BOUNDARY CONDITIONS FOR INTEGRABLE EQUATIONS

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In the study of the equations on a finite segment that are integrable with the help of the method of inverse problem it is convenient to impose periodic boundary conditions or their variants [1, 2]. Below we describe a new class of boundary conditions, compatible with complete integrability, for nonlinear equations that are integrable in the framework of ultralocal r-matrix scheme [2]. The idea of the method, proposed here, has been suggested to the author by the article [3] of Cherednik.

1. We fix a natural number D and a matrix-valued function r of a complex parameter u, $r(u) \in \mathrm{Mat}_D(C) \otimes \mathrm{Mat}_D(C)$, that satisfies the classical Young–Baxter equation [2] and the condition $r(-u) = -r(u)$. Let $\mathscr{A}$ be an algebra with Poisson bracket and T be a function from **C** into $\mathrm{Mat}_D(\mathscr{A})$ that satisfies the following well-known relation for the monodromy matrix [2]:

$$\{T^{(1)},\ T^{(2)}\} = [r_-,\ T^{(1)}T^{(2)}], \tag{1}$$

where $X^{(1)} \equiv X(u_1) \otimes \mathrm{id}_D$, $X^{(2)} \equiv \mathrm{id}_D \otimes X(u_2)$ $\forall X: C \to \mathrm{Mat}_D$, $r_\pm \equiv r(u_1 \pm u_2)$. Also, let K_α $(\alpha = \pm)$ be functions from C into $\mathrm{Mat}_D(C)$ that satisfy the relation

$$[r_-,\ K^{(1)}K^{(2)}] + K^{(1)}r_+K^{(2)} - K^{(2)}r_+K^{(1)} = 0. \tag{2}$$

Proposition 1. The function $\mathscr{T}: \mathbf{C} \to \mathrm{Mat}_D(\mathscr{A})$

$$\mathscr{T}(u) \equiv T(u)\, K_-(u)\, T^{-1}(-u) \tag{3}$$

satisfies the equation

$$\{\mathscr{T}^{(1)},\ \mathscr{T}^{(2)}\} = [r_-,\ \mathscr{T}^{(1)}\mathscr{T}^{(2)}] + \mathscr{T}^{(1)}r_+\mathscr{T}^{(2)} - \mathscr{T}^{(2)}r_+\mathscr{T}^{(1)}. \tag{4}$$

Proposition 2. The quantities $\tau(u)$

$$\tau(u) \equiv \mathrm{tr}\, K_+(u)\, \mathscr{T}(u) = \mathrm{tr}\, K_+(u)\, T(u)\, K_-(u)\, T^{-1}(-u) \tag{5}$$

are in involution: $\{\tau(u_1),\ \tau(u_2)\} = 0\ \forall u_1,\ u_2$.

Propositon 2 enables us to interpret $\tau(u)$ as the generating function of the commutating integrals of motion of a Hamiltonian system with the state space $\mathscr{A}$.

2. In the examples given below, T is constructed either as the product $T(u) = L_N(u) \ldots L_1(u)$ of L-operators [2] $L_n(u)$ that satisfy (1) after the substitution $T(u) = L_n(u)$ (the discrete case), or is determined from a differential equation [2]: $T(u) \equiv t(u;\ x_+,\ x_-)$, $\partial T(u;\ x,\ x_-)/\partial x = \mathscr{L}(u,\ x)T(u;\ x,\ x_-)$, $T(u;\ x_-,\ x_-) = \mathrm{id}_D$, where $\mathscr{L}(u,\ x)$ satisfies

$$\{\mathscr{L}^{(1)}(x_1),\ \mathscr{L}^{(2)}(x_2)\} = [r_-,\ \mathscr{L}^{(1)}(x_1) + \mathscr{L}^{(2)}(x_2)]\,\delta(x_1 - x_2) \tag{6}$$

(the continuous case). In the continuous case, the algebra $\mathscr{A}$ is formed by functions on the segment $[x_-,\ x_+]$ and is completely determined only after the specification of the boundary conditions for $x = x_\pm$ (cf. [2, Chap. 1, Sec. 1, pp. 19-22]). In this connection, we will suppose that there is given a local Hamiltonian $H_0 = \int_{x_-}^{x_+} h(x)\,dx$ such that $\{H_0,\ \mathrm{tr}\,T(u)\} = 0$, and the equation of motion $\partial\mathscr{L}(u,\ x)/\partial t = \{H_0, \mathscr{L}(u,\ x)\}$ (for periodic boundary conditions) can be expressed for $x \in (x_-,\ x_+)$ in the form

$$\partial\mathscr{L}(u,\ x)/\partial t = \partial\mathscr{M}(u,\ x)/\partial x + [\mathscr{M}(u,\ x),\ \mathscr{L}(u,\ x)], \tag{7}$$

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where $\mathcal{M}(u, x) \in \mathrm{Mat}_D(\mathcal{A})$ is the corresponding M-operator [2].

Proposition 3. Let the dynamics on $\mathcal{A}$ be given for the equation of motion (7) for $x \in (x_-, x_+)$ and the following boundary conditions for $x = x_\pm$:

$$K_\pm(u)\,\mathcal{M}(\pm u, x_\pm) = \mathcal{M}(\mp u, x_\pm)\,K_\pm(u), \tag{8}$$

and $\tau(u)$ be given by Eq. (5). Then $d\tau(u)/dt = 0\ \forall u$.

In all the examples considered below, it is immediately verified that the boundary conditions (8) are equivalent to the addition of the boundary terms $H_0 \to H = H_0 + H_b$, $\{H, \tau(u)\} = 0\ \forall u$ to the Hamiltonian H_0.

3. Examples. a) Nonlinear Schrödinger Equation [1, 2]: $i\psi_t = -\psi_{xx} + 2\kappa\bar{\psi}\psi^2$. The LM pair is $\mathcal{L}(u, x) = -iu\sigma_3/2 + i\kappa\psi(x)\sigma_+ - i\bar{\psi}(x)\sigma_-$, $\mathcal{M}(u, x) = i(u^2/2 + \kappa\bar{\psi}\psi)\sigma_3 + \kappa(\psi_x - iu\psi)\sigma_+ + (\bar{\psi}_x + iu\bar{\psi})\sigma_-$, where $\sigma_{1,2,3} \in sl(2)$ are the standard Pauli matrices, $\sigma_\pm \equiv (\sigma_1 \pm i\sigma_2)/2$. The Poisson structure is $\{\psi(x), \psi(y)\} = 0$, $\{\psi(x), \bar{\psi}(y)\} = i\delta(x - y)$, and the r-matrix $r(u) = -\varkappa \sum_\alpha \sigma_\alpha \otimes \sigma_\alpha/2u$. To the choice of the matrices $K_\pm(u) = u\sigma_3 + i\vartheta_\pm$, $\bar{\vartheta}_\pm = \vartheta_\pm$ there corresponds the Hamiltonian $H = \int_{x_-}^{x_+} (\bar{\psi}_x\psi_x + \varkappa\bar{\psi}^2\psi^2)\,dx + \sum_{\alpha=\pm} \vartheta_\alpha\bar{\psi}_\alpha\psi_\alpha$, where $\psi_\pm \equiv \psi(x_\pm)$, and the boundary conditions $\psi_\pm' \pm \vartheta_\pm\psi_\pm = 0$, which coincide with the standard mixed boundary conditions for the Strum–Liouville problem.

b) Sine-Gordon Equation [1, 2]: $\varphi_t = p$, $p_t = \varphi_{xx} - \sin\varphi$, $\mathcal{L}(u, x) = i(-\sigma_3 \operatorname{sh} u \cos(\varphi/2) - \sigma_1 p/2 + \sigma_2 \operatorname{ch} u \sin(\varphi/2))/2$, $\mathcal{M}(u, x) = i(\sigma_3 \operatorname{ch} u \cos(\varphi/2) - \sigma_1\varphi_x/2 - \sigma_2 \operatorname{sh} u \sin(\varphi/2))/2$. $\{\varphi(x), \varphi(y)\} = \{p(x), p(y)\} = 0$, $\{p(x), \varphi(y)\} = \delta(x - y)$, $r(u) = [\sigma_1 \otimes \sigma_1 \operatorname{cth} u + (\sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)\operatorname{sh}^{-1} u]/16$, $K_\pm(u) = \sigma_3 \operatorname{sh} u + i\vartheta_\pm$, $\bar{\vartheta}_\pm = \vartheta_\pm$,

$$H = \frac{1}{2}\int_{x_-}^{x_+} (p^2 + \varphi_x^2 - \cos\varphi)\,dx - \sum_{\alpha=\pm} 4\vartheta_\alpha \cos\frac{\varphi_\alpha}{2}, \quad \varphi_\pm' \pm 2\vartheta_\pm \sin\frac{\varphi_\pm}{2} = 0.$$

c) Landau–Lifshitz Equation [2]: $\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times \hat{J}\mathbf{S}$, $\mathbf{S} = (S_1, S_2, S_3)$, $(\mathbf{S}, \mathbf{S}) = 1$, $\hat{J} = \operatorname{diag}(J_1, J_2, J_3)$, $\mathcal{L}(u, x) = \sum_{\alpha=1}^{3} iw_\alpha(u) S_\alpha(x)\sigma_\alpha$, $\mathcal{M}(u, x) = i\sum_{\alpha\beta\gamma} (w_\alpha(u) \times \sigma_\alpha S_\beta S_\gamma' \varepsilon^{\alpha\beta\gamma} + S_\alpha\sigma_\alpha w_\beta(u) w_\gamma(u) |\varepsilon^{\alpha\beta\gamma}|)$, $w_1(u) = \rho/\operatorname{sn}(u, k)$, $w_2(u) = \rho\, \operatorname{dn}(u, k)/\operatorname{sn}(u, k)$, $w_3(u) = \rho\, \operatorname{cn}(u, k)/\operatorname{sn}(u, k)$, $\rho = (J_3 - J_1)^{1/2}/2$, $k = ((J_2 - J_1)/(J_3 - J_1))^{1/2}$, $\{S_\alpha(x), S_\beta(y)\} = -\sum_\gamma \varepsilon^{\alpha\beta\gamma} S_\gamma(x)\,\delta(x - y)$, $r(u) = \frac{1}{2}\sum_\alpha w_\alpha(u)\,\sigma_\alpha \otimes \sigma_\alpha$, $K_\pm(u) = 2w_3(u) + i\vartheta_\pm\sigma_3$, $\bar{\vartheta}_\pm = \vartheta_\pm$, $H = \frac{1}{2}\int_{x_-}^{x_+} [(\mathbf{S}_x, \mathbf{S}_x) + (\mathbf{S}, \hat{J}\mathbf{S})]\,dx + \sum_{\alpha=\pm} \vartheta_\alpha S_3(x_\alpha)$, $\mathbf{S}(x_\pm) \times (\vartheta_\pm \mathbf{e}_3 \pm \mathbf{S}'(x_\pm)) = 0$.

d) Toda Chain [1, 2]: $\{p_m, p_n\} = \{q_m, q_n\} = 0$, $\{p_m, q_n\} = \delta_{mn}$, $H = \sum_{n=1}^{N} p_n^2/2 + \sum^{N-1} e^{q_{n+1}-q_n} + \left(\alpha_1 e^{q_1} + \frac{\beta_1}{2} e^{2q_1}\right) + \left(\alpha_N e^{-q_N} + \frac{\beta_N}{2} e^{-2q_N}\right)$,

$$L_n(u) = \begin{pmatrix} u - p_n, & -e^{q_n} \\ e^{-q_n}, & 0 \end{pmatrix}, \quad K_-(u) = \begin{pmatrix} \alpha_1 & u \\ -\beta_1 u & \alpha_1 \end{pmatrix}, \quad K_+(u) = \begin{pmatrix} \alpha_N & \beta_N u \\ -u & \alpha_N \end{pmatrix}.$$

As far as the author knows, the complete integrability of the Hamiltonian H with four arbitrary constants α_1, β_1, α_N, and β_N has not been mentioned in the literature (see, e.g., [5]).

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GENERALIZED TODA FLOWS, RICCATI EQUATIONS ON THE GRASSMANIAN, AND THE QR-ALGORITHM

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The connection established in [1] between the QR-algorithm and a class of dynamical systems, called generalized Toda flows, has served as the impetus for a number of publications. As it turned out, the QR-algorithm proposed in [2, 3] is one of the most effective algorithms for finding the spectra of various classes of matrices [4]. Its modification by means of numerical methods of integration of dynamical systems, which leads to a significantly simplified computational scheme, is undoubtedly of interest. This note is devoted to a detailed study of the asymptotic behavior of the generalized Toda flows. Our approach is based on an observation of Kostant, Guillemin, and Sternberg [5], according to which this behavior is determined by the action of a one-parameter group of linear transformations on the manifold of complete flags, i.e., in the language of mathematical system theory, by the asymptotic behavior of a matrix Riccati differential equation (see, for example, [6-8]).

Given a complex n × n matrix Y, let $Y = Y_+ + Y_0 + Y_-$, where Y_+ and Y_- are strictly upper and, respectively, lower triangular matrices and $Y_0 = \operatorname{Re}(Y_0) = i\operatorname{Im}(Y_0)$ is a diagonal matrix. We put $\Pi_0(Y) = Y_- - Y_-^* + i\operatorname{Im}(Y_0)$, where Y_-^* denotes the Hermitian conjugate of the matrix Y_-. Given an n × n matrix X_0, let G be a complex analytic function defined in an open subset of C, which contains the spectrum of X_0. We call the phase flow of the dynamical system

$$\dot{X} = [X, \Pi_0(G(X))], \qquad X(0) = X_0 \tag{1}$$

a generalized Toda flow [9]. Here, [,] designates the usual commutator of matrices. The system (1) can be incorporated in the Kostant–Symes–Adler scheme [5]. Its solution has the form $X(t) = Q^*(t)X_0Q(t)$, where

$$\exp(tG(X_0)) = Q(t)R(t) \tag{2}$$

is the decomposition into the product of a unitary matrix Q(t) and an upper-triangular matrix R(t) with real nonnegative diagonal entries (the QR-decomposition). Taking $G(z) = \ln(z - c)$, with $c \in C$ a constant, we deduce that the values of the solution X(t) of problem (1) at t = 0, 1, ... are the successive iterations of the QR-algorithm with constant shift [9].

For a real number γ we let E(γ) denote the direct sum of the root subspaces of the matrix $G(X_0)$ corresponding to the eigenvalues λ with $\operatorname{Re}\lambda = \gamma$. Let $\gamma_1 < \gamma_2 < \ldots < \gamma_m$ be the list of all real numbers for which $E(\gamma_i) \neq 0$. We put $P_i = \Sigma\{E(\gamma_j)\colon j \leq i\}$, $N_i = \Sigma\{E(\gamma_j)\colon j \geq m + 1 - i\}$, $i \in [1, m]$. Also, let $\pi_i\colon C^n \to E(\gamma_i)$, $i \in [1, m]$ denote the projection onto $E(\gamma_i)$ parallel to $\Sigma\{E(\gamma_j)\colon \gamma \neq i\}$. Given a subspace $V \subset C^n$, we put $\Pi_+(V) = \Sigma\{\pi_i(V \cap P_i)\colon i \in [1, m]\}$ and $\Pi_-(V) = \Sigma\{\pi_i(V \cap N_i)\colon i \in [1, m]\}$. Further, let e_i, $i \in [1, n]$, be the standard basis in C^n and let V_i denote the subspace spanned by e_j, $j \in [1, i]$. We put $V_i^\pm = \Pi_\pm(V_i)$, $i \in [1, n]$. Clearly, $0 \subset V_1^\pm \subset V_2^\pm \subset \ldots \subset V_n^\pm = C^n$ are complete flags of subspaces in C^n. We choose in C^n orthonormal bases e_i^+, e_i^-, $i \in [1, n]$, such that the vectors e_j^+ (e_j^-) with $j \in [1, i]$ span the subspace V_i^+ (respectively, V_i^-), $i \in [1, n]$. Let $Q_\pm$ be the unitary transformations in C^n specified by the conditions $Q_\pm e_i = e_i^\pm$, $i \in [1, n]$. Finally, let $X_\pm = Q_\pm^* X_0 Q_\pm = \|x_{ij}^\pm\|$.

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