HILL'S OPERATOR WITH FINITELY MANY GAPS

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The goal of this paper is to give an effective description of those periodic potentials $q(x + T) = q(x)$, for which the number of gaps in the spectrum of Hill's operator $H = -D_x^2 + q(x)$, $x \in R^1$ is finite. Here and below D_t denotes differentiation with respect to t. Our interest in this problem was stimulated by an article of S. P. Novikov [1], who showed that such potentials, when taken as initial data in the Cauchy problem for the Kurteweg–de Vries (KV) equation, could be interpreted as periodic analogs of N-soliton solutions, and that furthermore, stationary periodic and almost-periodic solutions of the generalized KV-equations turned out to be potentials with a finite number of gaps.

We consider the differential equation

$$L\psi = 4\lambda D_x\psi, \quad L = -D_x^3 + 2(qD_x + D_xq). \tag{1}$$

<u>THEOREM 1.</u> In order that the operator H have precisely n nondegenerate gaps, it is necessary and sufficient that Eq. (1) have a solution of the form

$$\psi_n(x, \lambda) = (\lambda - \lambda_1(x))(\lambda - \lambda_2(x)) \ldots (\lambda - \lambda_n(x)), \tag{2}$$

where all $\lambda_i(x)$ are non-constant, and $\lambda_i(x + T) = \lambda_i(x)$. The largest value of the function $\lambda_i(x)$ coincides with the right endpoint of the i-th gap, its smallest value with the left endpoint. The set of solutions of Eq. (1) coincides with the set of products $f_1(x)f_2(x)$, where f_1 and f_2 are arbitrary solutions of the equation $Hf = \lambda f$.

We remark that the functions $\lambda_i(x)$ are eigenvalues of the following Sturm–Liouville problem: $D_\tau^2 y + q(x + \tau)y = \lambda y$, $y(0) = y(T) = 0$.

Let $\Omega_k(x)$ be the coefficient of λ^{n-k} in the polynomial ψ_n. By substituting this polynomial into (1) and collecting coefficients of powers of λ, one easily gets the following relations:

$$\Omega_k(x) = 4^{-k}\left[\prod_1^k (I_iL)\right]1, \quad k = 1, \ldots, n; \qquad L\left[\prod_1^n (I_iL)\right]1 = 0, \tag{3}$$

where the I_i are integration operators on the interval (c_i, x), and the c_i are constants. The second equation in (3) can be thought of as a differential equation for q(x).

Suppose the gaps in the spectrum of H are given by (α_i, β_i), $\beta_{i-1} < \alpha_i < \beta_i$, $i = 1, 2, \ldots, n$; β_0 denotes the lower bound of the spectrum of H. We define solutions of Hill's equation by the conditions $\varphi(x, \lambda)$, $\vartheta(x, \lambda)$. The notations $\varphi(0, \lambda) = 0$, $\varphi_x(0, \lambda) = 1$, $\vartheta(0, \lambda) = 1$, $\vartheta_x(0, \lambda) = 0$ will be used for the values of the functions $\varphi(\lambda)$, $\vartheta(\lambda)$, $\varphi'(\lambda)$, $\vartheta'(\lambda)$ and their derivatives with respect to x, evaluated at x = T. We set $F(\lambda) = (\vartheta(\lambda) + \varphi'(\lambda))/2$, $m_{1,2}(\lambda) = \varphi^{-1}(\lambda)[(\varphi'(\lambda) - \vartheta(\lambda))/2 \pm \sqrt{F^2(\lambda) - 1}]$. The solutions $\psi_{1,2}(x, \lambda) = \vartheta(x, \lambda) + m_{1,2}\varphi(x, \lambda)$ are linearly independent, if $F^2(\lambda) - 1 \neq 0$, and can be represented in the form [2] $\psi_{1,2}(x) = \exp[\pm x \ln(F(\lambda) + \sqrt{F^2(\lambda) - 1})]\chi_{1,2}(x, \lambda)$, where $\chi_{1,2}$ are periodic functions of x: $\chi_{1,2}(x + T, \lambda) = \chi_{1,2}(x, \lambda)$.

From Theorem 1 and the uniqueness (up to a constant factor) of the periodic solution of Eq. (1), we obtain the following representation for $\psi_1(x, \lambda)\psi_2(x, \lambda)$:

$$\psi_1(x, \lambda)\psi_2(x, \lambda) = \prod (\lambda - \lambda_i(x))/(\lambda - \lambda_i(0)). \tag{4}$$

It can furthermore be shown that for the potential q(x) under consideration, one has

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$$m_{1,2}(\lambda) = [Q(\lambda) \pm i\sqrt{P(\lambda)}] / \prod(\lambda - \lambda_i(0)),$$

where $Q(\lambda) = D_x\psi_n(x, \lambda)|_{x=0}/2$, $P(\lambda) = (\lambda - \beta_0)(\lambda - \alpha_1)(\lambda - \beta_1)\ldots(\lambda - \beta_n)$. Denote by σ the hyperelliptic surface of the function $\sqrt{P(\lambda)}$, thought of as a two-sheeted covering surface of the λ-plane. Then $\psi_{1,2}(x, \lambda)$ can be realized as different branches of a function which is analytic and single-valued on σ, namely $\psi(x, \lambda) = \vartheta(x, \lambda) + \varphi(x, \lambda)[Q(\lambda) + i\sqrt{P(\lambda)}] / \prod(\lambda - \lambda_i(0))$. By (4), $\psi(x, \lambda)$ as function on σ has n simple zeros at $\lambda_i(x)$ and n simple poles at $\lambda_i(0)$. For $\lambda \to \infty$, we have $\psi(x, \lambda) \sim \exp(ix\sqrt{\lambda})$.

According to standard procedure, we choose canonical cuts (a_i, b_i) on σ. By adopting arguments due to N. I. Akhieser [3], one can now show that for the determination of the functions $\lambda_i(x)$ it suffices to solve a Jacobi inversion problem:

$$\sum_{k=1}^{n} \int_{\infty}^{\lambda_k(x)} dU_\nu(\lambda) \equiv \frac{xG_\nu}{\pi} + \sum_{k=1}^{n} \int_{\infty}^{\lambda_k(0)} dU_\nu(\lambda), \quad G_i = -\int_{\alpha_i}^{\beta_i} d\omega(\lambda), \quad \nu = 1, 2, \ldots, n, \tag{5}$$

where $dU_\nu(\lambda)$ is a basis for the normed Abelian differentials of the first kind, $d\omega(\lambda)$ is an Abelian differential of the second kind with zero A-periods and a second-order pole at infinity. The integrals in (5) are taken along the sheet containing the corresponding $\lambda_i(x)$ and $\lambda_i(0)$. Define vectors e, l, g $\in C^n$ by the formulas

$$e_i(\lambda) = \int_{\infty}^{\lambda} dU_i(\lambda), \quad l_i = -\frac{G_i}{\pi}, \quad g_i = -\sum_{k=1}^{n} \int_{\infty}^{\lambda_k(0)} dU_i(\lambda) + \frac{i}{2} - \frac{1}{2}\sum_{k=1}^{n} B_{ki}, \tag{6}$$

where $B = \|B_{ik}\|$ is the matrix of B-periods of the differentials $dU_i(\lambda)$. With the matrix B one associates the well-known Riemann θ-function, $\theta(v) =$ sum $\exp\{\pi i(Bk, k) + 2\pi i(k, v)\}$, where the sum is extended over all integer n-tuples $k = \{k_1, \ldots, k_n\}$, $v \in C^n$, a $(k, v) = k_1v_1 + k_2v_2 + \ldots + k_nv_n$.

By using É. I. Zverovich's algorithm [4] for the solution of the Jacobi inversion problem, one easily finds a representation for the sum of the functions $\lambda_i(x)$:

$$\sum_{k=1}^{n} \lambda_k(x) = \sum_{k=1}^{n} \int_{a_k} \lambda dU_k(\lambda) - \operatorname*{res}_{\lambda=\infty} \{\lambda\, d\theta(e + xl + g)\}. \tag{7}$$

The left side of (7) differs only by a constant from $-q(x)/2$.* This follows from formula (3) for k = 1. Calculation of the residues on the right side of (7) leads one to the remarkable formula

$$q(x) = -2D_x^2 \ln\theta(xl + g) + C, \tag{8}$$

where C is a constant expressible through α_i and β_i.

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After finishing this work, we became aware of a paper by B. A. Dubrovin, the basic result of which is the statement that the solution of KV starting from an n-gap initial value has the form $q(x, t) = R(xl + tv + r)$, where R is a rational function of θ-functions, and the vectors l, v, r $\in C^n$ are independent of x and t. If this result is combined with formula (8), one gets the following representation of the solution q(x, t) of KV with initial condition (8):

$$q(x, t) = -2D_x^2 \ln\theta(xl + tv + g) + C.$$

LITERATURE CITED

1. S. P. Novikov, Funktsional. Analiz. i Ego Prilozhen., 8, No. 3, 54-66 (1974).
2. E. C. Titchmarsh, Eigenfunction Expansions Associated with Second-Order Differential Equations, Oxford Univ. Press.
3. N. I. Akhiézer, Dokl. Akad. Nauk SSSR, 141, No. 2, 263-266 (1961).
4. É. I. Zverovich, Uspekhi Mat. Nauk, 26, No. 1 113-180 (1971).
5. A. R. Its and V. B. Matveev, Teor. i Matem. Fiz., 23, No. 1 (1975).
6. H. Hochstadt, Arch. Rat. Mech. Anal., 19, 353-362 (1965).

*This particular fact is not new, having first been noted by Hochstadt [6]; a simpler proof can be based on the Gel'fand–Levitan trace formulas [5].