nu_1$; $\eta_1 = i\nu_2$; $\xi_2 = -i\nu_2$; and $\eta_2 = -i\nu_1$.

Then after simple calculations, we get

$$u = 2\frac{\partial^2}{\partial x^2}\ln\Delta \tag{7.10}$$

with

$$\Delta = 1 + ae^{-\delta(x+\nu t)}\cos\delta\nu y + \frac{a^2}{4(\nu^2-\delta^2)}e^{-2\delta x}, \tag{7.11}$$
$$\delta = \nu_1 - \nu_2;\ \nu = \nu_1 + \nu_2.$$

Solution (7.10)-(7.11) is periodic in y and decays for $x \to \pm\infty$; it was earlier obtained in [16] by a considerably more difficult method.

Our scheme permits also to find solutions of the Korteweg-deVries (KdV) equation; the latter is obtained from the KP-equation upon setting $u_y = 0$. One can obtain such solutions by taking

$$R_0(\lambda', \bar{\lambda}', \lambda, \bar{\lambda}) = R_0(\lambda, \bar{\lambda})\,\delta(\lambda + \lambda').$$

Thus, in the KdV case, function χ satisfies the compact relation

$$\frac{\partial\chi}{\partial\bar{\lambda}} = \frac{1}{2i}R_0(\lambda,\bar{\lambda})\,e^{-2i\lambda x + 2i\lambda^3 t}\chi(-\lambda, -\bar{\lambda}, x, t), \tag{7.12}$$

which, as a matter of fact, could be put at the basis of the theory of the KdV equation. In particular, the N-soliton solutions for KdV are given by formulas (7.8), (7.9) under the additional constraint $\xi_p = -\eta_p$. In the general case these solutions are complex and singular.

For $\delta \to 0$ and $a \to -1$ solution (7.10), (7.11) becomes rational and represents a localized two-dimensional soliton.

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