VARIETIES OF MODULES OF F-SHEAVES

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The present article is the first in a series of papers containing a detailed account of the proof of the part of Langland's conjecture for GL(2) over a global field k of characteristic p > 0 dealing with constructing a map $\Sigma_2 \to \Sigma_1$, where Σ_1 is the set of irreducible two-dimensional representations over $\overline{\mathbf{Q}}_l$, $\ell \neq p$, of the Weyl group of the field k, continuous in the ℓ-adic topology and having only finitely many branching points, and Σ_2 is the set of irreducible representations of GL(2) over adeles of the field k appearing in the space of the $\overline{\mathbf{Q}}_l$-valued parabolic forms. A brief account is contained in [2, 11]; see also [12, 13]. The existence of the inverse map $\Sigma_1 \to \Sigma_2$ has been proved in [4] and, by a different method, in [14, 15].

The general scheme of constructing the reciprocity map $\Sigma_2 \to \Sigma_1$ has been presented in [2, 11-13]. We only recall that the main role is played by varieties of modules of F-sheaves of rank 2: roughly speaking, the required ℓ-adic representations of Gal $(\overline{\mathbf{Q}}/\mathbf{Q})$, corresponding to parabolic representations π of the group GL $(2, \mathbf{A})$ such that π_∞ belongs to a discrete series are realized in cohomologies of varieties of modules of elliptic curves (cf. [16]).

The present article studies geometry of varieties of modules of F-sheaves. The consequent articles of the cycle will deal with computing the ζ-functions of these varieties (in the case of rank 2), constructing their compactifications (also for rank 2), and the proof of existence of the reciprocity map $\Sigma_2 \to \Sigma_1$.

I dedicate this article to my teacher Yu. I. Manin in connection with his 50th birthday.

The following notation is adopted throughout the article: k, fixed global field of characteristic p > 0; $\mathbf{F}_q$, field of constants of k; X, smooth projective irreducible curve over $\mathbf{F}_q$, corresponding to k; $\mathfrak{A}$ is the adele ring of k; O_v, completed local ring of a point $v \in X$; if D is a finite subscheme in X, then $A_D = H^0(D, \mathcal{O}_D)$; $|D|$ is the order of D. We write "a scheme" where one would write "a scheme over $\mathbf{F}_q$." If Y and Z are schemes, then $Y \times Z$ denotes the product of Y and Z over $\mathbf{F}_q$. Similar conventions are adopted for rings and their tensor products as well as for morphisms of schemes or rings. The Frobenius endomorphism of a scheme S (relative to $\mathbf{F}_q$) is denoted by Fr_S or Fr. If Y is a closed subscheme in Z, then we identify sheaves of $\mathcal{O}_Y$-modules with their direct images under the inclusion $Y \to Z$. In particular, if a sheaf of $\mathcal{O}_Z$-modules is referred to as an invertible sheaf on Y, then it is meant that it is a direct image of such a sheaf.

1. F-Sheaves

We recall the notion of an F-sheaf introduced in [2, 11] (note that F-sheaves are called Frobenius–Hecke sheaves or FH-sheaves in [12] and "shtukas" in [13]).

Definition. A left F-sheaf of rank d over a scheme S is a diagram

$$\mathcal{F} \begin{array}{l} \xrightarrow{\ i\ } \mathcal{L} \\ \xrightarrow{\ j\ } (id_X \times \mathrm{Fr}_S)^* \mathcal{L}, \end{array} \qquad (1.1)$$

where $\mathcal{L}$ and $\mathcal{F}$ are locally free sheaves of $\mathcal{O}_{X \times S}$-modules of rank d, i and j are inclusions, the cokernel of i is an invertible sheaf on the graph Γ_α of some morphism α: $S \to X$, the cokernel of j is an invertible sheaf on the graph Γ_β of some morphism β: $S \to X$.

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A right F-sheaf of rank d over a scheme S is a diagram

$$\begin{array}{ccc} \mathcal{L} & \overset{g}{\searrow} & \\ & & \mathcal{E}, \\ (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{L} & \underset{f}{\nearrow} & \end{array} \qquad (1.2)$$

where $\mathcal{L}$ and $\mathcal{E}$ are locally free sheaves of $\mathcal{O}_{X\times S}$-modules of rank d, f and g are inclusions, the cokernel of f is an invertible sheaf on the graph Γ_α of some morphism α: $S \to X$, the cokernel of g is an invertible sheaf on the graph Γ_β of some morphism β: $S \to X$.

We say that α is the zero of the F-sheaf and β is its pole.

Remarks. 1) We will often say "an F-sheaf $\mathcal{L}$" meaning the diagram (1.1) or (1.2).

2) Roughly speaking, an F-sheaf is a sheaf $\mathcal{L}$ together with an isomorphism φ between the restrictions of $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{L}$ and $\mathcal{L}$ to $(X \times S) - (\Gamma_\alpha \cup \Gamma_\beta)$ having singularities over $\Gamma_\alpha \cup \Gamma_\beta$ (φ is undefined over Γ_β and φ^{-1} is undefined over Γ_α). So we use the terms "zero" and "pole."

3) Roughly speaking, an F-sheaf is a locally free sheaf of $\mathcal{O}_{X\times S}$-modules $\mathcal{L}$ along with an isomorphism $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$, where $\mathcal{L}'$ is a modification of $\mathcal{L}$ of a certain type. By a modification of $\mathcal{L}$ is meant (in the case when S is a spectrum of a field) a locally free sheaf whose general fiber is identified with the general fiber of $\mathcal{L}$. The definition of an F-sheaf employs simplest nontrivial modifications of $\mathcal{L}$, preserving its degree: a "minimal decreasing" of $\mathcal{L}$ over Γ_α with a consequent "minimal increasing" over Γ_β (in the case of left F-sheaves) or, conversely, a "minimal increasing" over Γ_β with a consequent "minimal decreasing" over Γ_α (in the case of right F-sheaves).

4) It follows from the previous remark that if $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$, then the notions of a left and right F-sheaf essentially coincide. In this case, an F-sheaf can be understood as a d-dimensional locally free sheaf of $\mathcal{O}_{X\times S}$-modules $\mathcal{L}$, equipped with a morphism $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \cdot \mathcal{L} \to \mathcal{L}(\Gamma_\beta)$, whose cokernel is a direct sum of an invertible sheaf on Γ_α and a (d − 1)-dimensional locally free sheaf on Γ_β.

5) The notions of a left and right F-sheaf of rank 1 coincide even if $\Gamma_\alpha \cap \Gamma_\beta \neq \emptyset$: defining an F-sheaf of rank 1 is equivalent to defining an invertible sheaf $\mathcal{L}$ on $X \times S$ equipped with an isomorphism $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}(\Gamma_\beta - \Gamma_\alpha)$.

6) Sheaves $\mathcal{L}$ on $X \times S$ equipped with an isomorphism $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}$ (a regular one, i.e., with no singularities) are not very interesting objects, as is demonstrated by the following proposition.

Proposition 1.1. Let Y be a projective scheme over $\mathbf{F}_q$, L an algebraically closed field, and pr_Y: $Y \otimes L \to Y$ the projection. Then the functor $\mathcal{F} \mapsto (\mathrm{pr}_Y)^* \mathcal{F}$ is an equivalence between the category of coherent sheaves of $\mathcal{O}_Y$-modules and the category of coherent sheaves of $\mathcal{O}_{Y\otimes L}$-modules equipped with an isomorphism $(\mathrm{id}_Y \otimes \mathrm{Fr}_L)^* \mathcal{A} \xrightarrow{\sim} \mathcal{A}$.

Proof. We fix a very abundant invertible sheaf $\mathcal{O}(1)$ on Y. It is known that the category of coherent sheaves of $\mathcal{O}_Y$-modules is equivalent to the quotient of the category of finitely generated graded modules over $\bigoplus_{n\geqslant 0} H^0(Y, \mathcal{O}(n))$ relative to the subcategory of modules having only finitely many nonzero homogeneous components. The category of coherent sheaves of $\mathcal{O}_{Y\otimes L}$-modules admits a similar description. It remains to show that the functor $V \mapsto V \otimes L$ is an equivalence between the category of finite-dimensional vector spaces over $\mathbf{F}_q$ and the category of pairs (W, φ), where W is a vector space over L and φ: $W \to W$ is a bijective q-linear map (in other words, we have reduced the proof of the proposition to the special case of $Y = \mathrm{Spec}\, \mathbf{F}_q$). Indeed, it is easily seen that the above functor is completely strict, and its surjectivity follows from a corollary at the end of Sec. 14 in [7]. ■

Let $\mathcal{L}$ be a left or right F-sheaf over S and $D \subset X$ a finite subscheme such that the zero and the pole of $\mathcal{L}$ do not involve D (i.e., they are morphisms $S \to X - D$). The restriction of $\mathcal{L}$ to $D \times S$ is denoted by $\mathcal{L}_D$. In the studied situation the restrictions of the morphisms in the diagrams (1.1) or (1.2) to $D \times S$ are isomorphisms, so we obtain an isomorphism $(\mathrm{id}_D \times \mathrm{Fr}_S)^* \mathcal{L}_D \xrightarrow{\sim} \mathcal{L}_D$. A structure of level D on $\mathcal{L}$ is an isomorphism $\mathcal{L}_D \xrightarrow{\sim} \mathcal{O}^d_{D\times S}$ such that the diagram

$$\begin{array}{ccc}(\mathrm{id}_D \times \mathrm{Fr}_S)^* \mathcal{L}_D & \xrightarrow{\sim} & \mathcal{L}_D, \\ & \searrow \;\; \mathcal{O}^d_{D\times S} \;\; \swarrow & \end{array}$$

is commutative. If the zero or the pole of $\mathcal{L}$ involves D, then, by definition, there are no structures of level D on $\mathcal{L}$.

We will now describe several ways to construct F-sheaves starting from other F-sheaves.

Construction 1. The inverse image of an F-sheaf over S with a zero at $\alpha \in X(S)$ and a pole at $\beta \in X(S)$ relative to a morphism $f\colon S' \to S$ is an F-sheaf over S' with a zero at $f^*(\alpha)$ and a pole at $f^*(\beta)$.

Construction 2. The diagram obtained from a left (right) F-sheaf with a zero at α and a pole at β by replacing all sheaves and morphisms between them by their conjugates is a right (left) F-sheaf with a zero at β and a pole at α.

Construction 3a. If a diagram (1.1) is a left F-sheaf with a zero at α and a pole at β, then the diagram $\mathcal{F} \xrightarrow{j} (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{L} \xleftarrow{f} (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{F}$, where $f = (\mathrm{id}_X \times \mathrm{Fr}_S)^* i$, is a left F-sheaf with a zero at $\mathrm{Fr}\circ\alpha$ and a pole at β.

Construction 3b. If a diagram (1.2) is a right F-sheaf with a zero at α and a pole at β, then the diagram $\mathcal{E} \xleftarrow{f} (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{L} \xrightarrow{j} (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{E}$, where $j = (\mathrm{id}_X \times \mathrm{Fr}_S)^* g$, is a left F-sheaf with a zero at α and a pole at $\mathrm{Fr}\circ\beta$.

Note that a consecutive implementation of constructions 3a and 3b is equivalent to taking the inverse image relative to Fr: $S \xleftarrow{f} S$.

Construction 4. The determinant of an F-sheaf $\mathcal{L}$ is an F-sheaf of rank 1 with the same zero and pole as those of $\mathcal{L}$.

Construction 5. Let $\mathcal{L}$ be an F-sheaf on S, $\mathcal{M}$ an invertible sheaf on X, and $\tilde{\mathcal{M}}$ its inverse image relative to the projection $X \times S \to X$. Then $\mathcal{L} \otimes \tilde{\mathcal{M}}$ is also an F-sheaf (with the same zero and pole as those of $\mathcal{L}$).

Construction 6. Suppose that an F-sheaf $\mathcal{L}$ of rank d over S with a structure of level D and a subsheaf of $\mathcal{O}_D$-modules $\mathcal{Q} \subset \mathcal{O}^d_D$. are given. Let $\mathcal{L}'$ be the kernel of the composition $\mathcal{L} \to \mathcal{L}_D \xrightarrow{\sim} \mathcal{O}^d_{D\times S} \to \mathcal{O}^d_{D\times S}/\tilde{\mathcal{Q}}$, where $\tilde{\mathcal{Q}}$ is the inverse image of $\mathcal{Q}$ on $D \times S$. Then $\mathcal{L}'$ is an F-sheaf (with the same zero and pole as those of $\mathcal{L}$).

Constructions 1'-4'. A structure of level D on an F-sheaf $\mathcal{L}$ is naturally extended to F-sheaves obtained from it by applications of constructions 1-4.

Construction 5'. The same is also true for Construction 5 if the restriction of $\mathcal{M}$ to D is trivialized.

Construction 6. Suppose that in the situation of Construction 6 an epimorphism $\mathcal{Q} \to \mathcal{O}^d_{D'}$, is given, where D' is a subscheme of D. Then the composition $\mathcal{L}' \to \tilde{\mathcal{Q}} \to \mathcal{O}^d_{D'\times S}$ defines a structure of level D' on $\mathcal{L}'$.

In the remaining part of this section we will discuss a connection between F-sheaves and elliptic modules. We fix a closed point $\infty \in X$ and put $A = H^0(X - \{\infty\}, \mathcal{O}_X)$. The notion of an elliptic A-module of rank d was introduced, essentially, by Carlitz [10] and rediscovered by the author [5]. In [5, 6] elliptic modules were used to construct the restrictions of the reciprocity map $\Sigma_2 \to \Sigma_1$ (see the introduction to this paper) to the set of $\pi \in \Sigma_2$ in which the component π_∞ is caspidal or special: it has been shown that the required ℓ-adic representations of $\mathrm{Gal}(\bar{k}/k)$ are realized in cohomologies of varieties of modules of elliptic modules of rank 2 equipped with some additional structures. It has been shown in [3] (see also Sec. 3 in [17]) that defining an elliptic A-module of rank d over S, where S is a scheme over A, is equivalent to defining an increasing sequence of d-dimensional locally free sheaves of $\mathcal{O}_{X\times S}$-modules $\{\mathcal{F}_i\}$ and a compatible system of morphisms $t_i\colon (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{F}_i \to \mathcal{F}_{i+1}$ such that: 1) $\mathcal{F}_{i+dm} = \mathcal{F}_i(\infty)$, where m is the degree of the residue field of the point ∞ over $\mathbf{F}_q$; 2) the sheaves $\pi_*(\mathcal{F}_{i+1}/\mathcal{F}_i)$, where $\pi\colon X \times S \to S$ is the projection, are locally free and one-dimensional; 3) for each $s \in S$ the restriction of $\mathcal{F}_i$ to $X \times s$ has Euler characteristic $i + 1$; 4) the support of the cokernel of t_i is a graph of a structure morphism $S \to \operatorname{Spec} A$. A family $\{\mathcal{F}_i, t_i\}$, satisfying conditions 1)-4) is called an elliptic sheaf. If $\{\mathcal{F}_i, t_i\}$ is an elliptic sheaf, then the diagram

$$\mathcal{F}_i \subsetneq \mathcal{F}_{i+1} \leftarrow (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{F}_i \tag{1.3}$$

is an F-sheaf. If one applies Construction 3b to this F-sheaf, then one obtains the left F-sheaf $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{F}_{i+1} \leftarrow (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{F}_i \to \mathcal{F}_{i+1}$, corresponding to the right F-sheaf $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{F}_{i+1} \to \mathcal{F}_{i+2} \leftarrow \mathcal{F}_{i+1}$. In view of the fact that $\mathcal{F}_{i-dm} = \mathcal{F}_i(-\infty)$, we deduce that the elliptic sheaf $\{\mathcal{F}_i, t_i\}$ is uniquely reconstructed if the F-sheaf (1.3) is known for some i. Thus, elliptic sheaves (and thus, also elliptic A-modules) of rank d over S bijectively correspond to F-sheaves $\mathcal{L}$ of rank d over S with a zero at α and a pole at β such that: 1) α: S → Spec A ⊂ X is a structure morphism; 2) $\beta(S) = \{\infty\}$; 3) the (md)-fold application of Construction 3b to $\mathcal{L}$ yields the same result as a single application to $\mathcal{L}$ of Construction 5 with $\mathcal{M} = \mathcal{O}_X(\infty)$, (more exactly, there is an isomorphism, equal to the identity on $\mathcal{L}$ between the F-sheaves obtained from $\mathcal{L}$) by applying these constructions); 4) for each $s \in S$ the Euler characteristic of the restriction of $\mathcal{L}$ to X × s is equal to zero (of course, 0 could be replaced by any fixed integer).

2. Group Schemes Corresponding to an F-Sheaf

If G is an elliptic A-module of rank d over S, then to each nonzero ideal I ⊂ A corresponds a finite locally free group scheme G_I over S, namely, the annihilator of I in G. How can G_I be expressed in terms of the F-sheaf $\mathcal{L}$, corresponding to G?

Definition. A φ-sheaf on a scheme S is a finite-dimensional locally free sheaf of $\mathcal{O}_S$-modules $\mathcal{E}$ equipped with a homomorphism φ: $\mathrm{Fr}_S^*\mathcal{E} \to \mathcal{E}$.

Let E be the vector bundle corresponding to the φ-sheaf $\mathcal{E}$. On the one hand there is a linear map φ^*: $E^* \to \mathrm{Fr}_S{}^*E^*$; on the other hand, the Frobenius map Frob: $E^* \to \mathrm{Fr}_S{}^*E^*$, which is q-linear. Put $\mathrm{Gr}(\mathcal{E}) = \mathrm{Ker}(\varphi^* - \mathrm{Frob})$. Gr is a contravariant functor from the category of φ-sheaves on S to the category of group schemes over S equipped with an action of $\mathbf{F}_q$.

Suppose now that an F-sheaf $\mathcal{L}$ over S is given along with a finite subscheme D ⊂ X not involving the pole of $\mathcal{L}$. Then a morphism $(\mathrm{id}_D \times \mathrm{Fr}_S)^* \mathcal{L}_D \to \mathcal{L}_D$ arises. So the direct image $\mathcal{E}$ of the sheaf $\mathcal{L}_D$ relative to the projection D × S → S is a φ-sheaf. Put $(\mathcal{L}) = \mathrm{Gr}(\mathcal{E})$. The ring $A_D = H^0(D, \mathcal{O}_D)$ acts on $\mathcal{E}$ and, thus, also on $\mathrm{Gr}_D(\mathcal{L})$. It is readily seen that if $\mathcal{L}$ corresponds to an elliptic A-module G and D = Spec(A/I), then there is a canonical isomorphism $G_I \simeq \mathrm{Gr}_D(\mathcal{L})$ compatible with the action of the ring A_D = A/I. We omit this verification because the statement above is needed only to motivate the definition of $\mathrm{Gr}(\mathcal{E})$ and $\mathrm{Gr}_D(\mathcal{L})$.

We will prove some properties of the functor Gr.

Proposition 2.1. 1) For each φ-sheaf $\mathcal{E}$ the scheme $\mathrm{Gr}(\mathcal{E})$ is finite and locally free. If dim $\mathcal{E}$ = n, then the order of $\mathrm{Gr}(\mathcal{E})$ is equal to q^n.

2) The sheaf $\mathrm{Lie}^*\,\mathrm{Gr}(\mathcal{E})$ [i.e., the inverse image of the sheaf $\Omega^1_{\mathrm{Gr}(\mathcal{E})/S}$ relative to the zero section e: $S \to \mathrm{Gr}(\mathcal{E})$] is canonically isomorphic to the cokernel of the morphism φ: $\mathrm{Fr}_S^*\mathcal{E} \to \mathcal{E}$.

3) The scheme $\mathrm{Gr}(\mathcal{E})$ is étale over S if and only if φ is an isomorphism.

4) Gr maps short exact sequences into short exact sequences.

5) The functor Gr is completely strict.

6) Each finite etale group scheme over S equipped with an action of $\mathbf{F}_q$, is isomorphic to $\mathrm{Gr}(\mathcal{E})$ for some φ-sheaf $\mathcal{E}$.

Proof. If S = Spec B, $\mathcal{L} \simeq \mathcal{O}_S^n$, and (b_{ij}) is the matrix of the morphism φ: $\mathrm{Fr}_S^*\mathcal{E} \to \mathcal{E}$, then $\mathrm{Gr}(\mathcal{E})$ is the subscheme of $\mathbf{G}_a^n \otimes B$, given by the system of equations

$$x_j^q - \sum_{i=1}^n b_{ij} x_i = 0, \quad j = 1, \ldots, n. \tag{2.1}$$

It is easily seen that the quotient of $B[x_1, \ldots, x_n]$ over the ideal generated by the left-hand parts of (2.1) is a free B-module with a basis $x_1^{m_1} \ldots x_n^{m_n}$, $0 \leqslant m_i < q$. This immediately implies statement 1). Statements 2)-4) directly follow from the fact that $\mathrm{Gr}(\mathcal{E})$ is locally given by the system (2.1), and statement 5) is a consequence of the following lemma.

LEMMA. If $\mathrm{Gr}(\mathscr{E})$ is given by the system (2.1) and h is a function on $\mathrm{Gr}(\mathscr{E})$, defining a homomorphism of group schemes $\mathrm{Gr}(\mathscr{E}) \to \mathbf{G}_a \otimes B$, compatible with the action of $\mathbf{F}_q$, then $h = \sum_i c_i x_i$, $c_i \in B$.

Proof. We write h in the form $F(x_1, \ldots, x_n)$, where F is a polynomial whose degree in each variable is less than q. The polynomial $F(x_1 + y_1, \ldots, x_n + y_n) - F(x_1, \ldots, x_n) - F(y_1, \ldots, y_n)$ has, in each variable, a degree less than q and its restriction to $\mathrm{Gr}(\mathscr{E}) \times \mathrm{Gr}(\mathscr{E})$ is equal to zero. So this polynomial itself is zero. Similarly, for each $c \in \mathbf{F}_q$ the equality $F(cx_1, \ldots, cx_n) = cF(x_1, \ldots, x_n)$ holds in the polynomial ring. Since the degree of F in each variable is less than q, we conclude that F is a linear function. ■

Statement 5) which has already been proved and the descent theory allow us to reduce the proof of statement 6) to the case when the group scheme is equal to $S \times \mathbf{F}_q^n$, and in this case statement 6) is obvious. ■

Proposition 2.2. Suppose an F-sheaf $\mathscr{L}$ of rank d over S is given along with a subscheme $D \subset X$ not involving the pole of $\mathscr{L}$.

1) $\mathrm{Gr}_D(\mathscr{L})$ is a finite locally free scheme of order $q^{d \cdot |D|}$.

2) The scheme $\mathrm{Gr}_D(\mathscr{L})$ is etale over S if and only if D does not involve the zero of $\mathscr{L}$. In these cases the geometric fibers of $\mathrm{Gr}_D(\mathscr{L})$ are free A_D-modules of rank d.

3) Defining a structure of level D on $\mathscr{L}$ is equivalent to defining an isomorphism of group schemes $\mathrm{Gr}_D(\mathscr{L}) \widetilde{\to} S \times (A_D^*)^d$, compatible with the action of A_D, where A_D^* is the vector space over $\mathbf{F}_q$, dual to A_D.

Proof. Statement 1) and the first part of statement 2) follow from Proposition 2.1. It suffices to prove the second part of statement 2) in the case when $S = \mathrm{Spec}\, F$, F being an algebraically closed field. In this case $\mathrm{Gr}_D(\mathscr{L})$ can be viewed not as a group scheme but rather as a group equipped with an action of A_D, i.e., as an A_D-module M. It follows from the definition of $\mathrm{Gr}_D(\mathscr{L})$ that $H^0(D \otimes F, \mathscr{L}_D) = M^* \otimes F$, where $M^* = \mathrm{Hom}_{\mathbf{F}_q}(M, \mathbf{F}_q)$. Since $H^0(D \otimes F, \mathscr{L}_D)$ is a free d-dimensional module over $A_D \otimes F$, M^*, and thus also M, are free d-dimensional A_D-modules. Statement 3) follows from the equality $\mathrm{Gr}(\mathcal{O}_S \otimes A_D) = S \times A_D^*$ and the complete strictness of the functor Gr. ■

Remarks. 1) A_D^* is a one-dimensional free A-module.

2) In [5], a structure of level I on an elliptic module G of rank d over S is defined as an isomorphism $G_I \widetilde{\to} S \times (I^{-1}/A)^d$. On the other hand, defining a structure of level $D = \mathrm{Spec}(A/I)$ on the F-sheaf corresponding to G is equivalent to defining an isomorphism $G_I \widetilde{\to} S \times (A_D^*)^d$. Since $A_D^* = (A/I)^* = I^{-1}\Omega_A{}^{1}/\Omega_A{}^1$ (the matching between A/I and $I^{-1}\Omega_A{}^1/\Omega_A{}^1$ is determined by means of the "sum of residues" map from $I^{-1}\Omega_A{}^1/\Omega_A{}^1$ to $\mathbf{F}_q$), the two notions of level structures coincide whenever a generator of the one-dimensional free (A/I)-module $\Omega_A{}^1/I\Omega_A{}^1$ is fixed.

3) If Construction 6 from Sec. 1 is applied to the F-sheaf $\mathscr{L}$, corresponding to an elliptic module G with a fixed structure of level I and $D = \mathrm{Spec}(A/I)$, and then an elliptic module G' is constructed from the obtained F-sheaf $\mathscr{L}'$ then G' is the quotient of G relative to the subgroup

$$S \times \mathrm{Hom}_{\mathbf{F}_q}(H^0(D, \mathcal{O}_D^d/Q), \mathbf{F}_q) \subset G_I.$$

From Proposition 2.2 follows

Proposition 2.3. Suppose that an F-sheaf $\mathscr{L}$ of rank d over S is given along with a finite subscheme $D \subset X$ not involving the zero and the pole of $\mathscr{L}$, a subscheme $D' \subset D$, and a structure of level D' on $\mathscr{L}$. Then the functor assigning to an S-scheme S' the set of structures of level D on the inverse image of $\mathscr{L}$ relative to the morphism $S' \to S$ extending the inverse image of a given structure of level D' is representable by a scheme which is a principal fiber space over S with the group $G = \mathrm{Ker}(GL(d, A_D) \to GL(d, A_{D'}))$.

3. Schemes of Modules

We denote by $\mathrm{Fsh}_{D,d}$ (respectively, ${}_{D,d}\mathrm{Fsh}$) the functor assigning to a scheme S the set of isomorphism classes of right (respectively, left) F-sheaves of rank d over S equipped with a structure of level D. We will show that for each of these functors there exists a rough (and sometimes even actual) scheme of modules.

First, let d = 1. If $D \neq \emptyset$, then we denote by $\underline{\mathrm{Pic}}_D X$ the scheme representing the functor $S \mapsto$ {isomorphism classes of invertible sheaves on $X \times S$ trivialized over $D \times S$}. The representability of this functor for $D \neq \emptyset$ follows from the existence of the Picard scheme $\underline{\mathrm{Pic}}\,X$ and the lack of nontrivial automorphisms for an invertible sheaf on $X \times S$ which would be equal to the identity on $D \times S$. For $D = \emptyset$ the studied functor is not representable but it has a rough scheme of modules, namely $\underline{\mathrm{Pic}}\,X$. So we put $\underline{\mathrm{Pic}}_{\emptyset} X = \underline{\mathrm{Pic}}\,X$. The group operation in $\underline{\mathrm{Pic}}_D X$ will be written additively. We denote by $\underline{\mathrm{Pic}}_D^n X$ the clopen subset of $\underline{\mathrm{Pic}}_D X$ corresponding to invertible sheaves of degree n. $\underline{\mathrm{Pic}}_D^n X$ is a connected scheme of finite type (the connectedness follows from the presence of an epimorphism $\underline{\mathrm{Pic}}_D^0 X \to \underline{\mathrm{Pic}}^0 X$ with a connected kernel). Note that $\underline{\mathrm{Pic}}_D^0 X$ is nothing but a generalized Jacobian [8]. There is a natural morphism $X - D \to \underline{\mathrm{Pic}}_D^1 X$ assigning to a point $u \in X(\overline{\mathbf{F}}_q)$ the isomorphism class $\underline{u}$ of the sheaf $\mathcal{O}_{X \otimes \overline{\mathbf{F}}_q}(u)$, naturally trivialized over D.

Proposition 3.1. For the functor $\mathrm{Fsh}_{D,1} = {}_{D,1}\mathrm{Fsh}$ there exists a rough scheme of modules $M_{D,1}$. If $D \neq \emptyset$, then $M_{D,1}$ is an actual scheme of modules. There is a pullback

$$\begin{array}{ccc} M_{D,1} & \xrightarrow{\ f\ } & \mathrm{Pic}_D X \\ {\scriptstyle (g_1,\, g_2)}\downarrow & & \downarrow{\scriptstyle \mathrm{Fr-id}} \\ (X-D)\times(X-D) & \xrightarrow{(\alpha,\,\beta)\mapsto\bar{\beta}-\bar{\alpha}} & \mathrm{Pic}_D^0 X, \end{array}$$

where g_1 and g_2 assign to an F-sheaf its zero and its pole, respectively, and f is forgetting the F-structure on an invertible sheaf. ■

COROLLARY. $M_{D,1}$ is a disjoint union of nonempty schemes $M_{D,1}{}^n = f^{-1}(\underline{\mathrm{Pic}}_D^n X)$ which are finite and etale over $(X - D)^2$.

Note that one can restate Lang's geometrical theory of class fields [8] in terms of the schemes $M_{D,1}$; in a certain sense the restatement is more natural than the original statement.

If $d > 1$, then, in order to obtain a (rough) scheme of modules of finite type, one has, in addition to the degree, to "hold" another invariant of an F-sheaf. Namely, if $\mathcal{F}$ is a locally free sheaf of rank d on a smooth irreducible projective curve Y over an algebraically closed field L, then we denote by $h(\mathcal{F})$ the greatest degree of invertible subsheaves in $\mathcal{F}$ (these degrees are bounded above because if $\mathcal{A} \subset \mathcal{F}$ is an invertible subsheaf, then $\deg \mathcal{A} \leqslant \dim H^0(\mathcal{A}) + g - 1 \leq \dim H^0(\mathcal{F}) + g - 1$). In the case when L is not closed, $h(\mathcal{F}) \stackrel{\mathrm{def}}{=} h(\overline{\mathcal{F}})$, where $\overline{\mathcal{F}}$ is the inverse image of $\mathcal{F}$ on $Y \otimes \overline{L}$. A standard reasoning shows that $h(\mathcal{F})$ remains unchanged if the field L is extended and if there is a family of sheaves $\{\mathcal{F}_t\}$, then for each $m \in \mathbf{Z}$ the set $\{t \mid h(\mathcal{F}_t) \leq m\}$ is open. We denote by $\mathrm{Fsh}^n_{D,d}$ (respectively, $\mathrm{Fsh}^{m,n}_{D,d}$) the subfunctor of $\mathrm{Fsh}_{D,d}$ obtained by imposing the following condition on an F-sheaf $\mathcal{L}$ over S: for each $s \in S$ the degree of the restriction $\mathcal{L}_s$ of the sheaf $\mathcal{L}$ to $X \times s$ is equal to n (respectively, $\deg \mathcal{L}_s = n$ and $h(\mathcal{L}_s) \leqslant m$). Similarly, we define ${}^{n}_{D,d}\mathrm{Fsh}$ and ${}^{m,n}_{D,d}\mathrm{Fsh}$.

Proposition 3.2. There exists a number c (depending on d and X) such that if $|D| \geq dm - n + c$, then the functors $\mathrm{Fsh}^{m,n}_{D,d}$ and ${}^{m,n}_{D,d}\mathrm{Fsh}$ are representable by quasiprojective schemes $M^{m,n}_{D,d}$ and ${}^{m,n}_{D,d}M$ over $\mathbf{F}_q$.

Proof. A standard argument (see [18]) shows that there exists c [for instance $c = (d + 1)g + 1$, where g is the genus of X] such that: a) for each d-dimensional locally free sheaf $\mathcal{F}$ on $X \otimes \overline{\mathbf{F}}_q$ such that $\deg \mathcal{F} = n$, $h(\mathcal{F}) \leqslant m$, each automorphism of $\mathcal{F}$, equal to the identity over $D \otimes \overline{\mathbf{F}}_q$, is the identity itself; b) there exists a quasiprojective scheme $\mathrm{Bun}^{m,n}_{D,d}$ over $\mathbf{F}_q$ such that $\mathrm{Bun}^{m,n}_{D,d}(S)$ = {isomorphism classes of d-dimensional locally free sheaves $\mathcal{L}$ on $X \times S$ trivialized over $D \times S$ and such that $\deg \mathcal{L}_s = n$, $h(\mathcal{L}_s) \leqslant m$ for all $s \in S$}. To prove the representability of $\mathrm{Fsh}^{m,n}_{D,d}$ consider the functor $S \mapsto$ {isomorphism classes of diagrams of the form $\mathcal{L} \xrightarrow{g} \mathcal{E} \xleftarrow{f} \mathcal{L}'$, where $\mathcal{L}, \mathcal{E}, \mathcal{L}'$ are locally free sheaves of rank d, f, and g are are inclusions, Coker f and Coker g are invertible sheaves on graphs of some morphisms $S \to X - D$, the restriction of $\mathcal{L}$ to $D \times S$ is trivialized, for each $s \in S$ the relationships $\deg \mathcal{L}_s = n$, $h(\mathcal{L}_s) \leqslant m$, $h(\mathcal{L}'_s) \leqslant m$} hold. If $|D| \geq dm - n + c$, then this functor is representable by some quasiprojective scheme V over $\mathbf{F}_q$. There are morphisms ψ_1, ψ_2: $V \to \mathrm{Bun}^{m,n}_{D,d}$ assigning to a diagram $\mathcal{L} \to \mathcal{E} \leftarrow \mathcal{L}'$ the sheaves $\mathcal{L}$ and $\mathcal{L}'$ respectively. We denote by $M^{m,n}_{D,d}$ the preimage of the diagonal $\Delta \subset \mathrm{Bun}^{m,n}_{D,d} \times \mathrm{Bun}^{m,n}_{D,d}$ relative to the morphism $(\psi_1 \circ \mathrm{Fr}, \psi_2)$: $V \mapsto \mathrm{Bun}^{m,n}_{D,d} \times \mathrm{Bun}^{m,n}_{D,d}$. Then $M^{m,n}_{D,d}$ represents $\mathrm{Fsh}^{m,n}_{D,d}$. Similarly, one can construct ${}^{m,n}_{D,d}M$.

Remarks. 1) There are natural morphisms $M^{m,n}_{D,d} \to (X - D)^2$ and ${}^{m,n}_{D,d}M \to (X - D)^2$ assigning to an F-sheaf its zero and its pole.

2) On $M^{m,n}_{D,d}$ and ${}^{m,n}_{D,d}M$ there is a natural action of $GL(d, A_D)$.

We will now construct a rough scheme of modules for $Fsh_{D,d}$. First, we will fix $m, n \in \mathbf{Z}$ and show that there exists a rough scheme of modules for $Fsh^{m,n}_{D,d}$. Indeed, if $D' \subset X$ is a finite subscheme such that $D' \supset D$ and the functor $Fsh^{m,n}_{D,d}$ is representable, then, as follows from Proposition 2.3, the quotient of $M^{m,n}_{D,d}$ over the action of the group $G = \mathrm{Ker}(GL(d, A_{D'}) \to GL(d, A_D))$ is a rough scheme of modules for the functor $Fsh^{m,n}_{D,d} \times_{(X-D)^2}(X - D')^2$. We now choose finite subschemes D_1, D_2, $D_3 \subset X$ containing D in such a way that the functors $Fsh^{m,n}_{D_i,d}$, i = 1, 2, 3 would be representable and the equalities $D_1 \cap D_2 = D_1 \cap D_6 = D_2 \cap D_3 = D$ would hold, at least in the set-theoretic sense, then we glue together the rough schemes of modules for $Fsh^{m,n}_{D,d} \times_{(X-D)^2}(X - D_i)^2$. As a result, we obtain a scheme $M^{m,n}_{D,d}$ which is a rough scheme of modules for $Fsh^{m,n}_{D,d}$ [in verifying this fact, we use the condition $\bigcup_{i=1}^{3}(X - D_i)^2 = (X - D)^2$; note that $(X - D_1)^2 \cup (X - D_2)^2$ is not, generally, equal to $(X - D)^2$]. If $M' \geq m$, then $M^{m,n}_{D,d}$ is an open subscheme in $M^{m',n}_{D,d}$. Put $M^{n}_{D,d} = \bigcup_m M^{m,n}_{D,d}$. Then $M^{n}_{D,d}$ is a rough scheme of modules for $Fsh^{n}_{D,d}$. Finally, the disjoint union $M_{D,d}$ of the schemes $M^{n}_{D,d}$, $n \in \mathbf{Z}$, is a rough scheme of modules for $Fsh_{D,d}$. Similarly, we construct the rough schemes of modules ${}^{m,n}_{D,d}M$, ${}^{n}_{D,d}M$, and ${}_{D,d}M$. Clearly, $M^{m,n}_{D,d}$ and ${}^{m,n}_{D,d}M$ are schemes of finite type over $\mathbf{F}_q$, while $M^{n}_{D,d}$, ${}^{n}_{D,d}M$, $M_{D,d}$, and ${}_{D,d}M$ are schemes of locally finite type over $\mathbf{F}_q$. It is also clear that the schemes $M_{D,d}$ and ${}_{D,d}M$ are separable.

We will list structures present on $M_{D,d}$ and ${}_{D,d}M$.

1) Assigning to an F-sheaf its zero and its pole defines morphisms $M_{D,d} \to (X - D)^2$, ${}_{D,d}M \to (X - D)^2$.

2) $GL(d, A_D)$ acts on $M_{D,d}$ and ${}_{D,d}M$ preserving the morphisms $M_{D,d} \to X^2$, ${}_{D,d}M \to X^2$.

3) If $D' \supset D$, then there are natural morphisms $M_{D',d} \to M_{D,d}$ and ${}_{D',d}M \to {}_{D,d}M$ (forgetting the structure). They are morphisms of schemes over X^2 inducing the isomorphisms $G \setminus M_{D',d} \xrightarrow{\sim} M_{D,d} \times_{(X-D)^2}(X - D')^2$, $G \setminus {}_{D',d}M \xrightarrow{\sim} {}_{D,d}M \times_{(X-D)^2}(X - D')^2$, where $G = \mathrm{Ker}(GL(d, A_{D'}) \to GL(d, A_D))$.

4) The schemes $M_{D,d}$ and ${}_{D,d}M$ are canonically isomorphic over the complement to the diagonal $\Delta \subset X^2$. For d = 1, $M_{D,d} = {}_{D,d}M$.

5) Construction 2' from Sec. 1 defines an isomorphism $*$: $M_{D,d} \xrightarrow{\sim} {}_{D,d}M$ such that the diagram

$$\begin{array}{ccc} M_{D,d} & \xrightarrow{\ *\ } & {}_{D,d}M \\ \downarrow & & \downarrow \\ X^2 & \xrightarrow{(b,a)\mapsto(b,a)} & X^2 \end{array}$$

commutes. For d = 1, the morphism $*$ is involutive. For d > 1 it is involutive over the complement to the diagonal $\Delta \subset X^2$ (i.e., where $M_{D,d}$ does not differ from ${}_{D,d}M$).

6) Constructions (3a)' and (3b)' from Sec. 1 define morphisms F_1: ${}_{D,d}M \to M_{D,d}$, F_2: $M_{D,d} \to {}_{D,d}M$, such that the diagrams

$$\begin{array}{ccc} {}_{D,d}M & \xrightarrow{F_1} & M_{D,d} \\ \downarrow & & \downarrow \\ X^2 & \xrightarrow{\mathrm{Fr}\times\mathrm{id}} & X^2 \end{array} \qquad \begin{array}{ccc} M_{D,d} & \xrightarrow{F_2} & {}_{D,d}M \\ \downarrow & & \downarrow \\ X^2 & \xrightarrow{\mathrm{id}\times\mathrm{Fr}} & X^2 \end{array}$$

commute. Also, $F_1F_2 = \mathrm{Fr}$, $F_2F_1 = \mathrm{Fr}$, $*F_1* = F_2$.

7) Construction 4' from Sec. 1 defines morphisms det: $M_{D,d} \to M_{D,1}$, det: ${}_{D,d}M \to M_{D,1}$. They are morphisms of schemes over X^2.

8) Construction 5' from Sec. 1 defines an action on $M_{D,d}$ and ${}_{D,d}M$ of the group of isomorphism classes of invertible sheaves on X trivialized over D. We denote this group of $\mathrm{Pic}_D X$ (note that $\mathrm{Pic}_D X$ is the group of $\mathbf{F}_q$-points of the scheme $\underline{\mathrm{Pic}}_D X$). The action of $\mathrm{Pic}_D X$ preserves morphisms $M_{D,d} \to X^2$, ${}_{D,d}M \to X^2$.

9) Put $M_d = \varprojlim_D M_{D,d}$, ${}_dM = \varprojlim_D {}_{D,d}M$. We will define an action of the group $GL(d, \mathfrak{A})/k^*$ on M_d. Since M_d is an (actual) scheme of modules for the functor $\mathrm{Fsh}_d = \varprojlim_D \mathrm{Fsh}_{D,d}$, it suffices to define the action of $GL(d, \mathfrak{A})/k^*$ on Fsh_d. First, we will define an action of the semigroup $GL_-(d, \mathfrak{A}) = \{h \in GL(d, \mathfrak{A}) \mid h^{-1} \in \mathrm{Mat}(d, O)\}$, where $O \subset \mathfrak{A}$ is the ring of integral adeles. Suppose that $h \in GL_-(d, \mathfrak{A})$ is given along with an element of $\mathrm{Fsh}_d(S)$, i.e., a right F-sheaf $\mathcal{L}$ over S equipped with structures of all levels compatible with each other. Since $h \in GL(d, \mathfrak{A})$, we have $h^{-1}O^d \subset O^d$. There exists an open ideal $I \subset O$ such that $h^{-1}O^d \supset I \cdot O^d$. The ideal I is associated with a finite subscheme $D \subset X$ and the submodule $h^{-1}O^d/I \cdot O^d \subset (O/I)^d$ with a subsheaf $\mathcal{Q} \subset \mathcal{O}_D^d$. Applying Construction 6 from Sec. 1, we obtain an F-sheaf $\mathcal{L}'$, which, as can be easily seen, does not depend on the choice of I. In order to define on $\mathcal{L}'$ a structure of level D', where D' is an arbitrary finite subscheme in X, we choose I in such a way that $I \cdot O^d \subset J \cdot h^{-1}O^d$, where $J \subset O$ is the ideal corresponding to D'. The composition $h^{-1}O^d/I \cdot O^d \to h^{-1}O^d/J \cdot h^{-1}O^d \xrightarrow{h} (O/J)^d$ defines an epimorphism $\mathcal{Q} \to \mathcal{O}_{D'}^d$. Applying Construction 6' from Sec. 1, we obtain a structure of level D' on $\mathcal{L}'$ Thus, we have assigned to each $h \in GL_-(d, \mathfrak{A})$ a morphism $M_d \to M_d$. It is readily seen that we have obtained an action of $GL_-(d, \mathfrak{A})$ on M_d. On the other hand, the action of $\mathrm{Pic}_D X$ on $M_{D,d}$ turns, after taking the limit, into an action on M_d of the group $\varprojlim_D \mathrm{Pic}_D X = \mathfrak{A}^*/k^*$. It is easily seen that the restriction of this action to $\mathfrak{A}^* \cap GL_-(d, \mathfrak{A})$ coincides with the restriction of the action of $GL_-(d, \mathfrak{A})$. Since $GL(d, \mathfrak{A}) = \mathfrak{A}^* \cdot GL_-(d, \mathfrak{A})$, we obtain an action on M_d of the group $GL(d, \mathfrak{A})/k^*$. Similarly, one can define an action of this group on ${}_dM$.

Note that the action on M_d and ${}_dM$ of the subgroup $GL(d, O) \subset GL(d, \mathfrak{A})$ is obtained as a result of a limit passage over D from the action of $GL(d, A_D)$ on $M_{D,d}$ and ${}_{D,d}M$.

The action of $GL(d, \mathfrak{A})$ preserves morphisms $M_{D,d} \to X^2$, ${}_{D,d}M \to X^2$ and commutes with F_1 and F_2. Furthermore, for each $h \in GL(d, \mathfrak{A})$ the diagrams

$$\begin{array}{ccc} M_{D,d} & \xrightarrow{*} & {}_{D,d}M \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle (h^t)^{-1}} \\ M_{D,d} & \xrightarrow{*} & {}_{D,d}M \end{array} \qquad \begin{array}{ccccc} M_{D,d} & \xrightarrow{\det} & M_{D,1} & \xleftarrow{\det} & {}_{D,d}M \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \det h} & & \downarrow{\scriptstyle h} \\ M_{D,d} & \xrightarrow{\det} & M_{D,1} & \xleftarrow{\det} & {}_{D,d}M \end{array}$$

commute (here, h^t is the matrix transpose).

For each finite subscheme $D \subset X$ put $K_D = \mathrm{Ker}(GL(d, O) \to GL(d, A_D))$. Clearly, $K_D \setminus M_d = M_{D,d} \otimes_{X^2} (k \otimes k)$. Thus, the general fiber of the morphism $M_{D,d} \to X^2$ can be reconstructed if one knows the scheme M_d together with an action of $GL(d, \mathfrak{A})$. The information on closed fibers is lost in a passage to the limit in D. On the other hand, the passage to the limit in D is beneficial because it enables us to study the action of $GL(d, \mathfrak{A})$ instead of "Hecke correspondences" between the schemes $M_{D,d}$. In order to avoid loss of information on fibers of the schemes $M_{D,d}$ over a fixed closed point $u \in X^2$ and, at the same time, have an action of the adele group, it is convenient to introduce for each set T consisting of closed points of X the scheme $M_d^T = \varprojlim_{D \cap T = \varnothing} M_{D,d}$. In a manner similar to the action of $GL(d, \mathfrak{A})$ on M_d one can define an action on M_d^T of the group $GL(d, \mathfrak{A}^T)$, where $\mathfrak{A}^T$ is the adele ring without T-components. Here, elements of k^* having no zeros and poles at the points of T act trivially [it is assumed that $k^* \subset (\mathfrak{A}^T)^* \subset GL(d, \mathfrak{A}^T)$]. The natural morphism $M_d \to M_d^T$ is compatible with the action of $GL(d, \mathfrak{A}^T)$, and $M_d^T \otimes_{X^2} (k \otimes k) = G \setminus M_d$, where $G = \prod_{v \in T} GL(d, O_v)$. Of course, the above arguments are also true for varieties of modules of left F-sheaves.

<u>Remark</u>. The action on $M_{D,1}$ of the morphisms $*$, F_1, F_2 and of the group $\mathrm{Pic}_D X$ can be described quite explicitly. Namely, if points of $M_{D,1}(\bar{\mathbf{F}}_q)$ are written according to Proposition 3.1 in the form (α, β, a), where $\alpha, \beta \in (X - D)(\bar{\mathbf{F}}_q)$, $a \in \underline{\mathrm{Pic}}_D X(\bar{\mathbf{F}}_q)$, $\bar{\beta} - \bar{\alpha} = \mathrm{Fr}(a) - a$, then

$$*(\alpha, \beta, a) = (\beta, \alpha, -a), \tag{3.1}$$

$$F_1(\alpha, \beta, a) = (\mathrm{Fr}(\alpha), \beta, a - \bar{\alpha}), \quad F_2(\alpha, \beta, a) = (\alpha, \mathrm{Fr}(\beta), a + \bar{\beta}), \tag{3.2}$$

$$l(\alpha, \beta, a) = (\alpha, \beta, a + l), \quad l \in \mathrm{Pic}_D X. \tag{3.3}$$

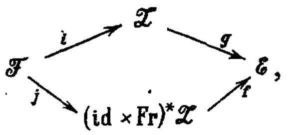
The action of $\mathrm{Pic}_D X$ on geometric fibers of the morphism $M_{D,1} \to (X - D)^2$ is free and transitive.

<u>Proposition 3.3</u>. There exists a number c such that if $|D| \geq dm - n + c$, then $M^{m,n}_{D,d}$ and ${}^{m,n}_{D,d}M$ are smooth schemes over X^2. The relative dimension of these schemes (if they are not empty) is equal to $2d - 2$.

Note that the fibers of the morphisms $M^{m,n}_{D,d} \to (X - D)^2$, ${}^{m,n}_{D,d}M \to (X - D)^2$ are, clearly, not empty if $dm \geq n$ (it suffices to consider F-sheaves of the form $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_d$, where $\mathcal{A}_1$ is an F-sheaf of rank 1 and $\mathcal{A}_2, \ldots, \mathcal{A}_d$ are inverse images of invertible sheaves on X with $\deg \mathcal{A}_i \leqslant m$ for all i and $\deg \mathcal{A}_1 + \ldots + \deg \mathcal{A}_d = n$).

<u>Proof</u>. Let c, $\mathrm{Bun}^{m,n}_{D,d}$, V, ψ_1, ψ_2 have the same meaning as in the proof of Proposition 3.2. Recall that $M^{m,n}_{D,d}$ is the preimage of the diagonal relative to the morphism $(\psi_1 \circ \mathrm{Fr}, \psi_2)$: $V \to \mathrm{Bun}^{m,n}_{D,d} \times \mathrm{Bun}^{m,n}_{D,d}$. A standard argument shows that $\mathrm{Bun}^{m,n}_{D,d}$ is a smooth variety. It is easily seen that V is also a smooth variety. Moreover, the morphisms (ψ_i, λ): $V \to \mathrm{Bun}^{m,n}_{D,d} \times X^2$, i = 1, 2, are smooth, where λ is the natural morphism $V \to X^2$. The fibers of these morphisms $V \to \mathrm{Bun}^{m,n}_{D,d} \times X^2$ have dimension $2d - 2$. This, along with the fact that the differential of the endomorphism Fr is equal to zero, implies the statement being proved for $M^{m,n}_{D,d}$. It is proved similarly for ${}^{m,n}_{D,d}M$. ∎

<u>Remarks</u>. 1) let $|D| \geq dm - n + c$. The birational isomorphism between $M^{m,n}_{D,d}$ and ${}^{m,n}_{D,d}M$ can be decomposed into σ-processes. To this end, consider "two-sided F-sheaves," i.e., commutative diagrams of the form



in which the left half is a left F-sheaf and the right half is a right F-sheaf with the same zero and pole as those of the left half. We denote by ${}^{m}_{D}M^{n}_{d}$ the analog of $M^{m,n}_{D,d}$ and ${}^{m,n}_{D,d}M$ for two-sided F-sheaves. It is easily deduced from the proof of Proposition 3.3 that the natural morphism ${}^{m}_{D}M^{n}_{d} \to M^{m,n}_{D,d}$ (respectively ${}^{m}_{D}M^{n}_{d} \to {}^{m,n}_{D,d}M$) is a σ-process with a center in a subscheme $Z \subset M^{m,n}_{D,d}$ (respectively, $Z' \subset {}^{m,n}_{D,d}M$) parametrizing F-sheaves $\mathcal{L}$, for which there exists an isomorphism $(\mathrm{id} \times \mathrm{Fr})^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}$, rendering the diagram (1.2) [respectively, (1.1)] commutative. It is easily seen that Z and Z' are smooth d-dimensional varieties, and the connected component of each of these varieties bijectively correspond to $\mathbf{F}_q$-points of the variety $\mathrm{Bun}^{m,n}_{D,d}$.

2) One can also construct the "lower bound" of $M^{m,n}_{D,d}$ and ${}^{m,n}_{D,d}M$, i.e., write the birational isomorphism between $M^{m,n}_{D,d}$ and ${}^{m,n}_{D,d}M$ in the form $\psi^{-1}\varphi$, where φ and ψ are proper morphisms from $M^{m,n}_{D,d}$ and ${}^{m,n}_{D,d}M$, respectively, into some variety U which are birational isomorphisms. In order to obtain U we have to introduce the following version of the notion of an F-sheaf over S: a d-dimensional locally free sheaf of $\mathcal{O}_{X \times S}$-modules $\mathcal{L}$, equipped with a morphism f: $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{L} \to \mathcal{L}(\Gamma_\beta)$, inducing an isomorphism $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \det \mathcal{L} \xrightarrow{\sim} (\det \mathcal{L})(\Gamma_\beta - \Gamma_\alpha)$ such that the rank of the restriction of f to Γ_β does not exceed 1. A point $u \in U$ is singular (for $d > 1$) if $\alpha = \beta$ and f induces the isomorphism $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}$. A formal neighborhood Φ of such a point is isomorphic to a formal neighborhood of the point $(0, 0) \in \mathbf{A}^1 \times Y$, where Y is the variety of matrices C of order d such that $\mathrm{rg}\, C \leq 1$. The natural morphism $\Phi \to X^2$ has the form $(z, C) \mapsto (z, z + \mathrm{Tr}\, C)$, where z is the coordinate in $\mathbf{A}^1$.

We will now show that in the absolute tangent bundle Θ to $M^{m,n}_{D,d}$, $|D| \geq dm - n + c$, whose dimension is equal to 2d, there is a natural d-dimensional subbundle Θ'. To this end, we note (see the proof of Proposition 3.3) that Θ is the restriction to $M^{m,n}_{D,d} \subset V$ of the relative tangent bundle of the morphism ψ_2: $V \to \mathrm{Bun}^{m,n}_{D,d}$. Recall that ψ_2 assigns to a diagram

$\mathcal{L}' \xrightarrow{f} \mathcal{E} \xleftarrow{g} \mathcal{L}$ the sheaf $\mathcal{L}'$. Thus, ψ_2 is written as a composition $V \xrightarrow{\varphi} W \xrightarrow{\psi} \mathrm{Bun}_{D,d}^{m,n}$, where W(S) = {d-dimensional locally free sheaves $\mathcal{E}$ on X × S trivialized over D × S with a selected subsheaf $\mathcal{L}' \subset \mathcal{E}$ such that $\mathcal{E}/\mathcal{L}'$ is an invertible sheaf on the graph of some morphism S → X − D and $\deg \mathcal{L}'_s = n$, $h(\mathcal{L}'_s) \leqslant m$ for all $s \in S$}. φ and ψ are smooth morphisms whose fibers are d-dimensional. So the relative tangent bundle of the morphism φ: V → W is a d-dimensional subbundle in the relative tangent bundle of the morphism ψ_2: $V \to \mathrm{Bun}_{D,d}^{m,n}$. The corresponding d-dimensional subbundle in Θ is denoted by Θ'. Similarly, one can define a d-dimensional distribution Θ" on ${}_{D,d}^{m,n}M$. It is easily seen that Θ' is invariant relative to GL(d, A_D), and the natural morphism $M_{D',d}^{m,n} \to M_{D,d}^{m,n}$, where D' ⊃ D, maps the distribution Θ' on $M_{D,d}^{m,n}$ into the distribution Θ' on $M_{D,d}^{m,n}$. This allows to define the distribution Θ' on M_d as well as on an open subscheme $U_{D,d} \subset M_{D,d}$ parametrizing F-sheaves with no nontrivial automorphisms identical over D. Similar statements are true for Θ". It is easily verified that: 1) the action of GL(d, $\mathfrak{A}$) on M_d (respectively, ${}_dM$) preserves Θ' (respectively, Θ"); 2) the kernel of the differential of the morphism ${}_{D,d}U \to U_{D,d}$ is equal to Θ" and its image is equal to Θ'; 3) the kernel of the differential of the morphism F_2: $U_{D,d} \to {}_{D,d}U$ is equal to Θ' and its image is equal to Θ"; 4) the morphism *: $U_{D,d} \rightleftarrows {}_{D,d}U$ maps Θ' onto Θ"; 5) the differential of the natural morphism $U_{D,d} \to X^2$ (respectively, ${}_{D,d}U \to X^2$) maps Θ' (respectively, Θ") into the relative tangent bundle of the first (respectively, second) projection $X^2 \to X$; 6) over the complement of the diagonal $\Delta \subset X^2$ (i.e., where $M_{D,d}^{m,n}$ does not differ from ${}_{D,d}^{m,n}M$ and, therefore, Θ' and Θ" are distributions on the same variety) Θ' is transversal to Θ".

Put $\Delta_r = (\mathrm{id}_X \times \mathrm{Fr}_X)^r(\Delta)$, where $\Delta \subset X^2$ is the diagonal, $\Lambda_i = X^2 - \bigcup_{|r| \leqslant i} \Delta_r$, $\Lambda = \varprojlim \Lambda_i$ (the projective limit exists because the morphisms $\Lambda_i \to X^2$ are affine).

The scheme Λ is Noetherian but not of finite type over $\mathbf{F}_q$. Put $\mathfrak{M}_{D,d} = M_{D,d} \times_{X^2} \Lambda = {}_{D,d}M \times_{X^2} \Lambda$. Similarly, we can define $\mathfrak{M}_{D,d}^n$, $\mathfrak{M}_{D,d}^{m,n}$, $\mathfrak{M}_d$. Henceforth, we will deal with $\mathfrak{M}_{D,d}$, rather than $M_{D,d}$ (this is sufficient for the proof of Langland's conjecture). Thus, we avoid problems related to the distinction between left and right F-sheaves while preserving all structures on schemes of modules listed above (in particular, there are no morphisms F_1: $\mathfrak{M}_{D,d} \to \mathfrak{M}_{D,d}$, F_2: $\mathfrak{M}_{D,d} \to \mathfrak{M}_{D,d}$, *: $\mathfrak{M}_{D,d} \rightleftarrows \mathfrak{M}_{D,d}$). Of course, effects associated with the diagonal in X^2 are also worth studying (an example of such an effect is the analytic theory of elliptic modules; see Secs. 3 and 6 in [5] and also [9, 19]).

For d = 2 there is another reason why it is more convenient to work with $\mathfrak{M}_{D,d}$ rather than with $M_{D,d}$ or ${}_{D,d}M$.

<u>Proposition 3.4</u>. Let h be an automorphism of an F-sheaf $\mathcal{L}$ of rank 2 over a field B which is the identity over D ⊗ B, where D is a nonempty subscheme in X. Suppose that $\mathrm{Fr}^m(\alpha) \neq \mathrm{Fr}^n(\beta)$ for all m, n, where $\alpha \in X(B)$ is the zero of $\mathcal{L}$, and $\beta \in X(B)$ is its pole. Then h = id.

This proposition will be proved in Sec. 4. It will imply, via standard reasoning, that if D ≠ ∅, then: 1) for each finite subscheme D' ⊃ D the group Ker(GL(2, $A_{D'}$) → GL(2, A_D)) acts freely on $\mathfrak{M}_{D',2}$ 2) $\mathfrak{M}_{D,2}$ is an actual scheme of modules for $\mathrm{Fsh}_{D,2} \times_{X^2} \Lambda$; 3) the statements on smoothness and structure of the tangent bundle stated above for $M_{D,d}^{m,n}$ and ${}_{D,d}^{m,n}M$ in the assumption that dm − n ≤ |D| − c are true for the entire scheme $\mathfrak{M}_{D,d}$ in the studied case of d = 2, D ≠ ∅.

<u>Remark</u>. Put $W = U_{D,d} \times_{X^2} \Lambda$ (if d = 2, D ≠ ∅, then $W = \mathfrak{M}_{D,d}$). The presence of "partial Frobenius endomorphisms" F_1, F_2: W → W and the decomposition of the tangent bundle to W into a direct sum of two d-dimensional subbundles mean that W resembles Y × Z, where Y and Z are schemes over X (in fact, in view of the existence of the automorphism *, even Y × Y). One can show that W is decomposed into a direct product at the "formal level." More exactly, the inverse image $\mathcal{J}$ of the sheaf of jets of functions (of finite order) on $M_{D,d}$ relative to the morphism $W \to M_{D,d}$ can be naturally written as a completed tensor product $\mathcal{J}_1 \hat{\otimes} \mathcal{J}_2$, where $\mathcal{J}_1$ and $\mathcal{J}_2$ are sheaves of algebras over $\mathcal{O}_W$. Here, F_1, F_2, and GL(d, $\mathfrak{A}$) preserve $\mathcal{J}_1$ and $\mathcal{J}_2$, and * reverses their places. Furthermore, the natural morphisms $F_1^*\mathcal{J}_2 \to \mathcal{J}_2$, $F_2^*\mathcal{J}_1 \to \mathcal{J}_1$ are isomorphisms.

4. Orispheric Curves

We fix an algebraically closed field B and points $\alpha \in X(B)$, $\beta \in X(B)$ such that the image of the morphism (α, β): $\operatorname{Spec} B \to X^2$ is contained in Λ, i.e.,

$$\mathrm{Fr}^n(\alpha) \neq \beta \text{ for each } n \in \mathbf{Z}. \tag{4.1}$$

Proposition 4.1. Let $\mathcal{L}$ be an F-sheaf of rank 1 over B with a zero at α and a pole at β. We denote by φ the embedding $(\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{L} \to \mathcal{L}(\beta)$. If $s \in H^0(X \otimes B, \mathcal{L})$, $\varphi((\mathrm{id}_X \times \mathrm{Fr}_B)^*s) = s$, then $s = 0$.

Henceforth, we write $\mathcal{L}(\beta)$ where we should write $\mathcal{L}(\Gamma_\beta)$ (cf. Sec. 1). In other words, we think of α and β as points of $X \otimes B$.

Proof. Suppose $s \neq 0$ and Y is the divisor of zeros of s. Then $(\mathrm{id}_X \times \mathrm{Fr}_B)^*Y + \alpha = \beta + Y$. But this equality (we emphasize that it involves divisors on $X \otimes B$ rather than classes of divisors) is impossible because, by (4.1), the number of points of the form $\mathrm{Fr}^n(\alpha)$, $n \in \mathbf{Z}$, in the left-hand part of the equality is greater by 1 than that in the right-hand part. ∎

Definition. An F-sheaf $\mathcal{L}$ of rank d over B with a zero at α and a pole at β is said to be reducible if there exists a nonzero subsheaf $\mathcal{A} \subset \mathcal{L}$ of rank $r < d$ such that the image of $(\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{A}$ in $\mathcal{L}(\beta)$ is contained in $\mathcal{A}(\beta)$.

Suppose that $\mathcal{L}$ is a reducible F-sheaf and the subsheaf $\mathcal{A}$, involved in the definition of reducibility is such that the sheaf $\mathcal{B} = \mathcal{L}/\mathcal{A}$ is locally free [note that if $\mathcal{B}$ is not locally free and $\mathcal{A}' \subset \mathcal{L}$ is the preimage of torsion of $\mathcal{B}$ under the homomorphism $\mathcal{L} \to \mathcal{B}$, then the image of $(\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{A}'$ in $\mathcal{L}(\beta)$ is contained in $\mathcal{A}'(\beta)$, so $\mathcal{A}$ can be replaced by $\mathcal{A}'$]. It is easily seen that one of two possibilities holds. Either $\mathcal{A}$ is an F-sheaf with a zero at α and a pole at β and the image of the morphism $(\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{B} \to \mathcal{B}(\beta)$ is equal to $\mathcal{B}$ and, therefore, $\mathcal{B}$ is the inverse image of a locally free sheaf $\mathcal{B}_0$ on X (see Proposition 1.1), or the image of the morphism $(\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{A} \to \mathcal{A}(\beta)$ is equal to $\mathcal{A}$ (so $\mathcal{A}$ is the inverse image of a locally free sheaf $\mathcal{A}_0$ on X) and $\mathcal{B}$ is an F-sheaf with a zero at α and a pole at β. An exact sequence $0 \to \mathcal{A} \to \mathcal{L} \to \mathcal{B} \to 0$ is called an F-decomposition of type 1 or 2 depending on which one of the two possibilities is materialized. Henceforth, we will consider the case of $d = 2$ (in this case $\mathcal{A}$ and $\mathcal{B}$ are invertible sheaves).

Proposition 4.2. Let $\mathcal{L}$ be an F-sheaf of rank 2 over B such that its zero α and its pole β satisfy (4.1). Then:

1) $\mathcal{L}$ possesses at most one F-decomposition of type 1 and at most one F-decomposition of type 2;

2) if an F-decomposition $0 \to \mathcal{A} \to \mathcal{L} \to \mathcal{B} \to 0$, is given, then each automorphism of the F-sheaf $\mathcal{L}$, inducing the identity automorphisms of the sheaves $\mathcal{A}$ and $\mathcal{B}$, is itself equal to the identity.

Proof. 1) Suppose F-decompositions $0 \to \mathcal{A} \to \mathcal{L} \to \mathcal{B} \to 0$ and $0 \to \mathcal{A}' \to \mathcal{L} \to \mathcal{B}' \to 0$, have the same type. The homomorphism $\mathcal{A} \to \mathcal{B}'$ determines a section $s \in H^0(X \otimes B, \mathcal{B}' \otimes \mathcal{A}^*)$. Since $\mathcal{B}' \otimes \mathcal{A}^*$ is an F-sheaf of rank 1 with a zero at α and a pole at β or with a zero at β and a pole at α (depending on the type of the F-decompositions) and s satisfies the hypothesis of Proposition 4.1, $s = 0$ and, therefore, $\mathcal{A} = \mathcal{A}'$.

2) Let h be an automorphism of $\mathcal{L}$, equal to the identity on $\mathcal{A}$ and $\mathcal{B}$. Then $h - 1$ determines a morphism $\mathcal{B} \to \mathcal{A}$ and, therefore, a section $s \in H^0(X \otimes B, \mathcal{B}^* \otimes \mathcal{A})$. By Proposition 4.1, $s = 0$ and, therefore, $h = 1$. ∎

Proof of Proposition 3.4. The trace and the determinant of the endomorphism $h - 1$ are functions on $X \otimes B$ equal to zero on $D \otimes B$. So they are equal to zero identically. Therefore, $(h - 1)^2 = 0$. Suppose that $h \neq 1$ and put $\mathcal{A} = \operatorname{Ker}(h - 1)$. Then the image of $(\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{A}$ in $\mathcal{L}(\beta)$ is contained in $\mathcal{A}(\beta)$ and the sheaf $\mathcal{L}/\mathcal{A}$ is invertible. It remains to apply statement 2) of Proposition 4.2 to the F-decomposition $0 \to \mathcal{A} \to \mathcal{L} \to \mathcal{L}/\mathcal{A} \to 0$. ∎

Let $P \in \Lambda(B)$ be the point determined by the pair (α, β) and $D \subset X$ a finite subscheme such that $\alpha, \beta \in (X - D)(B)$. The fiber of $M_{D,2}$ over P is denoted by $M_{D,P}$. We will identify the scheme $M_{D,P}$ with the set of its closed points. We will elucidate the structure of the set of points of $M_{D,P}$ corresponding to reducible F-sheaves. Suppose that a structure of level D is given on an F-sheaf $\mathcal{L}$ of rank 2 over B with a zero at α and a pole at β possessing an F-decomposition $0 \to \mathcal{A} \to \mathcal{L} \to \mathcal{B} \to 0$. Then the image of $\mathcal{A}_D$ under the isomorphism

$\mathcal{L}_D \rightrightarrows \mathcal{O}^2_{D\otimes B}$ is invariant relative to the isomorphism $(\mathrm{id}_D \times \mathrm{Fr}_B)^*\mathcal{O}^2_{D\otimes B} \simeq \mathcal{O}^2_{D\otimes B}$ and, therefore, is the inverse image of a one-dimensional free subsheaf of $\mathcal{O}_D$-modules $\mathcal{Q} \subset \mathcal{O}^2_D$. We tentatively fix an automorphism $h\colon \mathcal{O}^2_D \to \mathcal{O}^2_D$ such that $h(\mathcal{O}_D \oplus 0) = \mathcal{Q}$. Then h induces isomorphisms $\mathcal{O}_D \simeq \mathcal{Q}$, $\mathcal{O}_D \simeq \mathcal{O}^2_D/\mathcal{Q}$. If our F-decompositon has type 1, then $\mathcal{A}$ is an F-sheaf of rank 1 with a structure of level D and $\mathcal{B}$ is the inverse image of an invertible sheaf on X trivialized over D. Thus, we obtain an element $\omega \in \Omega^1_{D,P} \stackrel{\mathrm{def}}{=} [\mathrm{GL}(2, A_D) \times M_{D,1,P} \times \mathrm{Pic}_D X]/B_+(A_D)$, where $M_{D,1,P}$ is the fiber of $M_{D,1}$ over P, $B_+ \subset \mathrm{GL}(2)$ is the subgroup of upper triangular matrices, $B_+(A_D)$ is the set of A_D-points of the algebraic group B_+, and the right action of $B_+(A_D)$ on $M_{D,1,P} \times \mathrm{Pic}_D X$ is defined by the formula $(x,y)\cdot\begin{pmatrix} a_1 & b \\ 0 & a_2\end{pmatrix} = (\bar a_1^{-1}x, \bar a_2^{-1}y)$, where $x \in M_{D,1,P}$, $y \in \mathrm{Pic}_D X$, and $\bar a_i$ is the image of a_i under the natural homomorphism $A_D^* \to \mathrm{Pic}_D X$ (here, multiplicative notation of the group operation in $\mathrm{Pic}_D X$ is used temporarily). If the F-decomposition has type 2, then we obtain an element of the set $\Omega_{D,P}^2 = [\mathrm{GL}(2, A_D) \times M_{D,1,P} \times \mathrm{Pic}_D X]/B_+(A_D)$, differing from $\Omega_{D,P}^1$ by the fact that the action of $B_+(A_D)$ on $M_{D,1,P} \times \mathrm{Pic}_D X$ is defined by the formula $(x, y)\cdot\begin{pmatrix} a_1 & b \\ 0 & a_2\end{pmatrix} = (\bar a_2^{-1}x, \bar a_1^{-1}y)$, where $x \in M_{D,1,P}$, $y \in \mathrm{Pic}_D X$.

Let $\omega \in \Omega_{D,P}^1$ or $\omega \in \Omega_{D,P}^2$. The set of points of $M_{D,P}$ corresponding to a given ω is denoted by C_ω and called an orispheric curve of type 1, if $\omega \in \Omega_{D,P}^1$, or of type 2 if $\omega \in \Omega_{D,P}^2$. It will soon be proved (see Proposition 4.3) that these are indeed curves. The term "orispheric" has been borrowed from Chap. 3 in [1]. It follows from statement 1) of Proposition 4.2 that orispheric curves of the same type do not intersect each other. The union of all orispheric curves is the set of points of $M_{D,P}$ corresponding to reducible F-sheaves.

Proposition 4.3. 1) For each $\omega \in \Omega_{D,P}^1 \sqcup \Omega_{D,P}^2$ the set $C_\omega \subset M_{D,P}$ is closed and isomorphic to an affine line.

2) Suppose that $D \neq \emptyset$. Let $\Theta_P = \Theta_P' \oplus \Theta_P''$ be the decomposition of the tangent bundle Θ_P to $M_{D,P}$ induced by the decomposition $\Theta = \Theta' \oplus \Theta''$ considered in Sec. 3. Then C_ω is tangent to Θ_P' if $\omega \in \Omega_{D,P}^1$ and to Θ_P'' if $\omega \in \Omega_{D,P}^2$.

Note that C_ω is viewed as a reduced scheme.

Proof. If the F-sheaf $\mathcal{L}$ has an F-decomposition of type 2, then $\mathcal{L}^*$ has an F-decomposition of type 1. So the isomorphism $*\colon M_{D,P} \to M_{D,*(P)}$, where $*(P)$ is the point with the coordinates (β, α), maps orispheric curves of type 2 into orispheric curves of type 1. Thus, it suffices to prove the proposition for $\omega \in \Omega_{D,P}^1$. The actions of the groups $\mathrm{GL}(2, A_D)$ and $\mathrm{Pic}_D X$ reduce the argument to the case when ω corresponds to an element of $\mathrm{GL}(2, A_D) \times M_{D,1,P} \times \mathrm{Pic}_D X$ of the form $(E, \mathcal{A}, 0)$, where $\mathcal{A}$ is some F-sheaf of rank 1 with a structure of level D. Then C_ω is the set of F-sheaves $\mathcal{L}$ with a structure of level D possessing an F-decomposition

$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{L} \xrightarrow{g} \mathcal{O}_{X\otimes B} \to 0 \tag{4.2}$$

such that the diagram

$$\begin{array}{ccccccccc} 0 \to & \mathcal{A}_D & \xrightarrow{\quad f \quad} & \mathcal{L}_D & \xrightarrow{\quad g \quad} & \mathcal{O}_{D\otimes B} & \to 0 \\ & \downarrow\wr & & \downarrow\wr & & \| & \\ 0 \to & \mathcal{O}_{D\otimes B} & \xrightarrow{x \mapsto (x,0)} & \mathcal{O}^2_{D\otimes B} & \xrightarrow{(x,y)\mapsto y} & \mathcal{O}_{D\otimes B} & \to 0 \end{array}$$

commutes. If $D \neq \emptyset$, then the exact sequence (4.2) is unique, and if $D = \emptyset$, then there is arbitrariness in multiplying f and g by an element of $\mathbf{F}_q^*$. The kernel $\mathcal{L}'$ of the composition $\mathcal{L} \to \mathcal{L}_D \simeq \mathcal{O}^2_{D\otimes B} \to \mathcal{O}_{D\otimes B}$, where $\mathcal{O}^2_{D\otimes B} \to \mathcal{O}_{D\otimes B}$ is the projection onto the first factor, is an F-sheaf possessing an F-decomposition

$$0 \to \mathcal{A}(-D) \to \mathcal{L}' \to \mathcal{O}_{X\otimes B} \to 0. \tag{4.3}$$

Conversely, an element of C_ω is uniquely restored from an F-sheaf $\mathcal{L}'$ and an F-decomposition (4.6). The set of isomorphism classes of F-decompositions of the form (4.3) is made, in a usual way, into a vector space over $\mathbf{F}_q$, which we will denote by $\mathrm{Ext}_F(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D))$. Thus, we have constructed a bijection $\mathrm{Ext}_F(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D)) \to C_\omega$ for $D \neq \emptyset$ and $\mathrm{Ext}_F(\mathcal{O}_{X\otimes B}, \mathcal{A})/\mathbf{F}_q^* \to C_\omega$ for $D = \emptyset$.

Let $D' \supset D$ be a finite subscheme such that $\alpha, \beta \in (X - D')(B)$, and let $\omega' \in \Omega_{D',P}^1$ be one of the preimages of ω. Then the validity of the proposition for $C_{\omega'}$ implies its validity for C_ω. Indeed, C_ω is closed (as the image of $C_{\omega'}$ in $M_{D,P}$) and is isomorphic, as a scheme, to $H \setminus C_{\omega'}$, where $H \subset \mathrm{Ker}(GL(2, A_{D'}) \to GL(2, A_D))$ is the stationary subgroup of ω [the fact that $C_\omega \simeq H \setminus C_{\omega'}$ is not quite obvious for $D = \emptyset$ because, in this case, H does not act freely on $M_{D',P}$ and there could be the possibility that fibers of the natural morphism $H \setminus C_{\omega'} \to M_{D,P}$ have nilpotents; in order to exclude this possibility, it suffices to verify that the order of the stationary subgroup in $GL(2, A_D)$ of each point of $C_{\omega'}$ is not divisible by p, and this follows from Proposition 4.2].

Thus, we can assume without loss of generality that $D \neq \emptyset$ and $|D| \geq \deg \mathcal{A} + 2$. Then the natural map $\mathrm{Ext}_F(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D)) \to \mathrm{Ext}(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D))$ is an embedding. Indeed, if, for the exact sequence of $\mathcal{O}_{X\otimes B}$-modules $0 \to \mathcal{A}(-D) \to \mathcal{F} \to \mathcal{O}_{X\otimes B} \to 0$ there existed to distinct morphism $(\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{F} \to \mathcal{F}(\beta)$, inducing the given map $(\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{A}(-D) \to \mathcal{A}(\beta - D)$ and the identical inclusion $(\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{O}_{X\otimes B} \to \mathcal{O}_{X\otimes B}(\beta)$, then the difference of these morphisms would define a nonzero morphism $\mathcal{O}_{X\otimes B} = (\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{O}_{X\otimes B} \to \mathcal{A}(\beta - D)$, which is impossible because $\deg \mathcal{A}(\beta - D) < 0$. It is easily seen that the image of $\mathrm{Ext}_F(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D))$ in $\mathrm{Ext}(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D))$ is equal to $\mathrm{Ker}(\lambda - \psi)$, where λ is the natural map $\mathrm{Ext}(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D)) \to \mathrm{Ext}(\mathcal{O}_{X\otimes B}, \mathcal{A}(\beta - D))$, and ψ is the composition of the q-linear bijection $\mathrm{Ext}(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D)) \to \mathrm{Ext}(\mathcal{O}_{X\otimes B}, (\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{A}(-D))$ and the natural homomorphism $\mathrm{Ext}(\mathcal{O}_{X\otimes B}, (\mathrm{id}_X \times \mathrm{Fr}_B)^*\mathcal{A}(-D)) \to \mathrm{Ext}(\mathcal{O}_{X\otimes B}, \mathcal{A}(\beta - D))$.

LEMMA 1. Suppose that V and W are finite-dimensional vector spaces over B, $\dim V - \dim W = 1$, $\lambda: V \to W$ is a surjective linear map, $\psi: V \to W$ is a surjective q-linear map. Then the algebraic group $\mathrm{Ker}(\lambda - \psi)$ is isomorphic to $\mathbf{G}_a \times \mathbf{F}_q^r$, where r is the dimension over $\mathbf{F}_q$ of the space $\{\ell \in W^* / \ell(\lambda(x))^q = \ell(\psi(x)) \text{ for all } x \in V\}$.

Proof. We will show that V and W have bases $e_1, \ldots, e_s, v_1, \ldots, v_r$ and $f_1, \ldots, f_{s-1}, w_1, \ldots, w_r$, respectively, such that $\lambda(v_i) = \psi(v_i) = w_i$ and

$$\lambda(e_1) = 0,\ \lambda(e_{i+1}) = \psi(e_i) = f_i \text{ при } 1 \leqslant i \leqslant s-1,\ \psi(e_s) = 0. \tag{4.4}$$

The lemma is easily deduced from this.

We define subspaces $Y_n \subset V$ as follows: $Y_0 = 0$, $Y_{n+1} = \lambda^{-1}(\psi(Y_n))$. Clearly, $Y_n \subset Y_{n+1}$ and there exists s such that $Y_n = Y_s$ for $n \geq s$, $\dim Y_n = n$ for $n \leq s$. In Y_s and $\psi(Y_s)$ there exist bases $e_1', \ldots, e_s'$ and $f_1', \ldots, f_{s-1}'$, respectively, such that $\lambda(e_1') = 0$, $\lambda(e_{i+1}') = f_i$, $\psi(e_i) = f_i$ for $1 \leq i \leq s-1$. Putting $e_i = e_i' - \sum_{j=1}^{i-1} c_{j+s-i}^{q^{i-s-1}} e_j'$, $f_i = f_i' - \sum_{j=1}^{i-1} c_{j+s-i}^{q^{i-s}} f_j'$, where $c_1, \ldots, c_{s-1}$ are determined from the relationship $\psi(e_s') = \sum_{i=1}^{s-1} c_i f_i$, we make (4.4) hold. Since λ and ψ induce bijective maps $\bar{\lambda}, \bar{\psi}: V/Y_s \to W/\lambda(Y_s)$, there exists a basis $\bar{v}_1, \ldots, \bar{v}_r$ of the space V/Y_s such that $\lambda(\bar{v}_i) = \psi(\bar{v}_i)$ for all i. Since $\lambda - \psi$ induces a surjective map $Y_s \to \lambda(Y)$, $\bar{v}_i$ can be lifted to an element $v_i \in V$ such that $\lambda(v_i) = \psi(v_i)$. It remains to put $w_i = \lambda(v_i)$. ■

The maps $\lambda, \psi: \mathrm{Ext}(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D)) \to \mathrm{Ext}(\mathcal{O}_{X\otimes B}, \mathcal{A}(\beta - D))$ satisfy the hypotheses of Lemma 1. If $l \in \mathrm{Ext}(\mathcal{O}_{X\otimes B}, \mathcal{A}(\beta - D))^*$, $l(\lambda(x))^q = \ell(\psi(x))$ for all $x \in \mathrm{Ext}(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D))$, then the section of the F-sheaf $\mathcal{A}^{-1}(D - \beta) \otimes \Omega$ corresponding to ℓ by the Serre duality satisfies the condition of Proposition 2.1, whence $\ell = 0$. Thus, $\mathrm{Ext}_F(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D)) \simeq \mathbf{G}_a$.

It is easily seen that the map $f: \mathrm{Ext}_F(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D)) \to M_{D,P}$ constructed above is regular. A direct verification which we omit shows that the differential of f maps the tangent space to $x \in \mathrm{Ext}_F(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D))$ isomorphically onto the fiber of Θ_P' at the point f(x).

LEMMA 2. Let Y be a separable scheme over B, and let $f: \mathbf{A}^1 \to Y$ be a morphism injective in the set-theoretic sense and such that the differential of f at each point is also injective. Then either f is a closed embedding or f can be extended to a morphism $\mathbf{P}^1 \to Y$. ■

It remains to show that $f: \mathrm{Ext}_F(\mathcal{O}_{X\otimes B}, \mathcal{A}(-D)) \to M_{D,P}$ is not extended to a morphism $\tilde{f}: \mathbf{P}^1 \to M_{D,P}$. Otherwise, the point $\tilde{f}(\infty)$ would be fixed relative to $H = \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in A_D\right\}$, because $H \cdot C_\omega = C_\omega$. In this case the F-sheaf corresponding to $\tilde{f}(\infty)$ would possess a nonidentity automorphism h whose restriction to $D \otimes B$ would be unipotent. The impossibility of this situation is proved like Proposition 3.4. ■

Remarks. 1) It is readily seen that for an arbitrary F-sheaf $\mathcal{A}$ [it is not necessary that $H^0(X \otimes B, \mathcal{A}(\beta)) = 0$) $\mathrm{Ext}_F(\mathcal{O}_{X\otimes B}, \mathcal{A})$ are the one-dimensional hypercohomologies of the complex $0 \to \pi_*\mathcal{A} \to \pi_*\mathcal{A}(\beta) \to 0$, where π is the projection $X \otimes B \to X$ and the morphism $\pi_*\mathcal{A} \to$

$\pi_*\mathcal{A}(\beta)$ is the difference of the natural embedding and the composition $\pi_*\mathcal{A}\to\pi_*(\mathrm{id}_X\times\mathrm{Fr}_B)_*(\mathrm{id}_X\times\mathrm{Fr}_B)^*\mathcal{A}=\pi_*(\mathrm{id}_X\times\mathrm{Fr}_B)^*\mathcal{A}\to\pi_*\mathcal{A}(\beta)$.

2) It is easily seen that the groups $\mathrm{Ext}_F(\mathcal{O}_{X\otimes B},\mathcal{A}(-D))$, where $\mathcal{A}$ runs through the set of F-sheaves of rank 1 with a structure of level D in which the zero and the pole satisfy (4.1), are glued up into a group scheme G_D over $\mathfrak{M}_{D,1}$, and the morphisms $\mathrm{Ext}_F(\mathcal{O}_{X\otimes B},\mathcal{A}(-D))\to M_{D,P}$ (see the proof of Proposition 4.3) come from the morphism $f_D\colon G_D\to\mathfrak{M}_{D,2}$. One can show that the morphisms f_D, $D\neq\emptyset$, and the morphism $G_\emptyset/\mathbf{F}_q^*\to\mathfrak{M}_{\emptyset,2}$ are locally closed, but not closed, embeddings.

For all m, $n\in\mathbb{Z}$ we define ${}^{m,n}\Omega^i_{D,P}\subset\Omega^i_{D,P}$, i = 1, 2, as follows: ${}^{m,n}\Omega^1_{D,P}=[\mathrm{GL}(2,A_D)\times M^m_{D,1,P}\times\mathrm{Pic}^{n-m}_D X]/B_+(A_D)$, ${}^{m,n}\Omega^2_{D,P}=[\mathrm{GL}(2,A_D)\times M^{n-m}_{D,1,P}\times\mathrm{Pic}^m_D X]/B_+(A_D)$. This definition means that the union of the curves C_ω, $\omega\in{}^{m,n}\Omega^i_{D,P}$ consists of points $M_{D,P}$ which correspond to F-sheaves $\mathcal{L}$ of degree n possessing an F-decomposition $0\to\mathcal{A}\to\mathcal{L}\to\mathcal{B}\to 0$ of type i such that $\deg\mathcal{A}=m$.

<u>Proposition 4.4</u>. If $2m\geq n$, then $\mathfrak{M}^{m+1,n}_{D,|P|}-\mathfrak{M}^{m,n}_{D,P}$ is the union of disjoint curves C_ω, $\omega\in{}^{m+1,n}\Omega^1_{D,P}\bigsqcup{}^{m+1,n}\Omega^2_{D,P}$.

<u>Proof</u>. The point $\mathfrak{M}^{m+1,n}_{D,P}-\mathfrak{M}^{m,n}_{D,P}$ corresponds to an F-sheaf $\mathcal{L}$ of degree n having an invertible subsheaf $\mathcal{A}$ of degree m + 1 but no invertible subsheaves of higher degree. Since $\mathcal{A}$ is maximal, the sheaf $\mathcal{B}=\mathcal{L}/\mathcal{A}$ is invertible. The homomorphism $(\mathrm{id}_X\times\mathrm{Fr}_B)^*\mathcal{L}\to\mathcal{L}(\beta)$ induces the zero homomorphism $(\mathrm{id}_X\times\mathrm{Fr}_B)^*\mathcal{A}\to\mathcal{B}(\beta)$, because $\deg\mathcal{B}(\beta)=n-m<\deg\mathcal{A}$. So $0\to\mathcal{A}\to\mathcal{L}\to\mathcal{B}\to 0$ is an F-decomposition. If $\mathcal{A}'$ is a maximal invertible subsheaf in $\mathcal{L}$, distinct from $\mathcal{A}$, then the composition $\mathcal{A}'\to\mathcal{L}\to\mathcal{B}$ is distinct from zero and, therefore, $\deg\mathcal{A}'\leqslant\deg\mathcal{B}<m$. Thus, $\mathcal{A}$ is the only invertible subsheaf in $\mathcal{L}$ of degree m + 1. ■

It is easily seen that applying constructions 1-3, 5, 6 from Sec. 1 to reducible F-sheaves, and construction 2 reverses the type of an F-decomposition while the remaining constructions preserve it. The morphisms *, F_1, F_2 act on orispheric curves as follows. Put $F_1(P)=(\mathrm{Fr}(\alpha),\beta)$, $F_2(P)=(\alpha,\mathrm{Fr}(\beta))$, $*(P)=(\beta,\alpha)$. The endomorphisms *, F_1, F_2 of the scheme $\mathfrak{M}_{D,2}$ induce the morphisms $F_1\colon M_{D,P}\to M_{D,F_1'(P)}$, $F_2\colon M_{D,P}\to M_{D,F_2(P)}$, $*\colon M_{D,P}\to M_{D,*(P)}$. It is readily verified that $F_1(C_\omega)=C_{F_1(\omega)}$, $F_2(C_\omega)=C_{F_2(\omega)}$, $*(C_\omega)=C_{*(\omega)}$, where the map $F_i\colon\Omega^i_{D,P}\to\Omega^j_{D,F_i(P)}$, i, j = 1, 2 is induced by the map $F_i\colon M_{D,1,P}\to M_{D,1,F_i(P)}$, and the maps $*\colon\Omega^1_{D,P}\to\Omega^2_{D,*(P)}$, $*\colon\Omega^2_{D,P}\to\Omega^1_{D,*(P)}$ transform the class of a triple (h, x, y)$\in\mathrm{GL}(2,A_D)\times M_{D,1,P}\times\mathrm{Pic}_D X$ into the class of the triple $\left((h^t)^{-1}\cdot\begin{pmatrix}0&1\\-1&0\end{pmatrix},*(x),-y\right)$ (note that $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ could be replaced here by any matrix of the form $\begin{pmatrix}0&a\\b&0\end{pmatrix}$, $a,b\in\mathbf{F}_q^*$, because $\mathbf{F}_q^*$ lies in the kernel of the homomorphism $A_D^*\to\mathrm{Pic}_D X$).

We fix a finite set T of closed points of X such that the image of each morphism $\alpha^*\colon\mathrm{Spec}\,B\to X$, $\beta^*\colon\mathrm{Spec}\,B\to X$ either lies in T or is a common point of X. Put $M_P^T=\varprojlim M_{D,P}$, $M^T_{1,P}=\varprojlim M_{D,1,P}$, ${}^T\Omega^i_P=\varprojlim\Omega^i_{D,P}$, where D runs through the set of finite subschemes in X − T (the nonemptiness of these sets is guaranteed by the condition imposed on T). If $\omega\in{}^t\Omega^i_P$, then put $C_\omega=\varprojlim C_{\omega_D}$, where D runs through the set of finite subschemes in X − T and ω_D is the image of ω in $\Omega^i_{D,P}$. Since $\varprojlim_{D\subset X-T}\mathrm{Pic}_D X=(\mathfrak{A}^T)^*/\Gamma^T$, where Γ^T is the set of elements of k^* having no zeros or poles at the points of T, ${}^T\Omega^i_P=\{\mathrm{GL}(2,O^T)\times M^T_{1,P}\times[(\mathfrak{A}^T)^*/\Gamma^T]\}/B_+(O^T)$, where $O^T=\prod_{v\in T}O_v$, and the right action of $B_+(O^T)$ on $M^T_{1,P}\times[(\mathfrak{A}^T)^*/\Gamma^T]$ is given by the formula

$$(x,y)\cdot\begin{pmatrix}a_1&b\\0&a_2\end{pmatrix}=\begin{cases}(a_1^{-1}x,a_2^{-1}y), & i=1,\\(a_2^{-1}x,a_1^{-1}y), & i=2.\end{cases}\tag{4.5}$$

Since $\mathrm{GL}(2,O^T)\cdot B_+(\mathfrak{A}^T)=\mathrm{GL}(2,\mathfrak{A}^T)$, we have

$${}^T\Omega^i_P=\{\mathrm{GL}(2,\mathfrak{A}^T)\times M^T_{1,P}\times[(\mathfrak{A}^T)^*/\Gamma^T]\}/B_+(\mathfrak{A}^T),\tag{4.6}$$

where the action of $B_+(\mathfrak{A}^T)$ on $M^T_{1,P}\times[(\mathfrak{A}^T)^*/\Gamma^T]$ is still given by the formula (4.5). It is readily verified that the left action of $\mathrm{GL}(2,\mathfrak{A}^T)$ on ${}^T\Omega^i_P$ defined by formula (4.6) is "regular" in the sense that $C_{h\omega}=h\cdot C_\omega$ for $\omega\in{}^T\Omega^i_P$, $h\in\mathrm{GL}(2,\mathfrak{A}^T)$.

Recall that the action of $\mathrm{Pic}_D X$ on $M_{D,1,P}$ and thus also the action of $(\mathfrak{A}^T)^*/\Gamma_T$ on $M^T_{1,P}$ is free and transitive. So fixing a point $z \in M^T_{1,P}$ provides the isomorphism $M^T_{1,P} \xrightarrow{\sim} (\mathfrak{A}^T)^*/\Gamma^T$, and thus also the isomorphism

$$ {}^T\Omega^i_P \xrightarrow{\sim} \mathrm{GL}(2, \mathfrak{A}^T) \Big/ \left\{ \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \Big| \, a_1, a_2 \in \Gamma^T, b \in \mathfrak{A}^T \right\}. \tag{4.7} $$

Suppose now that α^* and β^* map Spec B into a common point of X and $T = \emptyset$. Instead of $M^{\emptyset}_P, M^{\emptyset}_{1,P}, {}^{\emptyset}\Omega^i_P$ we will write M_P, $M_{1,P}$, Ω^i_P. We denote by $\mathrm{Cr}_{D,P}$ the set of points of $M_{D,P}$ lying in two orispheric curves simultaneously. One of these curves is of type 1, another is of type 2, so we have maps $\mathrm{Cr}_{D,P} \to \Omega^i_{D,P}$, $i = 1, 2$. Put $\mathrm{Cr}_P = \varprojlim \mathrm{Cr}_{D,P}$, where D runs through the set of finite subschemes in X.

Proposition 4.5. There is a GL $(2, \mathfrak{A})$-equivariant bijection $\mathrm{Cr}_P \xrightarrow{\sim} \{\mathrm{GL}(2, \mathfrak{A}) \times M_{1,P} \times (\mathfrak{A}^*/k^*)\}/H$, where $H \subset \mathrm{GL}(2, \mathfrak{A})$ is the group of diagonal matrices and the right action of H on $M_{1,P} \times (\mathfrak{A}^*/k^*)$ is given by the formula $(x, y)\cdot\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = (a^{-1}x, b^{-1}y)$. The natural map $\{\mathrm{GL}(2, \mathfrak{A}) \times M_{1,P} \times (\mathfrak{A}^*/k^*)\}/H = \mathrm{Cr}_P \to \Omega^i_P = \{\mathrm{GL}(2, \mathfrak{A}) \times M_{1,P} \times (\mathfrak{A}^*/k^*)\}/B_+(\mathfrak{A})$ transforms the class of the triple $(h, x, y) \in \mathrm{GL}(2, \mathfrak{A}) \times M_{1,P} \times (\mathfrak{A}^*/k^*)$ to the class of the triple (h, x, y) for $i = 1$ and to the class of the triple $\left(h\cdot\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, x, y\right)$ for $i = 2$.

Proof. If we have an F-sheaf $\mathcal{A}$ of rank 1 over B with a structure of level D and an invertible sheaf $\mathcal{B}_0$ on X trivialized over D, then $\mathcal{A} \oplus \mathcal{B}$, where $\mathcal{B}$ is the inverse image of $\mathcal{B}_0$ on $X \otimes B$, is an F-sheaf with a structure of level D. Here, $\mathcal{A} \oplus \mathcal{B}$ possesses F-decompositions of both types. A map $M_{D,1,P} \times \mathrm{Pic}_D X \to \mathrm{Cr}_{D,P}$ arises. Upon a passage to the limit with respect to D, we obtain $f: M_{1,P} \times (\mathfrak{A}^*/k^*) \to \mathrm{Cr}_P$. It is easily seen that f is injective and the equality $hx = y$, where $x, y \in \mathrm{Im}\, f$, $h \in \mathrm{GL}(2, \mathfrak{A})$, implies that $h \in H$. It remains to show that if $x \in \mathrm{Cr}_P$, then there exists $h \in \mathrm{GL}(2, \mathfrak{A})$ such that $hx \in \mathrm{Im}\, f$. Suppose that the point x corresponds to an F-sheaf $\mathcal{L}$, having F-decompositions $0 \to \mathcal{A} \to \mathcal{L} \to \mathcal{B} \to 0$ and $0 \to \mathcal{A}' \to \mathcal{L}' \to \mathcal{B}' \to 0$ of types 1 and 2, respectively. We denote by Δ the support of the sheaf $\mathcal{L}/(\mathcal{A} \oplus \mathcal{A}') = \mathrm{Coker}(\mathcal{A}' \to \mathcal{B})$. Since $\mathcal{A} \neq \mathcal{A}'$, the scheme Δ is finite, and the commutativity of the diagram

$$ \begin{array}{ccc} (\mathrm{id}_X \times \mathrm{Fr}_B)^* \mathcal{A}' & \xrightarrow{\sim} & \mathcal{A}', \\ \downarrow & & \downarrow \\ (\mathrm{id}_X \times \mathrm{Fr}_B)^* \mathcal{B} & \xrightarrow{\sim} & \mathcal{B} \end{array} $$

implies that Δ is invariant relative to $\mathrm{id}_X \times \mathrm{Fr}_B$. So $\Delta = D \otimes B$, where D is a finite subscheme in X. We have: $\mathcal{L}/(\mathcal{A} \oplus \mathcal{A}') = \mathcal{L}_D/\mathcal{R}$, where $\mathcal{R}$ is a subsheaf of $\mathcal{L}_D$, invariant relative to $\mathrm{id}_D \times \mathrm{Fr}_B$. The image of $\mathcal{R}$ under the isomorphism $\mathcal{L}_D \xrightarrow{\sim} \mathcal{O}^2_{D\otimes B}$, defining the structure of level D on $\mathcal{L}$, is the inverse image of some subsheaf $\mathcal{Q} \subset \mathcal{O}^2_D$. So for some $g \in \mathrm{GL}(2, \mathfrak{A})$ the point gx corresponds to the F-sheaf $\mathcal{A} \oplus \mathcal{A}'$. ∎

Remark. Fixing a point $z \in M_{1,P}$ we obtain a bijection

$$ \mathrm{Cr}_P \xrightarrow{\sim} \mathrm{GL}(2, \mathfrak{A}) \Big/ \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \Big| \, a_1, a_2 \in k^* \right\}. \tag{4.8} $$

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KÄHLER STRUCTURES ON K-ORBITS OF THE GROUP OF DIFFEOMORPHISMS OF A CIRCLE

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1. Formulation of the Problem

Among the infinite-dimensional symplectic manifolds which arise as orbits of the coadjoint representation of an infinite-dimensional Lie group, one of the simplest and at the same time most important examples is the manifold (cf. [2, 3])

$$M = \mathrm{Diff}_+ (S^1)/\mathrm{Rot}\,(S^1). \qquad (1)$$

Here $\mathrm{Diff}_+(S^1)$ means the subgroup of diffeomorphisms of the unit circle S^1, preserving orientation, and $\mathrm{Rot}(S^1)$ is the subgroup of rotations of the circle. I have more than once already stated the conjecture that on M there exists a $\mathrm{Diff}_+(S^1)$-invariant complex structure, which together with the symplectic structure on M turns M into a homogeneous Kähler manifold. It will be shown below that this is really so.

The original version of this paper is contained in [1].

We recall how one usually constructed a complex structure on a homogeneous manifold X = G/H, where G and H are ordinary (finite-dimensional) Lie groups (cf. [4]). Let x_0 be the initial point in X, corresponding to the coset H, $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the groups G and H, respectively; $\mathfrak{g}^c$ and $\mathfrak{h}^c$ be their complexifications. The space $\mathfrak{g}^c/\mathfrak{h}^c$ can be identified naturally with the complexification $T_{x_0}{}^cX$ of the tangent space to X at x_0. The group H acts on this space.

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